Complexity of the isomorphism problem for subshifts

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Subshifts

Definition

Let A be a finite set of symbols. A *one-dimensional subshift on* A is a closed subset of $A^{\mathbb{Z}}$ which is invariant under the left shift operator S, where $S(x)(n) = x(n+1)$. Two subshifts X on A and Y on B are *isomorphic* if there is a homeomorphism $\varphi : X \to Y$ which commutes with S.

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A subshift may be defined by a set of *forbidden words* $W \subseteq A^{<\mathbb{N}}$ (where $A^{<\mathbb{N}}$ is the set of finite sequences from A), where W determines the subshift

$$X_W = \{x \in A^{\mathbb{Z}} : \forall w \in W \ (w \not\sqsubseteq x)\}$$

(and $w \sqsubseteq x$ means that w occurs as a subword of x).
A subshift is said to be of *finite type* when W is finite.

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- One can ask how complicated a set of complete invariants for isomorphism needs to be.
- We will consider this equivalence relation from the standpoint of descriptive set theory, and determine its complexity among Borel equivalence relations under the relation of Borel reducibility, $\leq_B$.
- This will allow us to gauge the difficulty of this classification problem, and compare it to other classification problems.

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We say that an equivalence relation E on a space X is *Borel reducible* to an equivalence relation F on the space Y, $E \leq_B F$, if there is a Borel-measurable function $f : X \to Y$ such that for all $x_1, x_2 \in X$ we have $x_1 \; E \; x_2$ if and only if $f(x_1) \; F \; f(x_2)$.
We write $E \sqsubseteq_B F$ when the we can find such an f which is injective.

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We write $E \sqsubseteq_B F$ when the we can find such an f which is injective.

Note that having $E \leq_B F$ amounts to having a definable injection from the quotient space X/E into the quotient Y/F, so that when $E \leq_B F$ we may view the relation F as being at least as complicated as E.

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- If a complicated equivalence relation E is Borel reducible to an equivalence relation F, then there can be no set of complete invariants for F which are fundamentally simpler a set of complete invariants for E. Hence F is at least as complicated as E.

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- If a complicated equivalence relation E is Borel reducible to an equivalence relation F, then there can be no set of complete invariants for F which are fundamentally simpler a set of complete invariants for E. Hence F is at least as complicated as E.
- Several canonical examples of equivalence relations are well-understood, and these can be used to gauge the complexity of other relations.

Smooth equivalence relations

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An equivalence relation E on X is *smooth* if it is Borel reducible to the identity relation on some Polish space Y, i.e. there is a Borel-measurable function $f : X \to Y$ such that $x_1 \mathrel{E} x_2$ if and only if $f(x_1) = f(x_2)$.

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- For the sort of equivalence relations we consider here, being smooth is equivalent to admitting a Borel selector which picks one representative from each equivalence class.
- Smooth equivalence relations are the simplest equivalence relations, and are also referred to as "concretely classifiable."

Hyperfinite equivalence relations

- A canonical example of a non-smooth countable Borel equivalence relation is the relation E_0 of eventual equality on the Cantor space $2^{\mathbb{N}}$ defined by setting $x \, E_0 \, y$ iff $x(n) = y(n)$ for all but finitely many n.

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A universal countable Borel equivalence relations is of maximum complexity among countable Borel equivalence relations. In particular, it is not smooth.

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Our main result:

Theorem (C.)

Isomorphism of one-dimensional subshifts is a universal countable Borel equivalence relation.

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Our main result:

Theorem (C.)

Isomorphism of one-dimensional subshifts is a universal countable Borel equivalence relation.

In particular it is not smooth, and it will not admit complete invariants fundamentally simpler than the equivalence classes under isomorphism.

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These are more complicated than countable Borel equivalence relations.

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- Isometry of Polish metric spaces is universal for orbit equivalence relations of actions of Polish groups (C., Gao-Kechris).
- Conjugacy of Borel automorphisms is more complicated than any Borel equivalence relation (C.) or analytic equivalence relation (Gao).

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Proof.

