Isomorphism of subshifts and countable Borel equivalence relations

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Subshifts

Definition

Let A be a finite set of symbols. A *one-dimensional subshift on* A is a closed subset of $A^{\mathbb{Z}}$ which is invariant under the left shift operator S, where $S(x)(n) = x(n+1)$. Two subshifts X on A and Y on B are *isomorphic* if there is a homeomorphism $\varphi : X \to Y$ which commutes with S.

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A subshift may be defined by a set of *forbidden words* $W \subseteq A^{<\mathbb{N}}$ (where $A^{<\mathbb{N}}$ is the set of finite sequences from A), where W determines the subshift

$$X_W = \{x \in A^{\mathbb{Z}} : \forall w \in W \ (w \not\sqsubseteq x)\}$$

(and $w \sqsubseteq x$ means that w occurs as a subword of x).
A subshift is said to be of *finite type* when W is finite.

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- One can ask how complicated a set of complete invariants for isomorphism needs to be.
- We will consider this equivalence relation from the standpoint of descriptive set theory, and determine its complexity among Borel equivalence relations under the relation of Borel reducibility, $\leq_B$.
- This will allow us to gauge the difficulty of this classification problem, and compare it to other classification problems.

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We say that an equivalence relation E on a space X is *Borel reducible* to an equivalence relation F on the space Y, $E \leq_B F$, if there is a Borel-measurable function $f : X \to Y$ such that for all $x_1, x_2 \in X$ we have $x_1 \; E \; x_2$ if and only if $f(x_1) \; F \; f(x_2)$.
We write $E \sqsubseteq_B F$ when the we can find such an f which is injective.

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Note that having $E \leq_B F$ amounts to having a definable injection from the quotient space X/E into the quotient Y/F, so that when $E \leq_B F$ we may view the relation F as being at least as complicated as E.

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- If a complicated equivalence relation E is Borel reducible to an equivalence relation F, then there can be no set of complete invariants for F which are fundamentally simpler a set of complete invariants for E. Hence F is at least as complicated as E.

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- If a complicated equivalence relation E is Borel reducible to an equivalence relation F, then there can be no set of complete invariants for F which are fundamentally simpler a set of complete invariants for E. Hence F is at least as complicated as E.
- Several canonical examples of equivalence relations are well-understood, and these can be used to gauge the complexity of other relations.

Smooth equivalence relations

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An equivalence relation E on X is *smooth* if it is Borel reducible to the identity relation on some Polish space Y, i.e. there is a Borel-measurable function $f : X \to Y$ such that $x_1 \mathrel{E} x_2$ if and only if $f(x_1) = f(x_2)$.

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- For the sort of equivalence relations we consider here, being smooth is equivalent to admitting a Borel selector which picks one representative from each equivalence class.
- Smooth equivalence relations are the simplest equivalence relations, and are also referred to as "concretely classifiable."

Hyperfinite equivalence relations

- A canonical example of a non-smooth countable Borel equivalence relation is the relation E_0 of eventual equality on the Cantor space $2^{\mathbb{N}}$ defined by setting $x \mathrel{E_0} y$ iff $x(n) = y(n)$ for all but finitely many n.

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A universal countable Borel equivalence relations is of maximum complexity among countable Borel equivalence relations. In particular, it is not smooth.

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Our main result:

Theorem (C.)

Isomorphism of one-dimensional subshifts is a universal countable Borel equivalence relation.

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Our main result:

Theorem (C.)

Isomorphism of one-dimensional subshifts is a universal countable Borel equivalence relation.

In particular it is not smooth, and it will not admit complete invariants fundamentally simpler than the equivalence classes under isomorphism.

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These are more complicated than countable Borel equivalence relations.

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- Isometry of Polish metric spaces is universal for orbit equivalence relations of actions of Polish groups (C., Gao-Kechris).
- Conjugacy of Borel automorphisms is more complicated than any Borel equivalence relation (C.) or analytic equivalence relation (Gao).

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Proof.

