# Isomorphism of subshifts is a universal countable Borel equivalence relation

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# Subshifts

## Definition

Let $A$ be a finite set of symbols. A *one-dimensional subshift on $A$* is a closed subset of $A^{\mathbb{Z}}$ which is invariant under the shift operator $S$, where $S(x)(n) = x(n+1)$. Two subshifts $X$ on $A$ and $Y$ on $B$ are *isomorphic* if there is a homeomorphism $\varphi : X \to Y$ which commutes with $S$.

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## Definition

A subshift may be defined by a set of *forbidden words* $W \subseteq A^{<\mathbb{N}}$ (where $A^{<\mathbb{N}}$ is the set of finite sequences from $A$), where $W$ determines the subshift

$$X_W = \{x \in A^{\mathbb{Z}} : \forall w \in W\ (w \not\sqsubseteq x)\}$$

(and $w \sqsubseteq x$ means that $w$ occurs as a subsequence of $x$).
A subshift is said to be of *finite type* when $W$ is finite.

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- One can ask how complicated a set of complete invariants for isomorphism needs to be.
- We will consider this equivalence relation from the standpoint of descriptive set theory, and determine its complexity among Borel equivalence relations under the relation of Borel reducibility, $\leq_B$.
- This will allow us to gauge the difficulty of this classification problem, and compare it to other problems.

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We say that an equivalence relation $E$ on a space $X$ is *Borel reducible* to an equivalence relation $F$ on the space $Y$, $E \leq_B F$, if there is a Borel-measurable function $f : X \to Y$ such that for all $x_1, x_2 \in X$ we have $x_1 \, E \, x_2$ if and only if $f(x_1) \, F \, f(x_2)$.
We write $E \sqsubseteq_B F$ when the we can find such an $f$ which is injective.

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We write $E \sqsubseteq_B F$ when the we can find such an $f$ which is injective.

Note that having $E \leq_B F$ amounts to having a definable injection from the quotient space $X/E$ into the quotient $Y/F$, so that when $E \leq_B F$ we may view the relation $F$ as being at least as complicated as $E$.

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- Recursive homeomorphism of elements of $5^{\mathbb{N}}$.
- Conjugacy of subgroups of $\mathbb{F}_2$.

Any two of these are bi-reducible.

# Borel reducibility (cont.)

## Definition

An equivalence relation $E$ on $X$ is *smooth* if it is Borel reducible to the identity relation on some Polish space $Y$, i.e. there is a Borel-measurable function $f : X \to Y$ such that $x_1 \, E \, x_2$ if and only if $f(x_1) = f(x_2)$.

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Smooth equivalence relations are those which admit a single element of some Polish space as a definable complete invariant.

A standard example of a non-smooth countable Borel equivalence relation is the relation $E_0$ of eventual equality on the Cantor space $2^{\mathbb{N}}$.

In particular, any universal countable Borel equivalence relation is non-smooth.

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- If a complicated equivalence relation $E$ is Borel reducible to an equivalence relation on a suitable class of structures, then there can be no complete invariants for the second relation which are fundamentally simpler than complete invariants for $E$.
- If we can reduce $E_0$ to some equivalence relation, then this relation does not admit a single real (or even a finite set of reals) as a complete invariant.
- We will show that isomorphism of subshifts is a universal countable Borel equivalence relation.
- In particular it is not smooth, and it will not admit complete invariants fundamentally simpler than the equivalence classes under isomorphism.

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For $n \geq 2$, the space $\mathcal{F}_n^S$ is the set of closed, $S$-invariant subsets of $n^{\mathbb{Z}}$. We view this as a subspace of the compact Polish space $\mathcal{F}(n^{\mathbb{Z}}) = K(n^{\mathbb{Z}})$ in the Vietoris topology.

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**Lemma**

*$\mathcal{F}_n^S$ is a closed subspace of $K(n^{\mathbb{Z}})$.*

**Proof.**

It suffices to check that the function $\tilde{S} : K(n^{\mathbb{Z}}) \to K(n^{\mathbb{Z}})$ given by $\tilde{S}(X) = S[X]$ is a homeomorphism, since then $\mathcal{F}_n^S = \{X \in K(n^{\mathbb{Z}}) : \tilde{S}(X) = X\}$ is closed.

