Complemented Sets and Difference Sets

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A Question from Ergodic Theory

Definition 1. *Let T be a measure-preserving transformation on a measure space $(X, \mathcal{B}, \mu)$. A sequence Ω in $[\omega]^\omega$ is a weakly wandering sequence for T if there is a measurable set A of positive measure such that for all n and m in Ω with $n \neq m$, we have:*

$$T^n[A] \cap T^m[A] = \emptyset$$

We say Ω is exhaustive if there is such an A which also satisfies:

$$X = \bigcup_{n \in \Omega} T^n[A]$$

We can also consider sequences which are subsets of $\mathbb{Z}$.

We say T is *ergodic* if for any T-invariant Borel set A, either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Definition 2. *We consider the following sets:*

$$\begin{aligned}
\mathcal{WW} &= \{\Omega : \Omega \text{ is w.w. for some } T\} \\
\mathcal{WW}_0 &= \{\Omega : \Omega \text{ is w.w. for some ergodic } T\} \\
\mathcal{EWW} &= \{\Omega : \Omega \text{ is e.w.w. for some } T\} \\
\mathcal{EWW}_0 &= \{\Omega : \Omega \text{ is e.w.w. for some ergodic } T\}
\end{aligned}$$

Note $\mathcal{EWW}_0 \subseteq \mathcal{EWW}, \mathcal{WW}_0 \subseteq \mathcal{WW}$; these inclusions turn out to be proper.
Question 3. *How complicated are these sets?*

There are known characterizations of these sets which we will use in establishing their descriptive complexity.

These characterizations involve complemented sets and difference sets, which leads us to consider the complexity of such sets.

Complemented Sets and Difference Sets

We denote the *sum* of two sets $A, B \subseteq \mathbb{N}$ or $\mathbb{Z}$ by

$$A + B = \{a + b : a \in A, b \in B\}$$

The set $A - B$ is defined similarly.

We say A and B have a *direct sum* if for every $a_1 \neq a_2$ in A and every $b_1 \neq b_2$ in B we have $a_1 + b_1 \neq a_2 + b_2$. We denote this by writing $A + B = A \oplus B$.

A and B have direct sum $\mathbb{Z}$ if, in addition, every integer equals $a + b$ for some a in A and b in B. We denote this by $A \oplus B = \mathbb{Z}$.

We say a set $A \subseteq \mathbb{N}$ or $\mathbb{Z}$ is *complemented* (in $\mathbb{Z}$) if there is a set $B \subseteq \mathbb{Z}$ with $A \oplus B = \mathbb{Z}$. Thus, A is complemented iff $\exists B \subseteq \mathbb{Z}$ such that

$$(\forall k \in \mathbb{Z})(\exists! a \in A)(\exists! b \in B)[k = a + b]$$

Complemented sets (in $\mathbb{N}$) were first studied by DeBruijn.

For $A \subseteq \mathbb{N}$ or $\mathbb{Z}$, let:

$$\begin{aligned} D(A) &= \{a_2 - a_1 : a_1, a_2 \in A \;\&\; a_2 \geq a_1\} \\ S(A) &= \{a_1 + a_2 : a_1, a_2 \in A\} \end{aligned}$$

Then $A + B = A \oplus B$ iff $D(A) \cap D(B) = \{0\}$.

We say B is a *difference set* (from $\mathbb{N}$, resp. $\mathbb{Z}$) if $B = D(A)$ for some $A \subseteq \mathbb{N}$ (resp. $\mathbb{Z}$).