It suffices to observe that the function $\tilde{S} : K(n^{\mathbb{Z}}) \to K(n^{\mathbb{Z}})$ given by $\tilde{S}(X) = S[X]$ is a homeomorphism, since then $\mathcal{F}_n^S = \{X \in K(n^{\mathbb{Z}}) : \tilde{S}(X) = X\}$ is closed. □

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$$X_W = \{x \in n^{\mathbb{Z}} : \forall i \, (W(i) \neq \emptyset \Rightarrow W(i) \not\sqsubseteq x)\}.$$

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- Every subshift on n is represented by an element $W \in Z_n$.
- Several different sets of forbidden words may produce the same subshift; we wish to see that this is not problematic.

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The relation $\sim$ is a smooth Borel equivalence relation on Z_n. □

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Hence it will not matter which representation we use for subshifts (from the standpoint of Borel reducibility).

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A basic result which we will utilize is the following:

Theorem (Curtis-Hedlund-Lyndon)

Suppose X and Y are subshifts on A and B, respectively, and $\varphi : X \to Y$ is a morphism of subshifts, i.e., a map commuting with the shift. Then φ is given by a block code, i.e., there is a natural number r and a function $\pi : A^{\{-r,\dots,r\}} \to B$ such that

$$\varphi(x)(n) = \pi(S^n(x) \restriction \{-r, \dots, r\}).$$

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Note that there are only countably many such block codes, and any isomorphism of subshifts is computable.

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$$\begin{aligned} X\,E\,Y \Leftrightarrow\ & \exists\pi\exists\tilde{\pi}\forall x \in n^{\mathbb{Z}}\ [(x \in X \Rightarrow f_\pi(x) \in Y)\wedge \\ & \qquad (x \in Y \Rightarrow f_{\tilde{\pi}} \in X) \wedge (x \in X \Rightarrow f_{\tilde{\pi}} \circ f_\pi(x) = x)] \\ \Leftrightarrow\ & \exists\pi\exists\tilde{\pi}\neg\exists x \in n^{\mathbb{Z}}\ [(x \in X \setminus f_\pi^{-1}[Y])\vee \\ & \qquad (x \in Y \setminus F_{\tilde{\pi}}^{-1}[X]) \vee (x \in X \wedge f_{\tilde{\pi}}(f_\pi(x)) \neq x)] \end{aligned}$$

The matrix is K_σ so the projection is also K_σ; as π and $\tilde{\pi}$ range over a countable set this relation is Σ_3^0 and hence Borel. □

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- We will need the following technical lemma:

Lemma

There is an $\mathbb{F}_2$-invariant Borel set $A \subseteq 2^{\mathbb{F}_2}$ such that:

- $E(\mathbb{F}_2, 2) \sqsubseteq_B E(\mathbb{F}_2, 2) \restriction A$, *so* $E(\mathbb{F}_2, 2) \restriction A$ *is a universal countable Borel equivalence relation.*
- *For* $x, y \in A$ *with* $\neg x \; E(\mathbb{F}_2, 2) \; y$, *there are infinitely many* $g \in \mathbb{F}_2$ *with* $x(g) \neq y(g)$. □

Complexity of the isomorphism relation (cont.)

Theorem

The isomorphism relation on one-dimensional subshifts is a universal countable Borel equivalence relation.

Complexity of the isomorphism relation (cont.)

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Proof.

- We will show that $E(\mathbb{F}_2, 2) \restriction A \leq_B E$, where A is the set from the previous lemma and E is the isomorphism relation for subshifts on the alphabet 2.

Complexity of the isomorphism relation (cont.)

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Proof.

- We will show that $E(\mathbb{F}_2, 2) \restriction A \leq_B E$, where A is the set from the previous lemma and E is the isomorphism relation for subshifts on the alphabet 2.
- Given a point $x \in 2^{\mathbb{F}_2}$, we will produce a subshift $\varphi(x) = X_x$.

Complexity of the isomorphism relation (cont.)

Theorem

The isomorphism relation on one-dimensional subshifts is a universal countable Borel equivalence relation.