# The space of subshifts (cont.)

## Proof (cont.).

- Let $d$ be the compatible metric on $n^{\mathbb{Z}}$ given by

$$d(x,y) = 2^{-\inf\{|k| : x(k) \neq y(k)\}} \text{ for } x \neq y.$$

  Then $d(S(x), S(y)) \leq 2d(x,y)$.

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  Then $d(S(x), S(y)) \leq 2d(x, y)$.
- Let $d_H$ be the Hausdorff metric on $K(n^{\mathbb{Z}})$ given by

$$d_H(X, Y) = \sup_{x \in X} \inf_{y \in Y} d(x, y) \quad \text{for } X, Y \neq \emptyset$$
$$d_H(X, \emptyset) = 1 \quad \text{for } X \neq \emptyset.$$

  This is a compatible metric for $K(n^{\mathbb{Z}})$.

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$$d_H(\tilde{S}(X), \tilde{S}(Y)) = \sup_{x \in \tilde{S}(X)} \inf_{y \in \tilde{S}(Y)} d(x, y) = \sup_{x \in X} \inf_{y \in Y} d(S(x), S(y))$$
$$\leq 2 \sup_{x \in X} \inf_{y \in Y} d(x, y) = 2 d_H(X, Y).$$

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Hence $\tilde{S}$ is continuous.
$\tilde{S}^{-1}$ is similar, so $\tilde{S}$ is a homeomorphism. □

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## Definition

For $n \geq 2$, let $Z_n$ be the set of countable sequences of finite sequences in $n$, i.e. $Z_n = (n^{<\mathbb{N}})^{\mathbb{N}}$, with the product topology. For $W \in Z_n$ we associate the space $X_W \in \mathcal{F}_n^S$ given by

$$X_W = \{x \in n^{\mathbb{Z}} : \forall i \, (W(i) \neq \emptyset \Rightarrow W(i) \not\sqsubseteq x)\}.$$

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- Then every subshift on $n$ is represented by an element $W \in Z_n$.
- Several different sets of forbidden words may produce the same subshift; we wish to see that this is not problematic.

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Since $W_1 \sim W_2$ if and only if $F(W_1) = F(W_2)$, we thus have:

**Corollary**

*The relation $\sim$ is a smooth Borel equivalence relation on $Z_n$.* $\square$

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Hence it will not matter which representation we use for subshifts (from the standpoint of Borel reducibility).

# The isomorphism relation on subshifts

We now consider the isomorphism relation on the spaces $\mathcal{F}_n^S$.

**Definition**

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A basic result which we will utilize is the following:

### Theorem (Curtis-Hedlund-Lyndon)

*Suppose $X$ and $Y$ are subshifts on $A$ and $B$, respectively, and $\varphi : X \to Y$ is a morphism of subshifts, i.e. a map commuting with the shift. Then $\varphi$ is given by a block code, i.e. there is a natural number $r$ and a function $\pi : A^{\{-r,\ldots,r\}} \to B$ such that*

$$\varphi(x)(n) = \pi(S^n(x) \restriction \{-r, \ldots, r\}).$$

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Note that there are only countably many such block codes, and any isomorphism of subshifts is recursive.

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Each $E$ class is countable since there are only countable many block codes $\pi$ and each determines $Y$ uniquely from $X$.

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Each $E$ class is countable since there are only countable many block codes $\pi$ and each determines $Y$ uniquely from $X$.
To see that $E$ is Borel, let $f_\pi$ be the continuous function given by a block code $\pi : n^{\{-r,\dots,r\}} \to n$. We then have:

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& (x \in Y \Rightarrow f_{\tilde{\pi}} \in X) \wedge (x \in X \Rightarrow f_{\tilde{\pi}} \circ f_\pi(x) = x)] \\
\Leftrightarrow\ & \exists\pi\exists\tilde{\pi}\neg\exists x \in n^{\mathbb{Z}}\ [(x \in X \setminus f_\pi^{-1}[Y])\vee \\
& (x \in Y \setminus F_{\tilde{\pi}}^{-1}[X]) \vee (x \in X \wedge f_{\tilde{\pi}}(f_\pi(x)) \neq x)]
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The matrix is $K_\sigma$ so the projection is also $K_\sigma$; as $\pi$ and $\tilde{\pi}$ range over a countable set this relation is $\Sigma_3^0$ and hence Borel. □

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This is a universal countable Borel equivalence relation.
We will need the following technical lemma:

**Lemma**

*There is an $\mathbb{F}_2$-invariant Borel set $A \subseteq 2^{\mathbb{F}_2}$ such that:*