Define the following sets:

$$\begin{aligned} \mathcal{COMP} &= \{A \subseteq \mathbb{N} : \exists B \subseteq \mathbb{Z}(A \oplus B = \mathbb{Z})\} \\ \mathcal{DF} &= \{A \subseteq \mathbb{N} : \text{ } A \text{ is a difference set}\} \\ \mathcal{DF}_\infty &= \{A \subseteq \mathbb{N} : \text{ } A \text{ is an infinite} \\ &\qquad \text{difference set}\} \\ \mathcal{CDF} &= \{A \subseteq \mathbb{N} : \text{ } A \text{ contains an infinite} \\ &\qquad \text{difference set}\} \end{aligned}$$

$\mathcal{COMP}$, $\mathcal{DF}$, $\mathcal{DF}_\infty$, and $\mathcal{CDF}$ are all $\mathbf{\Sigma}^1_1$.

Define the *shift* of a set A by an integer n:

$$A + n = \{a + n : a \in A\}$$

We say this is a shift to the right if $n > 0$ and to the left if $n < 0$.

$D(A + n) = D(A)$, so shifts of A and B have a direct sum iff A and B do. If $A \oplus B = \mathbb{Z}$, then also $(A + n) \oplus B = \mathbb{Z}$.

We say that one sequence *occurs* in another if some shift of the first sequence forms a consecutive subsequence of the second.

An increasing sequence $H = \langle \ldots h_{-1}, h_0, h_1, \ldots \rangle$ is a *hitting sequence* if each finite consecutive subsequence $h_{-n}, \ldots, h_0, \ldots h_n$ occurs shifted to both the right and left in H.

The following characterization is due to Eigen and Hajian, extending work of Kamae:

Theorem 4 (Eigen and Hajian, Kamae).

1. *$\Omega \in \mathcal{WW} \iff$ there is an infinite $H \subseteq \mathbb{Z}$ such that $H + \Omega = H \oplus \Omega$.*

2. *$\Omega \in \mathcal{WW}_0 \iff$ there is a hitting sequence $H \subseteq \mathbb{Z}$ such that $H + \Omega = H \oplus \Omega$.*

3. *$\Omega \in \mathcal{EWW} \iff$ there is $H \subseteq \mathbb{Z}$ such that $H \oplus \Omega = \mathbb{Z}$.*

4. *$\Omega \in \mathcal{EWW}_0 \iff$ there is a hitting sequence $H \subseteq \mathbb{Z}$ such that $H \oplus \Omega = \mathbb{Z}$.*

In particular, $\mathcal{EWW} = \mathcal{COMP}$.

Note that this gives $\mathbf{\Sigma}^1_1$ definitions for each of these sets. We will show that they are $\mathbf{\Sigma}^1_1$-complete, as are $\mathcal{COMP}$ and $\mathcal{CDF}$.

Let $\mathcal{T}$ be the set of trees on ω, and $\mathcal{WF}$ the set of well-founded trees. It is well-known that $\mathcal{T} \setminus \mathcal{WF}$ is a $\mathbf{\Sigma}^1_1$-complete subset of $\mathcal{T}$.

The following is the main technical lemma:
Main Lemma. *There is a continuous function* $f : \mathcal{T} \to [\omega]^\omega$ *sending* T *to* Ω_T *such that:*

1. *If* T *is ill-founded, then there is a hitting sequence* H *such that* $H \oplus \Omega_T = \mathbb{Z}$*.*
2. *If* T *is well-founded then* $D(\Omega_T) \setminus \{0\}$ *meets every infinite difference set.*

From this we get:
Theorem 5. *If* X *is any set with* $\mathcal{EWW}_0 \subseteq X \subseteq \mathcal{WW}$*, then* X *is* $\mathbf{\Sigma}^1_1$*-hard. Hence the sets* $\mathcal{WW}$*,* $\mathcal{WW}_0$*,* $\mathcal{EWW}$*, and* $\mathcal{EWW}_0$ *are all* $\mathbf{\Sigma}^1_1$*-complete.*

Proof: With f as above we have $f[\mathcal{T} \setminus \mathcal{WF}] \subseteq \mathcal{EWW}_0$ and $f[\mathcal{WF}] \cap \mathcal{WW} = \emptyset$, so if $\mathcal{EWW}_0 \subseteq X \subseteq \mathcal{WW}$ then $\mathcal{T} \setminus \mathcal{WF} = f^{-1}[X]$. $\square$