Proof.

- We will show that $E(\mathbb{F}_2, 2) \restriction A \leq_B E$, where A is the set from the previous lemma and E is the isomorphism relation for subshifts on the alphabet 2.
- Given a point $x \in 2^{\mathbb{F}_2}$, we will produce a subshift $\varphi(x) = X_x$.
- The idea is to encode the orbit of x under the shift action of $\mathbb{F}_2$ into the subshift X_x, and use periodic points to recover the orbit of x from the isomorphism type of X_x.

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Theorem

The isomorphism relation on one-dimensional subshifts is a universal countable Borel equivalence relation.

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Complexity of the isomorphism relation (cont.)

Theorem

The isomorphism relation on one-dimensional subshifts is a universal countable Borel equivalence relation.

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- The idea is to encode the orbit of x under the shift action of $\mathbb{F}_2$ into the subshift X_x, and use periodic points to recover the orbit of x from the isomorphism type of X_x.
- Let $s \frown t$ denote the concatenation of two finite sequences s and t, and let $|s|$ be the length of a finite sequence.
- Let $\mathbb{F}_2$ be generated by a and b.

Complexity of the isomorphism relation (cont.)

Proof (cont.).

Define a map $\rho : \mathbb{F}_2 \to 2^{<\mathbb{N}}$ by letting:

$$\begin{aligned} \rho(e) &= 11000011 \\ \rho(a) &= 11100011 \\ \rho(a^{-1}) &= 11010011 \\ \rho(b) &= 11001011 \\ \rho(b^{-1}) &= 11000111 \end{aligned}$$

$$\begin{aligned} \rho(w) = \rho(w_0) \frown \cdots \frown \rho(w_n) & \\ & \text{for } w = w_0 \frown \cdots \frown w_n \neq e \text{ a reduced word} \\ & \text{with each } w_i \in \{a, a^{-1}, b, b^{-1}\} \text{ and } w_{i+1} \neq w_i^{-1}. \end{aligned}$$

Note that for each $w \in \mathbb{F}_2$, $|\rho(w)| = 8k$ for some $k > 0$.

Complexity of the isomorphism relation (cont.)

Proof (cont.).

- Define the map $d : 2 \to 2^{<\mathbb{N}}$ by:

$$d(0) = 101$$
$$d(1) = 111.$$

Note that both of these sequences have length 3.

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- For a finite sequence $s \in 2^{<\mathbb{N}}$, let $\bar{0} \frown s \frown \bar{0}$ denote the element of $2^{\mathbb{Z}}$ whose negative coordinates are all 0, followed by s, followed by all 0's.

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- Let $\bar{s}$ be the periodic element of $2^{\mathbb{Z}}$ with period $|s|$ induced by s (starting at coordinate 0).

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- Let $\bar{s}$ be the periodic element of $2^{\mathbb{Z}}$ with period $|s|$ induced by s (starting at coordinate 0).
- Finally, let $p(i)$ denote the i-th prime.

Complexity of the isomorphism relation (cont.)

Proof (cont.).

For an element $x \in 2^{\mathbb{F}^2}$ define the countable set A_x:

$$
\begin{aligned}
A_x = \{&\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\} \cup \\
&\{\overline{d(0) \frown 0^{p(2n+2)-3}} : n \in \mathbb{N}\} \cup \\
&\{\overline{d(1) \frown 0^{p(2n+3)-3}} : n \in \mathbb{N}\}
\end{aligned}
$$

Let $\varphi(x) = X_x$ where:

$$X_x = \overline{\bigcup_{n \in \mathbb{Z}} S^n[A_x]}.$$

Thus, X_x is the smallest subshift containing A_x.

Complexity of the isomorphism relation (cont.)

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Let $\varphi(x) = X_x$ where:

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Thus, X_x is the smallest subshift containing A_x.
It is straightforward to check that φ is Borel, using the previous lemma about the map $W_x \mapsto X_x$.

Complexity of the isomorphism relation (cont.)

Proof (cont.).