- *$E(\mathbb{F}_2, 2) \sqsubseteq_B E(\mathbb{F}_2, 2) \restriction A$, so $E(\mathbb{F}_2, 2) \restriction A$ is a universal countable Borel equivalence relation.*
- *For $x, y \in A$ with $\neg x \; E(\mathbb{F}_2, 2) \; y$, there are infinitely many $g \in \mathbb{F}_2$ with $x(g) \neq y(g)$.*

# Complexity of the isomorphism relation (cont.)

**Proof.**

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For $x \in 2^{\mathbb{F}_2}$ let $f(x)(w) =$

$$\begin{cases} x(u) & \text{if } w = \rho(u) \text{ for some } u \text{ or} \\ & \quad w = \rho(u)abv \text{ for some } u \text{ and some reduced word} \\ & \qquad v \text{ whose leftmost symbol is not } b^{-1} \\ 0 & \text{otherwise} \end{cases}$$

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Then $A$ is Borel since $f$ is injective.

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It is straightforward to check that $A$ is as desired. □

# Complexity of the isomorphism relation (cont.)

**Theorem**

*The isomorphism relation on one-dimensional subshifts is a universal countable Borel equivalence relation.*

# Complexity of the isomorphism relation (cont.)

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## Proof.

- We will show that $E(\mathbb{F}_2, 2) \restriction A \leq_B E$, where $A$ is the set from the previous lemma and $E$ is the isomorphism relation for subshifts on the alphabet 2.

# Complexity of the isomorphism relation (cont.)

## Theorem

*The isomorphism relation on one-dimensional subshifts is a universal countable Borel equivalence relation.*

## Proof.

- We will show that $E(\mathbb{F}_2, 2) \restriction A \leq_B E$, where $A$ is the set from the previous lemma and $E$ is the isomorphism relation for subshifts on the alphabet 2.
- Given a point $x \in 2^{\mathbb{F}_2}$, we will produce a subshift $\varphi(x) = X_x$.

# Complexity of the isomorphism relation (cont.)

**Theorem**

*The isomorphism relation on one-dimensional subshifts is a universal countable Borel equivalence relation.*

**Proof.**

- We will show that $E(\mathbb{F}_2, 2) \restriction A \leq_B E$, where $A$ is the set from the previous lemma and $E$ is the isomorphism relation for subshifts on the alphabet 2.
- Given a point $x \in 2^{\mathbb{F}_2}$, we will produce a subshift $\varphi(x) = X_x$.
- Let $s \frown t$ denote the concatenation of two finite sequences $s$ and $t$.

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**Theorem**

*The isomorphism relation on one-dimensional subshifts is a universal countable Borel equivalence relation.*

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- Given a point $x \in 2^{\mathbb{F}_2}$, we will produce a subshift $\varphi(x) = X_x$.
- Let $s \frown t$ denote the concatenation of two finite sequences $s$ and $t$.
- Let $|s|$ be the length of a finite sequence.
- Let $\mathbb{F}_2$ be generated by $a$ and $b$.

# Complexity of the isomorphism relation (cont.)

**Proof (cont.).**

Define a map $\rho : \mathbb{F}_2 \to 2^{<\mathbb{N}}$ by letting:

$$\begin{aligned}
\rho(e) &= 11000011 \\
\rho(a) &= 11100011 \\
\rho(a^{-1}) &= 11010011 \\
\rho(b) &= 11001011 \\
\rho(b^{-1}) &= 11000111
\end{aligned}$$

$$\begin{aligned}
\rho(w) = \rho(w_0) &\frown \cdots \frown \rho(w_n) \\
&\text{for } w = w_0 \frown \cdots \frown w_n \neq e \text{ a reduced word} \\
&\text{with each } w_i \in \{a, a^{-1}, b, b^{-1}\} \text{ and } w_{i+1} \neq w_i^{-1}.
\end{aligned}$$

Note that for each $w \in \mathbb{F}_2$, $|\rho(w)| = 8k$ for some $k > 0$.

# Complexity of the isomorphism relation (cont.)

## Proof (cont.).

- Define the map $d : 2 \longrightarrow 2^{<\mathbb{N}}$ by:

$$d(0) = 101$$
$$d(1) = 111.$$

  Note that both of these sequences have length 3.