Corollary 6. *The set $\mathcal{COMP}$ is $\mathbf{\Sigma}^1_1$-complete.*

We get the following as a corollary to the proof of the Main Lemma:

Corollary 7 (Mannsfield). *The set $\mathcal{CDF}$ is $\mathbf{\Sigma}^1_1$-complete.*

Proof: The proof continuously assigns Ω_T to T, as well as a set β_T with $\beta_T \cap D(\Omega_T) = \{0\}$ and $\beta_T \cup D(\Omega_T) = \mathbb{N}$. When T is ill-founded, $\beta_T \supseteq D(H)$ for some hitting sequence H, so $\beta_T \in \mathcal{CDF}$. When T is ill-founded, $D(\Omega_T)$ meets every infinite difference set, so $\beta_T \notin \mathcal{CDF}$. So the function $g : T \mapsto \beta_T$ works. □

A related theorem is due to Schmerl:

Theorem 8 (Schmerl). *$\mathcal{DF}$ is $\mathbf{\Sigma}^1_1$-complete, as is $\mathcal{DF}_\infty$.*

Question 9 (Schmerl). *Is every set X with $\mathcal{DF}_\infty \subseteq X \subseteq \mathcal{CDF}$ $\mathbf{\Sigma}^1_1$-hard?*

There are results characterizing when certain types of sets are complemented. The above theorem shows that in a descriptive context no simpler classification is possible for arbitrary sets than the definition itself.

In contrast, consider sets $A \subseteq \mathbb{N}$ which are complemented in $\mathbb{N}$, i.e., where there is a $B \subseteq \mathbb{N}$ such that $A \oplus B = \mathbb{N}$. Here the classification is much simpler.

We have that A is complemented in $\mathbb{N}$ iff there is a $B \subseteq \mathbb{N}$ such that

$$(\forall k \in \mathbb{N})(\exists! a \leq k \in A)(\exists! b \leq k \in B)[k = a + b]$$

Thus $\mathcal{COMP}(\mathbb{N})$ is the projection of a compact set and hence closed.

The point is that the set of possible b's to achieve a particular sum k is bounded.

Proof of the Main Lemma

We define the map $T \mapsto \Omega_T$. We may assume that T contains infinitely many nodes.

$t_i \prec t_j$ means t_i is a predecessor of t_j in T.
$t_i \perp t_j$ means t_i and t_j are incomparable.
$|t|$ denotes the length of t.

Let $\langle t_i \rangle_{i \in \omega}$ enumerate T so that if $t_i \prec t_j$ then $i < j$. In particular, t_0 is the root of the tree.

We build $\Omega = \Omega_T$ as the union of finite sequences ω_i, and simultaneously construct potential witnesses H_i which will complement Ω when T contains an infinite branch.

We proceed in stages, defining at stage n the finite sets $\omega_n \subseteq \mathbb{N}$ and $H_n \subseteq \mathbb{Z}$. We set
$A_n = D(\omega_n)$, $D_n = D(H_n)$, and $\beta_n = \bigcup_{i \leq n} D_i$.

We say B is an *end-extension* of A, $A \sqsubseteq B$, if $B \cap [\min(A), \max(A)] = A$.

We construct Ω to satisfy the following nine conditions at each stage n for all $i < n$:

1. $\omega_i \sqsubset \omega_n$ and $\beta_i \sqsubset \beta_n$
2. $A_n \cap \beta_n = \{0\}$
3. $A_n \cup \beta_n \supseteq [0, \max(\beta_n)]$
4. If $t_i \prec t_n$ then $H_i \sqsubset H_n$.
5. Let t_k be the maximal predecessor of t_i and t_n. Then $D_i \cap D_n \subseteq D_k$.