It suffices to observe that the function $\tilde{S} : K(n^{\mathbb{Z}}) \to K(n^{\mathbb{Z}})$ given by $\tilde{S}(X) = S[X]$ is a homeomorphism, since then $\mathcal{F}_n^S = \{X \in K(n^{\mathbb{Z}}) : \tilde{S}(X) = X\}$ is closed. □

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$$X_W = \{x \in n^{\mathbb{Z}} : \forall i \; (W(i) \neq \emptyset \Rightarrow W(i) \not\sqsubseteq x)\}.$$

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- Every subshift on n is represented by an element $W \in Z_n$.
- Several different sets of forbidden words may produce the same subshift; we wish to see that this is not problematic.

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Since $W_1 \sim W_2$ if and only if $F(W_1) = F(W_2)$, we thus have:

Corollary

The relation $\sim$ is a smooth Borel equivalence relation on Z_n. $\square$

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Hence it will not matter which representation we use for subshifts (from the standpoint of Borel reducibility).

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A basic result which we will utilize is the following:

Theorem (Curtis-Hedlund-Lyndon)

Suppose X and Y are subshifts on A and B, respectively, and $\varphi : X \to Y$ is a morphism of subshifts, i.e., a map commuting with the shift. Then φ is given by a block code, i.e., there is a natural number r and a function $\pi : A^{\{-r,\dots,r\}} \to B$ such that

$$\varphi(x)(n) = \pi(S^n(x) \restriction \{-r, \dots, r\}).$$

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Note that there are only countably many such block codes, and any isomorphism of subshifts is computable.

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$$\begin{aligned} X\,E\,Y \Leftrightarrow\ & \exists\pi\exists\tilde{\pi}\forall x \in n^{\mathbb{Z}}\ [(x \in X \Rightarrow f_\pi(x) \in Y)\wedge \\ & (x \in Y \Rightarrow f_{\tilde{\pi}} \in X) \wedge (x \in X \Rightarrow f_{\tilde{\pi}} \circ f_\pi(x) = x)] \\ \Leftrightarrow\ & \exists\pi\exists\tilde{\pi}\neg\exists x \in n^{\mathbb{Z}}\ [(x \in X \setminus f_\pi^{-1}[Y])\vee \\ & (x \in Y \setminus F_{\tilde{\pi}}^{-1}[X]) \vee (x \in X \wedge f_{\tilde{\pi}}(f_\pi(x)) \neq x)] \end{aligned}$$

The matrix is K_σ so the projection is also K_σ; as π and $\tilde{\pi}$ range over a countable set this relation is Σ_3^0 and hence Borel. □

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- We will need the following technical lemma:

Lemma

There is an $\mathbb{F}_2$-invariant Borel set $A \subseteq 2^{\mathbb{F}_2}$ such that:

- $E(\mathbb{F}_2, 2) \sqsubseteq_B E(\mathbb{F}_2, 2) \restriction A$*, so* $E(\mathbb{F}_2, 2) \restriction A$ *is a universal countable Borel equivalence relation.*
- *For* $x, y \in A$ *with* $\neg x \; E(\mathbb{F}_2, 2) \; y$*, there are infinitely many* $g \in \mathbb{F}_2$ *with* $x(g) \neq y(g)$*.* □

Complexity of the isomorphism relation (cont.)

Theorem

The isomorphism relation on one-dimensional subshifts is a universal countable Borel equivalence relation.

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- We will show that $E(\mathbb{F}_2, 2) \restriction A \leq_B E$, where A is the set from the previous lemma and E is the isomorphism relation for subshifts on the alphabet 2.

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The isomorphism relation on one-dimensional subshifts is a universal countable Borel equivalence relation.

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- Given a point $x \in 2^{\mathbb{F}_2}$, we will produce a subshift $\varphi(x) = X_x$.
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- Let $s \frown t$ denote the concatenation of two finite sequences s and t, and let $|s|$ be the length of a finite sequence.

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- Let $s \frown t$ denote the concatenation of two finite sequences s and t, and let $|s|$ be the length of a finite sequence.
- Let $\mathbb{F}_2$ be generated by a and b.

Complexity of the isomorphism relation (cont.)

Proof (cont.).