It remains to check that φ witnesses that $E(\mathbb{F}_2, 2) \upharpoonright A \leq_B E$.

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It remains to check that φ witnesses that $E(\mathbb{F}_2, 2) \restriction A \leq_B E$.
Recall A_x:

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It remains to check that φ witnesses that $E(\mathbb{F}_2, 2) \restriction A \leq_B E$.
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&\{\overline{d(1) \frown 0^{p(2n+3)-3}} : n \in \mathbb{N}\}
\end{aligned}
$$

The periodic points of X_x are (shifts of) the following points:

$\bar{0}$	period 1
$\overline{d(0) \frown 0^{p(2n+2)-3}}$	period $p(2n+2)$
$\overline{d(1) \frown 0^{p(2n+3)-3}}$	period $p(2n+3)$
$\overline{\rho(w)}$ for $w \in \mathbb{F}_2$	period $8 \cdot \lvert w \rvert$ (where $\lvert e \rvert = 1$)

Complexity of the isomorphism relation (cont.)

Proof (cont.).

For every x, X_x also includes the following limit points:

$$\bar{0} \frown d(0) \frown \bar{0}$$
$$\bar{0} \frown d(1) \frown \bar{0}$$
$$\bar{0} \frown \rho(w) \frown \bar{0} \quad \text{for } w \in \mathbb{F}_2$$

all left-, right- or bi-infinite sequences of $\rho(w)$'s (with other coordinates 0).

Depending on the orbit of x (for various i, k and right-infinite words w^* from $\mathbb{F}_2$) it may also include points of the form:

$$\bar{0} \frown d(i) \frown 0^{10+k^2} \frown \rho[w^*].$$

Complexity of the isomorphism relation (cont.)

Proof (cont.).

Suppose first that $x, y \in 2^{\mathbb{F}_2}$ with $x \, E(\mathbb{F}_2, 2) \, y$.
We show that $(X_x, S) \cong (Y_x, S)$ (we do not need $x, y \in A$).

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- Since a and b generate $\mathbb{F}_2$, it suffices to show this when $y = a \cdot x$ or $y = b \cdot x$.

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We show that $(X_x, S) \cong (Y_x, S)$ (we do not need $x, y \in A$).

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- Consider the case $y = a \cdot x$ (the case of $y = b \cdot x$ is identical).
- Let P be the set:

$$\{\overline{d(0) \frown 0^{p(2n+2)-3}} : n \in \mathbb{N}\} \cup \{\overline{d(1) \frown 0^{p(2n+3)-3}} : n \in \mathbb{N}\}$$

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- Then we have:

$$A_x = P \; \cup \; \{\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\}$$

Complexity of the isomorphism relation (cont.)

Proof (cont.).

Compare this to A_y when $y = a \cdot x$:

$$\begin{aligned} A_y &= P \cup \{\bar{0} \frown d(w_i \cdot y(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\} \\ &= P \cup \{\bar{0} \frown d(w_i a \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\} \\ &= P \cup \{\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i a^{-1}) \frown \bar{0} : i, k \in \mathbb{N}\} \\ A_x &= P \cup \{\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\} \end{aligned}$$

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\end{aligned}$$

- We want a map from A_x to A_y which sends each point

$$\begin{aligned}
\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} \quad &\mapsto \\
\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i a^{-1}) \frown \bar{0}.&
\end{aligned}$$

Complexity of the isomorphism relation (cont.)

Proof (cont.).

Compare this to A_y when $y = a \cdot x$:

$$\begin{aligned}
A_y &= P \cup \{\bar{0} \frown d(w_i \cdot y(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\} \\
&= P \cup \{\bar{0} \frown d(w_i a \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\} \\
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\end{aligned}$$

- All other sequences are left unchanged.

Complexity of the isomorphism relation (cont.)

Proof (cont.).

Compare this to A_y when $y = a \cdot x$:

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$$= P \cup \{\bar{0} \frown d(w_i a \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\}$$
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$$\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} \quad \mapsto$$
$$\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i a^{-1}) \frown \bar{0}.$$

- All other sequences are left unchanged.
- This induces an S-invariant homeomorphism of X_x with X_y.