6. If $t_i \perp t_n$, $a \in D_i \setminus \beta_{i-1}$, and $b \in D_n \setminus \beta_{n-1}$, then $|b-a| \in A_n$.

7. For t_m the immediate predecessor of t_n, left and right shifts of H_m occur in H_n.

8. Let $r_k = (-1)^{k+1} \lfloor \frac{k+1}{2} \rfloor$. Then we require $r_{|t_n|} \in \omega_n \oplus H_n$.

9. For $i \leq n$, $D(D_i)$ and $S(D_i)$ are disjoint from $\beta_n \setminus D_i$.

At the end of the construction we set:

$$\begin{aligned} \Omega &= \bigcup_i \omega_i \\ A &= \bigcup_i A_i = D(\Omega) \\ \beta &= \bigcup_i \beta_i \end{aligned}$$

Note that $A \cap \beta = \{0\}$ and $A \cup \beta = \mathbb{N}$.

Assuming the construction has been carried out, we show it completes the lemma.

$T \mapsto \Omega_T$ is continuous by conditions (1)–(3).

Suppose T has an infinite branch $\langle t_{m_i} \rangle_{i \in \omega}$. Let $H = \bigcup_i H_{m_i}$. Since $H_{m_i} \sqsubseteq H_{m_{i+1}}$ for each i by condition (4), we have:

$$D(H) = \bigcup_i D(H_{m_i}) = \bigcup_i D_{m_i} \subseteq \beta$$

Since $A \cap \beta = \{0\}$ by condition (2), we have $D(\Omega) \cap D(H) = \{0\}$, so Ω has a direct sum with H. Condition (7) guarantees that H is a hitting sequence. Finally, since each t_{m_i} has length i, each r_i will occur in $\omega_{m_i} \oplus H_{m_i}$ by condition (8), and hence $\Omega \oplus H = \mathbb{Z}$.

Conversely, suppose $D(\Omega)\setminus\{0\}$ is disjoint from some infinite difference set. Any infinite difference set from $\mathbb{Z}$ contains one from $\mathbb{N}$, so let $C \subseteq \mathbb{N}$ be such that $D(C) \cap D(\Omega) = \{0\}$. By shifting, we may assume $C \subseteq \beta$. Let C_i be the first i elements of C. Let m_i be the least such that $C_i \subseteq \beta_{m_i}$. Then $m_i \leq m_{i+1}$ and $m_i \to \infty$.

We claim that $t_{m_i} \preceq t_{m_{i+1}}$ for all i. If not, let i be such that $t_{m_i} \perp t_{m_{i+1}}$ and let t_m be the maximal predecessor of t_{m_i} and $t_{m_{i+1}}$. Then $m < m_i$. By the minimality of m_i and m_{i+1} there are $a \in C_i$ and $b \in C_{i+1}$ such that:

$$a \in \beta_{m_i} \setminus \beta_{m_i-1} = D_{m_i} \setminus \beta_{m_i-1}$$
$$b \in \beta_{m_{i+1}} \setminus \beta_{m_{i+1}-1} = D_{m_{i+1}} \setminus \beta_{m_{i+1}-1}$$

Condition (5) ensures $a \neq b$ since $D_{m_{i+1}} \cap D_{m_i} \subseteq D_m \subseteq \beta_{m_i-1}$. By condition (6) we have $|b-a| \in A_{m_{i+1}} \subseteq A$. But $|b-a| \in D(C)$, contradicting that $D(C) \cap A = \{0\}$. Thus, each $t_{m_i} \preceq t_{m_{i+1}}$, so T has an infinite branch.

The Construction of $\Omega_{\mathbf{T}}$

Stage 0: Set $\omega_0 = \{0\}$ and $H_0 = \{0\}$. Then $A_0 = D_0 = \beta_0 = \{0\}$. We have satisfied all the conditions, since $|t_0| = 0$ and $r_0 = 0$.