Define a map $\rho : \mathbb{F}_2 \to 2^{<\mathbb{N}}$ by letting:

$$\begin{aligned}
\rho(e) &= 11000011 \\
\rho(a) &= 11100011 \\
\rho(a^{-1}) &= 11010011 \\
\rho(b) &= 11001011 \\
\rho(b^{-1}) &= 11000111
\end{aligned}$$

$$\begin{aligned}
\rho(w) = \rho(w_0) \frown \cdots \frown \rho(w_n) \\
&\text{for } w = w_0 \frown \cdots \frown w_n \neq e \text{ a reduced word} \\
&\text{with each } w_i \in \{a, a^{-1}, b, b^{-1}\} \text{ and } w_{i+1} \neq w_i^{-1}.
\end{aligned}$$

Note that for each $w \in \mathbb{F}_2$, $|\rho(w)| = 8k$ for some $k > 0$.

Complexity of the isomorphism relation (cont.)

Proof (cont.).

- Define the map $d : 2 \to 2^{<\mathbb{N}}$ by:

$$d(0) = 101$$
$$d(1) = 111.$$

  Note that both of these sequences have length 3.

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- Let $\bar{s}$ be the periodic element of $2^{\mathbb{Z}}$ with period $|s|$ induced by s (starting at coordinate 0).

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- Let $\bar{s}$ be the periodic element of $2^{\mathbb{Z}}$ with period $|s|$ induced by s (starting at coordinate 0).
- Finally, let $p(i)$ denote the i-th prime.

Complexity of the isomorphism relation (cont.)

Proof (cont.).

For an element $x \in 2^{\mathbb{F}^2}$ define the countable set A_x:

$$A_x = \{\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\} \cup \\ \{\overline{d(0) \frown 0^{p(2n+2)-3}} : n \in \mathbb{N}\} \cup \\ \{\overline{d(1) \frown 0^{p(2n+3)-3}} : n \in \mathbb{N}\}$$

Let $\varphi(x) = X_x$ where:

$$X_x = \overline{\bigcup_{n \in \mathbb{Z}} S^n[A_x]}.$$

Thus, X_x is the smallest subshift containing A_x.

Complexity of the isomorphism relation (cont.)

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Thus, X_x is the smallest subshift containing A_x.
It is straightforward to check that φ is Borel, using the previous lemma about the map $W_x \mapsto X_x$.

Complexity of the isomorphism relation (cont.)

Proof (cont.).

It remains to check that φ witnesses that $E(\mathbb{F}_2, 2) \restriction A \leq_B E$.

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It remains to check that φ witnesses that $E(\mathbb{F}_2, 2) \upharpoonright A \leq_B E$. Recall A_x:

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The periodic points of X_x are (shifts of) the following points:

$\bar{0}$	period 1
$\overline{d(0) \frown 0^{p(2n+2)-3}}$	period $p(2n+2)$
$\overline{d(1) \frown 0^{p(2n+3)-3}}$	period $p(2n+3)$
$\overline{\rho(w)}$ for $w \in \mathbb{F}_2$	period $8 \cdot \lvert w \rvert$ (where $\lvert e \rvert = 1$)

Complexity of the isomorphism relation (cont.)

Proof (cont.).

For every x, X_x also includes the following limit points:

$$\bar{0} \frown d(0) \frown \bar{0}$$

$$\bar{0} \frown d(1) \frown \bar{0}$$

$$\bar{0} \frown \rho(w) \frown \bar{0} \quad \text{for } w \in \mathbb{F}_2$$

all left-, right- or bi-infinite sequences of $\rho(w)$'s

(with other coordinates 0).

Depending on the orbit of x (for various i, k and right-infinite words w^* from $\mathbb{F}_2$) it may also include points of the form:

$$\bar{0} \frown d(i) \frown 0^{10+k^2} \frown \rho[w^*].$$

Complexity of the isomorphism relation (cont.)

Proof (cont.).

Suppose first that $x, y \in 2^{\mathbb{F}_2}$ with $x \; E(\mathbb{F}_2, 2) \; y$.
We show that $(X_x, S) \cong (Y_x, S)$ (we do not need $x, y \in A$).

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- Then we have:

$$A_x = P \cup \{\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\}$$

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Proof (cont.).

Compare this to A_y when $y = a \cdot x$:

$$
\begin{aligned}
A_y &= P \cup \{\bar{0} \frown d(w_i \cdot y(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\} \\
&= P \cup \{\bar{0} \frown d(w_i a \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\} \\
&= P \cup \{\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i a^{-1}) \frown \bar{0} : i, k \in \mathbb{N}\} \\
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\end{aligned}$$

- We want a map from A_x to A_y which sends each point

$$\begin{aligned}
&\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} \qquad \mapsto \\
&\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i a^{-1}) \frown \bar{0}.
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\end{aligned}$$

- All other sequences are left unchanged.