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- For a finite sequence $s \in 2^{<\mathbb{N}}$, let $\bar{0} \frown s \frown \bar{0}$ denote the element of $2^{\mathbb{Z}}$ whose negative coordinates are all 0, followed by $s$, followed by all 0's.

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- Let $\bar{s}$ be the periodic element of $2^{\mathbb{Z}}$ with period $|s|$ induced by $s$ (starting at coordinate 0).

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- Let $\bar{s}$ be the periodic element of $2^{\mathbb{Z}}$ with period $|s|$ induced by $s$ (starting at coordinate 0).
- Finally, let $p(i)$ denote the $i$-th prime.

# Complexity of the isomorphism relation (cont.)

**Proof (cont.).**

For an element $x \in 2^{\mathbb{F}^2}$ define the countable set $A_x$:

$$A_x = \{\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\} \cup$$
$$\{\overline{d(0) \frown 0^{p(2n+2)-3}} : n \in \mathbb{N}\} \cup$$
$$\{\overline{d(1) \frown 0^{p(2n+3)-3}} : n \in \mathbb{N}\}$$

# Complexity of the isomorphism relation (cont.)

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For an element $x \in 2^{\mathbb{F}^2}$ define the countable set $A_x$:

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Let $\varphi(x) = X_x$ where:

$$X_x = \overline{\bigcup_{n \in \mathbb{Z}} S^n[A_x]}.$$

Thus, $X_x$ is the smallest subshift containing $A_x$.

# Complexity of the isomorphism relation (cont.)

**Proof (cont.).**

We first check that $\phi$ is a Borel-measurable map from $2^{\mathbb{F}_2}$ to $\mathcal{F}_2^s$.

# Complexity of the isomorphism relation (cont.)

**Proof (cont.).**

We first check that $\phi$ is a Borel-measurable map from $2^{\mathbb{F}_2}$ to $\mathcal{F}_2^s$. To see this, first note that the intermediate map $x \mapsto W_x$ is Borel, since:

$$s \in W_x \text{ if and only if}$$

$$\neg\exists i \exists k \exists n \, (s \sqsubseteq 0^m \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown 0^m) \wedge$$

$$\neg\exists n \exists m \, (s \sqsubseteq (d(0) \frown 0^{p(2n+2)-3})^m) \wedge$$

$$\neg\exists n \exists m \, (s \sqsubseteq (d(1) \frown 0^{p(2n+3)-3})^m).$$

# Complexity of the isomorphism relation (cont.)

**Proof (cont.).**

We first check that $\phi$ is a Borel-measurable map from $2^{\mathbb{F}_2}$ to $\mathcal{F}_2^s$. To see this, first note that the intermediate map $x \mapsto W_x$ is Borel, since:

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Then $\varphi$ is this map followed by the Borel map given earlier which sends $W_x$ to $X_x$.

# Complexity of the isomorphism relation (cont.)

**Proof (cont.).**

It remains to check that $\varphi$ witnesses that $E(\mathbb{F}_2, 2) \restriction A \leq_B E$.

# Complexity of the isomorphism relation (cont.)

**Proof (cont.).**

It remains to check that $\varphi$ witnesses that $E(\mathbb{F}_2, 2) \restriction A \leq_B E$. Recall $A_x$:

$$A_x = \{\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\} \cup \{\overline{d(0) \frown 0^{p(2n+2)-3}} : n \in \mathbb{N}\} \cup \{\overline{d(1) \frown 0^{p(2n+3)-3}} : n \in \mathbb{N}\}$$

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It remains to check that $\varphi$ witnesses that $E(\mathbb{F}_2, 2) \restriction A \leq_B E$. Recall $A_x$:

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The periodic points of $X_x$ are (shifts of) the following points:

| | |
|---|---|
| $\bar{0}$ | period 1 |
| $\overline{d(0) \frown 0^{p(2n+2)-3}}$ | period $p(2n+2)$ |
| $\overline{d(1) \frown 0^{p(2n+3)-3}}$ | period $p(2n+3)$ |
| $\overline{\rho(w)}$ for $w \in \mathbb{F}_2$ | period $8 \cdot \lvert w \rvert$ (where $\lvert e \rvert = 1$) |

# Complexity of the isomorphism relation (cont.)

## Proof (cont.).

For every $x$, $X_x$ also includes the following limit points:

$$\bar{0} \frown d(0) \frown \bar{0}$$
$$\bar{0} \frown d(1) \frown \bar{0}$$
$$\bar{0} \frown \rho(w) \frown \bar{0} \quad \text{for } w \in \mathbb{F}_2$$

all left-, right- or bi-infinite sequences of $\rho(w)$'s (with other coordinates 0).