Complexity of the isomorphism relation (cont.)

Proof (cont.).

We define a block code $\pi : 2^{\{-r,\ldots,r\}} \to 2$ with $r = 16$ so that f_π is an isomorphism of X_x and X_y.

Complexity of the isomorphism relation (cont.)

Proof (cont.).

We define a block code $\pi : 2^{\{-r,\ldots,r\}} \to 2$ with $r = 16$ so that f_π is an isomorphism of X_x and X_y.
We want the following subsequences mapped as shown:

$$
\begin{aligned}
0 \frown \rho(e) \frown 0 &\quad\mapsto\quad 0 \frown \rho(a^{-1}) \frown 0 \\
0 \frown \rho(a) \frown 0 &\quad\mapsto\quad 0 \frown \rho(e) \frown 0 \\
\rho(a) \frown \rho(a) \frown 0 &\quad\mapsto\quad \rho(a) \frown 0^8 \frown 0 \\
\rho(b) \frown \rho(a) \frown 0 &\quad\mapsto\quad \rho(b) \frown 0^8 \frown 0 \\
\rho(b^{-1}) \frown \rho(a) \frown 0 &\quad\mapsto\quad \rho(b^{-1}) \frown 0^8 \frown 0 \\
\rho(a^{-1}) \frown 0^8 \frown 0 &\quad\mapsto\quad \rho(a^{-1}) \frown \rho(a^{-1}) \frown 0 \\
\rho(b) \frown 0^8 \frown 0 &\quad\mapsto\quad \rho(b) \frown \rho(a^{-1}) \frown 0 \\
\rho(b^{-1}) \frown 0^8 \frown 0 &\quad\mapsto\quad \rho(b^{-1}) \frown \rho(a^{-1}) \frown 0
\end{aligned}
$$

and all other sequences should be left unchanged.

Complexity of the isomorphism relation (cont.)

Proof (cont.).

- None of the listed sequences overlaps a shift of another.

Complexity of the isomorphism relation (cont.)

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- None of the listed sequences overlaps a shift of another.
- Each pair above have the same length.

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- None of the listed sequences overlaps a shift of another.
- Each pair above have the same length.
- So our requirements can be ensured by choosing the block code π appropriately.
- Taking $r = 16$ is sufficient for each image digit to “see enough” of the input to tell if and wherein it is part of one of the listed sequences.

Complexity of the isomorphism relation (cont.)

Proof (cont.).

- None of the listed sequences overlaps a shift of another.
- Each pair above have the same length.
- So our requirements can be ensured by choosing the block code π appropriately.
- Taking $r = 16$ is sufficient for each image digit to “see enough” of the input to tell if and wherein it is part of one of the listed sequences.
- This gives a shift-invariant homeomorphism of A_x with A_y which extends to an isomorphism of the subshifts X_x and X_y.

Complexity of the isomorphism relation (cont.)

Proof (cont.).

For the other direction, suppose $x, y \in A$ and $X_x \cong X_y$ via f_π induced by the block code $\pi : 2^{\{-r,\ldots,r\}} \to 2$.

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For the other direction, suppose $x, y \in A$ and $X_x \cong X_y$ via f_π induced by the block code $\pi : 2^{\{-r,\dots,r\}} \to 2$.
We will show that $x \, E(\mathbb{F}_2, 2) \, y$.

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We will show that $x \; E(\mathbb{F}_2, 2) \; y$.
The periods of the periodic points must be preserved by f_π, so:

Complexity of the isomorphism relation (cont.)

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For the other direction, suppose $x, y \in A$ and $X_x \cong X_y$ via f_π induced by the block code $\pi : 2^{\{-r,\dots,r\}} \to 2$.
We will show that $x\ E(\mathbb{F}_2, 2)\ y$.
The periods of the periodic points must be preserved by f_π, so:

- $f_\pi(\bar{0}) = \bar{0}$ as this is the unique point of period 1.
  Hence $\pi(0^{2r+1}) = 0$.