Stage $\mathbf{n+1}$: Suppose $\omega_0, \ldots, \omega_n$ and $H_0, \ldots, H_n$ satisfy the given conditions. We define ω_{n+1} and H_{n+1} to satisfy them at stage $n+1$.

Let t_m be the immediate predecessor of t_{n+1} in T. We proceed in three steps:

Step I: End-extend H_m to $\widetilde{H}_{n+1}$ to satisfy the shift condition (7) while preserving the other conditions.

We choose two numbers Δ_l and Δ_r in $\mathbb{N}$, and define $\widetilde{H}_{n+1}$ to be the union of H_m and two shifts of H_m:

$$\widetilde{H}_{n+1} = (H_m - \Delta_l) \sqcup H_m \sqcup (H_m + \Delta_r)$$

We must choose Δ_l and Δ_r to preserve the other conditions, which amounts to controlling new differences.

We keep new differences out of particular sets by choosing Δ_l and Δ_r sufficiently large.

We keep new differences in particular sets by also using condition (9), which is the necessary inductive assumption.

Condition (9) can be ensured by choosing Δ_l and Δ_r sufficiently large.

Step II: If necessary, end-extend ω_n to $\widetilde{\omega}_{n+1}$ and extend $\widetilde{H}_{n+1}$ to H_{n+1} to satisfy the fullness condition (8); otherwise let $H_{n+1} = \widetilde{H}_{n+1}$ and $\widetilde{\omega}_{n+1} = \omega_n$. We again preserve the other conditions.

We ensure that $r = r_{|t_{n+1}|} \in \omega_{n+1} \oplus H_{n+1}$.

If r is already in $\omega_n \oplus \widetilde{H}_{n+1}$ we do nothing.

Otherwise, we add an element $h < 0$ to $\widetilde{H}_{n+1}$ and add $w = r - h$ to ω_n.

We need to choose h to preserve the other conditions. This is done (using condition (9)) by choosing h sufficiently small.

Step III: End-extend $\widetilde{\omega}_{n+1}$ to ω_{n+1} to satisfy condition (3) while preserving condition (2).

We add elements to $\widetilde{\omega}_{n+1}$. Let $a_0, \ldots, a_{l-1}$ enumerate $\max(\beta_{n+1}) \setminus (\beta_{n+1} \cup D(\widetilde{\omega}_{n+1}))$. We successively pick pairs (w_i, w_i') for $i < l$ such that $w_i' - w_i = a_i$. We will then let

$$\omega_{n+1} = \widetilde{\omega}_{n+1} \cup \{w_i, w_i' : i < l\}$$

We need to keep new differences out of β_{n+1}. We can do this by making the w_i's and w_i''s sufficiently large.

Constructing Particular Sequences

The techniques in the proof of the Main Lemma can be used to show that all of the obvious inclusions among the sets $\mathcal{WW}$, $\mathcal{WW}_0$, $\mathcal{EWW}$, and $\mathcal{EWW}_0$ are proper. Eigen and Hajian ask: If Ω is an exhaustive weakly wandering sequence for some transformation T, must Ω be an exhaustive weakly wandering sequence for some *ergodic* transformation T'? In particular, the answer to this is no:
Theorem 10. *There is a sequence $\Omega \in [\omega]^\omega$ such that $\Omega \in \mathcal{EWW}$ but $\Omega \notin \mathcal{WW}_0$.*

Modifications

We can in fact show that the four sets $\mathcal{WW} \setminus (\mathcal{WW}_0 \cup \mathcal{EWW})$, $\mathcal{EWW} \setminus \mathcal{WW}_0$, $\mathcal{WW}_0 \setminus \mathcal{EWW}$, and $(\mathcal{WW}_0 \cap \mathcal{EWW}) \setminus \mathcal{EWW}_0$ are all $\mathbf{\Sigma}^1_1$-hard.

In all of the constructions, we built sequences Ω as subsets of $\mathbb{N}$. We can get the same results for sequences which are subsets of $\mathbb{Z}$ unbounded in both directions.

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