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\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i a^{-1}) \frown \bar{0}.&
\end{aligned}$$

- All other sequences are left unchanged.
- This induces an S-invariant homeomorphism of X_x with X_y.

Complexity of the isomorphism relation (cont.)

Proof (cont.).

We define a block code $\pi : 2^{\{-r,\ldots,r\}} \to 2$ with $r = 16$ so that f_π is an isomorphism of X_x and X_y.

Complexity of the isomorphism relation (cont.)

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We want the following subsequences mapped as shown:

$$
\begin{array}{rcl}
0 \frown \rho(e) \frown 0 & \mapsto & 0 \frown \rho(a^{-1}) \frown 0 \\
0 \frown \rho(a) \frown 0 & \mapsto & 0 \frown \rho(e) \frown 0 \\
\rho(a) \frown \rho(a) \frown 0 & \mapsto & \rho(a) \frown 0^8 \frown 0 \\
\rho(b) \frown \rho(a) \frown 0 & \mapsto & \rho(b) \frown 0^8 \frown 0 \\
\rho(b^{-1}) \frown \rho(a) \frown 0 & \mapsto & \rho(b^{-1}) \frown 0^8 \frown 0 \\
\rho(a^{-1}) \frown 0^8 \frown 0 & \mapsto & \rho(a^{-1}) \frown \rho(a^{-1}) \frown 0 \\
\rho(b) \frown 0^8 \frown 0 & \mapsto & \rho(b) \frown \rho(a^{-1}) \frown 0 \\
\rho(b^{-1}) \frown 0^8 \frown 0 & \mapsto & \rho(b^{-1}) \frown \rho(a^{-1}) \frown 0
\end{array}
$$

and all other sequences should be left unchanged.

Complexity of the isomorphism relation (cont.)

Proof (cont.).

- None of the listed sequences overlaps a shift of another.

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- None of the listed sequences overlaps a shift of another.
- Each pair above have the same length.
- So our requirements can be ensured by choosing the block code π appropriately.
- Taking $r = 16$ is sufficient for each image digit to "see enough" of the input to tell if and wherein it is part of one of the listed sequences.
- This gives a shift-invariant homeomorphism of A_x with A_y which extends to an isomorphism of the subshifts X_x and X_y.

Complexity of the isomorphism relation (cont.)

Proof (cont.).

For the other direction, suppose $x, y \in A$ and $X_x \cong X_y$ via f_π induced by the block code $\pi : 2^{\{-r,\dots,r\}} \to 2$.

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The periods of the periodic points must be preserved by f_π, so:

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We will show that $x \, E(\mathbb{F}_2, 2) \, y$.
The periods of the periodic points must be preserved by f_π, so:

- $f_\pi(\bar{0}) = \bar{0}$ as this is the unique point of period 1.
  Hence $\pi(0^{2r+1}) = 0$.

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  Hence $\pi(0^{2r+1}) = 0$.
- Considering the points of P for large n we have:

$$f_\pi(\bar{0} \frown d(0) \frown \bar{0}) = S^{n(0)}(\bar{0} \frown d(0) \frown \bar{0})$$
$$f_\pi(\bar{0} \frown d(1) \frown \bar{0}) = S^{n(1)}(\bar{0} \frown d(1) \frown \bar{0})$$

  for some $n(0), n(1) \in \{-r, \dots, r\}$.

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  for some $n(0), n(1) \in \{-r, \ldots, r\}$.
- There is also a sequence s_0 and $m_0 \in \{-r, \ldots, r\}$ such that $f_\pi(\bar{0} \frown \rho(e) \frown \bar{0}) = S^{m_0}(\bar{0} \frown s_0 \frown \bar{0})$.

Complexity of the isomorphism relation (cont.)

Proof (cont.).