# Complexity of the isomorphism relation (cont.)

**Proof (cont.).**

For every $x$, $X_x$ also includes the following limit points:

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(with other coordinates 0).

Depending on the orbit of $x$ (for various $i$, $k$ and right-infinite words $w^*$ from $\mathbb{F}_2$) it may also include points of the form:

$$\bar{0} \frown d(i) \frown 0^{10+k^2} \frown \rho[w^*].$$

# Complexity of the isomorphism relation (cont.)

**Proof (cont.).**

Suppose first that $x, y \in 2^{\mathbb{F}_2}$ with $x \, E(\mathbb{F}_2, 2) \, y$.
We show that $(X_x, S) \cong (Y_x, S)$ (we do not need $x, y \in A$).

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- Since $a$ and $b$ generate $\mathbb{F}_2$, it suffices to show this when $y = a \cdot x$ or $y = b \cdot x$.

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- Since $a$ and $b$ generate $\mathbb{F}_2$, it suffices to show this when $y = a \cdot x$ or $y = b \cdot x$.
- Consider the case $y = a \cdot x$ (the case of $y = b \cdot x$ is identical).

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- Consider the case $y = a \cdot x$ (the case of $y = b \cdot x$ is identical).
- Let $P$ be the set:

$$\{\overline{d(0) \frown 0^{p(2n+2)-3}} : n \in \mathbb{N}\} \cup \{\overline{d(1) \frown 0^{p(2n+3)-3}} : n \in \mathbb{N}\}$$

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- Then we have:

$$A_x = P \;\cup\; \{\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\}$$

# Complexity of the isomorphism relation (cont.)

## Proof (cont.).

Compare this to $A_y$ when $y = a \cdot x$:

$$\begin{aligned} A_y &= P \cup \{\bar{0} \frown d(w_i \cdot y(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\} \\ &= P \cup \{\bar{0} \frown d(w_i a \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\} \\ &= P \cup \{\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i a^{-1}) \frown \bar{0} : i, k \in \mathbb{N}\} \\ A_x &= P \cup \{\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\} \end{aligned}$$

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&= P \cup \{\bar{0} \frown d(w_i a \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\} \\
&= P \cup \{\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i a^{-1}) \frown \bar{0} : i, k \in \mathbb{N}\} \\
A_x &= P \cup \{\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\}
\end{aligned}$$

- We want a map from $A_x$ to $A_y$ which sends each point

$$\begin{aligned}
\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} \quad &\mapsto \\
\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i a^{-1}) \frown \bar{0}.&
\end{aligned}$$

# Complexity of the isomorphism relation (cont.)

## Proof (cont.).

Compare this to $A_y$ when $y = a \cdot x$:

$$
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&= P \cup \{\bar{0} \frown d(w_i a \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\} \\
&= P \cup \{\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i a^{-1}) \frown \bar{0} : i, k \in \mathbb{N}\} \\
A_x &= P \cup \{\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\}
\end{aligned}
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&\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i a^{-1}) \frown \bar{0}.
\end{aligned}
$$

- All other sequences are left unchanged.

# Complexity of the isomorphism relation (cont.)

## Proof (cont.).

Compare this to $A_y$ when $y = a \cdot x$:

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A_y &= P \cup \{\bar{0} \frown d(w_i \cdot y(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\} \\
&= P \cup \{\bar{0} \frown d(w_i a \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\} \\
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\end{aligned}$$

- We want a map from $A_x$ to $A_y$ which sends each point

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&\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} \qquad \mapsto \\
&\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i a^{-1}) \frown \bar{0}.
\end{aligned}$$

- All other sequences are left unchanged.
- This induces an $S$-invariant homeomorphism of $X_x$ with $X_y$.

# Complexity of the isomorphism relation (cont.)

## Proof (cont.).

We define a block code $\pi : 2^{\{-r,\ldots,r\}} \to 2$ with $r = 16$ so that $f_\pi$ is an isomorphism of $X_x$ and $X_y$.