Complexity of the isomorphism relation (cont.)

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- $f_\pi(\bar{0}) = \bar{0}$ as this is the unique point of period 1.
  Hence $\pi(0^{2r+1}) = 0$.
- Considering the points of P for large n we have:

$$f_\pi(\bar{0} \frown d(0) \frown \bar{0}) = S^{n(0)}(\bar{0} \frown d(0) \frown \bar{0})$$
$$f_\pi(\bar{0} \frown d(1) \frown \bar{0}) = S^{n(1)}(\bar{0} \frown d(1) \frown \bar{0})$$

  for some $n(0), n(1) \in \{-r, \ldots, r\}$.

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$$f_\pi(\bar{0} \frown d(1) \frown \bar{0}) = S^{n(1)}(\bar{0} \frown d(1) \frown \bar{0})$$

  for some $n(0), n(1) \in \{-r, \ldots, r\}$.
- There is also a sequence s_0 and $m_0 \in \{-r, \ldots, r\}$ such that $f_\pi(\bar{0} \frown \rho(e) \frown \bar{0}) = S^{m_0}(\bar{0} \frown s_0 \frown \bar{0})$.

Complexity of the isomorphism relation (cont.)

Proof (cont.).

For sufficiently large k (i.e. $k > r$), consider the points:

$$\bar{0} \frown d(x(w_k)) \frown 0^{10+k^2} \frown \rho(e) \frown \bar{0} \text{ in } X_x.$$

Complexity of the isomorphism relation (cont.)

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Thus $\rho(w_{i_k}) = s_0$ for sufficiently large k, so there is some fixed $w \in \mathbb{F}_2$ such that $s_0 = \rho(w)$.

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Hence $y \, E(\mathbb{F}_2, 2) \, x$ and we are done. $\square$

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Definition

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- The reverse is also true:

Higher dimensional subshifts (cont.)

Theorem

For each $n \geq 1$, we have that $E \leq_B E_n$, where E is isomorphism of one-dimensional subshifts on 2.
In particular all the relations E_n are all universal countable Borel equivalence relations, and hence they are all mutually bi-reducible.

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Proof.

Fix $n \geq 1$, and consider the injection $f : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}^n}$ given by

$$f(x)(i_1, \ldots, i_n) = x(i_1).$$

This induces the map $\tilde{f} : \mathcal{F}_2^S \to \mathcal{F}_2^{S,n}$ given by $\tilde{f}(X) = f[X]$.
It is straightforward to check that $\tilde{f}$ is a reduction of E to E_n. $\square$

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- The above theorem shows that their classification up to isomorphism is no more difficult than for the one-dimensional case, at least from the standpoint of Borel reducibility.
- In particular, for each two-dimensional subshift X we can assign a one-dimensional subshift X' in a Borel-measurable way, so that $X \cong Y$ if and only if $X' \cong Y'$.

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What is the complexity of the isomorphism relation restricted to subshifts with no periodic points?

Questions (cont.)

For subshifts X and Y, write $X \geq Y$ if there is a shift homomorphism $f : X \to Y$. We write $X \geq_{\text{inj}} Y$ when f is injective, and $X \geq_{\text{surj}} Y$ when f is surjective.

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We can then define the following three equivalence relations:

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X \, E_{\text{hom}} \, Y &\quad\Leftrightarrow\quad X \geq Y \wedge Y \geq X \\
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We can then define the following three equivalence relations:

$$X \, E_{\text{hom}} \, Y \iff X \geq Y \wedge Y \geq X$$
$$X \, E_{\text{inj}} \, Y \iff X \geq_{\text{inj}} Y \wedge Y \geq_{\text{inj}} X$$
$$X \, E_{\text{surj}} \, Y \iff X \geq_{\text{surj}} Y \wedge Y \geq_{\text{surj}} X$$

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All of these contain the isomorphism relation.
E_{hom} contains the other two.
These no longer have all equivalence classes countable.