For sufficiently large k (i.e. $k > r$), consider the points:

$$\bar{0} \frown d(x(w_k)) \frown 0^{10+k^2} \frown \rho(e) \frown \bar{0} \text{ in } X_x.$$

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These must map to (shifts of) the points:

$$\bar{0} \frown d(x(w_k)) \frown 0^{10+k^2+n(x(w_k))-m_0} \frown s_0 \frown \bar{0} \text{ in } X_y.$$

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$$\bar{0} \frown d(x(w_k)) \frown 0^{10+k^2} \frown \rho(e) \frown \bar{0} \text{ in } X_x.$$

These must map to (shifts of) the points:

$$\bar{0} \frown d(x(w_k)) \frown 0^{10+k^2+n(x(w_k))-m_0} \frown s_0 \frown \bar{0} \text{ in } X_y.$$

Each of these must be a shift of a point of the form:

$$\bar{0} \frown d(w_{i_k} \cdot y(w_{j_k})) \frown 0^{10+{j_k}^2} \frown \rho(w_{i_k}) \frown \bar{0} \text{ with } i_k, j_k \in \mathbb{N}.$$

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Thus $\rho(w_{i_k}) = s_0$ for sufficiently large k, so there is some fixed $w \in \mathbb{F}_2$ such that $s_0 = \rho(w)$.

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- Since $\rho(w_{i_k}) = s_0 = \rho(w)$ for sufficiently large k, we must have $w_{i_k} = w$ for all but finitely many k.

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Hence $y \, E(\mathbb{F}_2, 2) \, x$ and we are done. □

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Definition

An *n-dimensional subshift* on an alphabet A is a closed subset of $A^{\mathbb{Z}^n}$ which is invariant under the shift maps $S_1, \ldots, S_n$, where $S_k(x)(i_1, \ldots, i_n) = x(i_1, \ldots, i_{k-1}, i_k + 1, i_{k+1}, \ldots, i_n)$.

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- The reverse is also true:

Higher dimensional subshifts (cont.)

Theorem

For each $n \geq 1$, we have that $E \leq_B E_n$, where E is isomorphism of one-dimensional subshifts on 2.
In particular all the relations E_n are all universal countable Borel equivalence relations, and hence they are all mutually bi-reducible.

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Proof.

Fix $n \geq 1$, and consider the injection $f : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}^n}$ given by

$$f(x)(i_1, \ldots, i_n) = x(i_1).$$

This induces the map $\tilde{f} : \mathcal{F}_2^S \to \mathcal{F}_2^{S,n}$ given by $\tilde{f}(X) = f[X]$.
It is straightforward to check that $\tilde{f}$ is a reduction of E to E_n. □

Higher dimensional subshifts (cont.)

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- The above theorem shows that their classification up to isomorphism is no more difficult than for the one-dimensional case, at least from the standpoint of Borel reducibility.
- In particular, for each two-dimensional subshift X we can assign a one-dimensional subshift X' in a Borel-measurable way, so that $X \cong Y$ if and only if $X' \cong Y'$.

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The proof above made fundamental use of the periodic points of the subshift X_x. We could ask to what extent this is necessary.

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The proof above made fundamental use of the periodic points of the subshift X_x. We could ask to what extent this is necessary.

Question

What is the complexity of the isomorphism relation restricted to subshifts with no periodic points?

Questions (cont.)

For subshifts X and Y, write $X \geq Y$ if there is a shift homomorphism $f : X \to Y$. We write $X \geq_{\text{inj}} Y$ when f is injective, and $X \geq_{\text{surj}} Y$ when f is surjective.

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We can then define the following three equivalence relations:

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\begin{aligned}
X\, E_{\text{hom}}\, Y &\quad\Leftrightarrow\quad X \geq Y \wedge Y \geq X \\
X\, E_{\text{inj}}\, Y &\quad\Leftrightarrow\quad X \geq_{\text{inj}} Y \wedge Y \geq_{\text{inj}} X \\
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We can then define the following three equivalence relations:

$$X \, E_{\text{hom}} \, Y \iff X \geq Y \wedge Y \geq X$$
$$X \, E_{\text{inj}} \, Y \iff X \geq_{\text{inj}} Y \wedge Y \geq_{\text{inj}} X$$
$$X \, E_{\text{surj}} \, Y \iff X \geq_{\text{surj}} Y \wedge Y \geq_{\text{surj}} X$$

Question

What are the complexities of the above three equivalence relations?

All of these contain the isomorphism relation.
E_{hom} contains the other two.
These no longer have all equivalence classes countable.