# Complexity of the isomorphism relation (cont.)

## Proof (cont.).

We define a block code $\pi : 2^{\{-r,\dots,r\}} \to 2$ with $r = 16$ so that $f_\pi$ is an isomorphism of $X_x$ and $X_y$.
We want the following subsequences mapped as shown:

$$
\begin{aligned}
0 \frown \rho(e) \frown 0 &\quad\mapsto\quad 0 \frown \rho(a^{-1}) \frown 0 \\
0 \frown \rho(a) \frown 0 &\quad\mapsto\quad 0 \frown \rho(e) \frown 0 \\
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and all other sequences should be left unchanged.

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- Taking $r = 16$ is sufficient for each image digit to "see enough" of the input to tell if and wherein it is part of one of the listed sequences.
- This gives a shift-invariant homeomorphism of $A_x$ with $A_y$ which extends to an isomorphism of the subshifts $X_x$ and $X_y$.

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For the other direction, suppose $x, y \in A$ and $X_x \cong X_y$ via $f_\pi$ induced by the block code $\pi : 2^{\{-r,\ldots,r\}} \rightarrow 2$.

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  Hence $\pi(0^{2r+1}) = 0$.

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- There is also a sequence $s_0$ and $m_0 \in \{-r, \ldots, r\}$ such that $f_\pi(\bar{0} \frown \rho(e) \frown \bar{0}) = S^{m_0}(\bar{0} \frown s_0 \frown \bar{0})$.

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For sufficiently large $k$ (i.e. $k > r$), consider the points:

$$\bar{0} \frown d(x(w_k)) \frown 0^{10+k^2} \frown \rho(e) \frown \bar{0} \text{ in } X_x.$$

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Thus $\rho(w_{i_k}) = s_0$ for sufficiently large $k$, so there is some fixed $w \in \mathbb{F}_2$ such that $s_0 = \rho(w)$.

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- Since $\rho(w_{i_k}) = s_0 = \rho(w)$ for sufficiently large $k$, we must have $w_{i_k} = w$ for all but finitely many $k$.

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Hence $y \, E(\mathbb{F}_2, 2) \, x$ and we are done. □

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**Definition**

An *n-dimensional subshift* on an alphabet $A$ is a closed subset of $A^{\mathbb{Z}^n}$ which is invariant under the shift maps $S_1, \ldots, S_n$, where $S_k(x)(i_1, \ldots, i_n) = x(i_1, \ldots, i_{k-1}, i_k + 1, i_{k+1}, \ldots, i_n)$.

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- The reverse is also true:

# Higher dimensional subshifts (cont.)

**Theorem**

*For each $n \geq 1$, we have that $E \leq_B E_n$, where $E$ is isomorphism of one-dimensional subshifts on $2$.*
*In particular all the relations $E_n$ are all universal countable Borel equivalence relations, and hence they are all mutually bi-reducible.*

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## Proof.

Fix $n \geq 1$, and consider the injection $f : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}^n}$ given by

$$f(x)(i_1, \ldots, i_n) = x(i_1).$$

This induces the map $\tilde{f} : \mathcal{F}_2^S \to \mathcal{F}_2^{S,n}$ given by $\tilde{f}(X) = f[X]$.
It is straightforward to check that $\tilde{f}$ is a reduction of $E$ to $E_n$. □

# Higher dimensional subshifts (cont.)

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- The above theorem shows that their classification up to isomorphism is no more difficult than for the one-dimensional case, at least from the standpoint of Borel reducibility.
- In particular, for each two-dimensional subshift $X$ we can assign a one-dimensional subshift $X'$ in a Borel-measurable way, so that $X \cong Y$ if and only if $X' \cong Y'$.

# Questions

Here we have considered the two-sided shift on $A^{\mathbb{Z}}$.
We could also consider the one-sided shift on $A^{\mathbb{N}}$.

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The proof above made fundamental use of the periodic points of the subshift $X_x$. We could ask to what extent this is necessary.

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**Question**

*What is the complexity of the isomorphism relation restricted to subshifts with no periodic points?*

# Questions (cont.)

For subshifts $X$ and $Y$, write $X \geq Y$ if there is a shift morphism $f : X \to Y$. We write $X \geq_{\text{inj}} Y$ when $f$ is injective, and $X \geq_{\text{surj}} Y$ when $f$ is surjective.

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We can then define the following three equivalence relations:

$$\begin{aligned}
X \, E_{\text{morph}} \, Y &\Leftrightarrow X \geq Y \wedge Y \geq X \\
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*What are the complexities of the above three equivalence relations?*

All of these contain the isomorphism relation.
$E_{\text{morph}}$ contains the other two.
These no longer have all equivalence classes countable.