Complemented sets, difference sets, and weakly wandering sequences

John D. Clemens

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Abstract

We consider the descriptive complexity of several sets of sequences of natural numbers, and show that the following are all complete analytic sets: the set of complemented sequences, the set of sequences containing an infinite difference set, the set of sequences which are weakly wandering sequences for some transformation, and several variants of these. We then use the same techniques to produce weakly wandering sequences with special properties, such as a sequence which is exhaustive weakly wandering for some transformation but which is not weakly wandering for any ergodic transformation.

In this paper we consider descriptive aspects of *weakly wandering sequences*. These sequences are isomorphism invariants for measure-preserving transformations or Borel automorphisms introduced by Hajian and Kakutani in [6]. We first consider how difficult it is to determine whether some sequence can be a weakly wandering sequence for some transformation, an exhaustive weakly wandering sequence, etc. We show that these are all complete analytic ($\mathbf{\Sigma}^1_1$-complete) questions. We also examine the connection between such sequences and complemented sequences or sequences containing an infinite difference set. We then use the techniques developed to construct some particular sequences to show that all of the above classes of weakly wandering sequences are distinct; for instance, we construct a sequence which is an exhaustive weakly wandering sequence for some transformation T, but for no *ergodic* T.

The paper is organized as follows. In Section 1 we introduce several classes of weakly wandering sequences and determine their complexity. In Section 2 we define complemented sets and difference sets and similarly consider their complexity. Section 3 presents the Main Lemma, which is used to show that the various classes of sequences defined in the first two sections are all complete analytic sets. In Section 4 we construct weakly wandering sequences with special properties, and in the final section we present several modifications of the preceding techniques.

1 Weakly wandering sequences

Weakly wandering sequences are increasing sequences of natural numbers associated to measure-preserving transformations of a measure space $(X, \mathcal{B}, \mu)$. They are motivated by the notion of a *wandering set*, where a set $B \subseteq X$ is *wandering* for a transformation T if B meets each orbit of T in exactly one point, i.e., $\{T^n[B] : n \in \mathbb{Z}\}$ forms a partition of X a.e.. A *weakly wandering set* is one for which we only require that certain translates of the set be disjoint, and a weakly wandering sequence consists of the mutually disjoint translates. We use $[\mathbb{N}]^{\mathbb{N}}$ to denote the set of increasing sequences of natural numbers.

Definition 1. Let T be a measure-preserving transformation of a measure space $(X, \mathcal{B}, \mu)$. An increasing sequence of natural numbers Ω in $[\mathbb{N}]^{\mathbb{N}}$ is a (positive) *weakly wandering sequence* for T if there is a measurable set $A \subseteq X$ of positive measure such that for all n and m in Ω with $n \neq m$ we have

$$T^n[A] \cap T^m[A] = \emptyset.$$

We say Ω is an *exhaustive weakly wandering sequence* for T if there is such an A which also satisfies:

$$X = \bigcup_{n \in \Omega} T^n[A].$$

We can also consider weakly wandering sequences which are subsets of $\mathbb{Z}$ (where we demand that the sequence extend infinitely in both directions); we will restrict ourselves here to subsets of $\mathbb{N}$ since the results we obtain will also hold for subsets of $\mathbb{Z}$. We discuss the necessary modifications for bi-infinite sequences in Section 5.

A transformation T is said to be *ergodic* if it admits an invariant ergodic measure μ, where a measure μ is *invariant* for T if $\mu(T[A]) = \mu(A)$ for any measurable set A, and it is *ergodic* if for any T-invariant measurable set A, either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$. This is equivalent to saying that the associated orbit equivalence relation generated by T is non-smooth. We can now define the key sets in this paper.

Definition 2. With the notation given, let:

$$\begin{aligned}
\mathcal{WW} &= \{\Omega : \Omega \text{ is weakly wandering for some transformation } T\} \\
\mathcal{WW}_0 &= \{\Omega : \Omega \text{ is weakly wandering for some ergodic } T\} \\
\mathcal{EWW} &= \{\Omega : \Omega \text{ is exhaustive weakly wandering for some } T\} \\
\mathcal{EWW}_0 &= \{\Omega : \Omega \text{ is exhaustive weakly wandering for some ergodic } T\}.
\end{aligned}$$

We see that $\mathcal{WW}_0$ and $\mathcal{EWW}$ are subsets of $\mathcal{WW}$, and that $\mathcal{EWW}_0 \subseteq \mathcal{EWW} \cap \mathcal{WW}_0$. Our first goal will be to give a descriptive set-theoretic classification of these four sets, and show that they are all complete analytic sets. We will then see in Section 4 that all of the above inclusions are proper.

There are known characterizations of these sets which we will use in establishing their descriptive complexity. We must first introduce some terminology. For two sets of integers A and B, we let $A+B$ denote their *sum*, $A+B = \{a+b : a \in A, b \in B\}$. We say that two sets of integers A and B have a *direct sum*, and denote this by writing $A+B = A \oplus B$, if for every $a_1 \neq a_2$ in A and every $b_1 \neq b_2$ in B we have $a_1 + b_1 \neq a_2 + b_2$. We say that A and B have *direct sum* $\mathbb{Z}$ if, in addition, every integer is equal to $a+b$ for some a in A and b in B. Thus:

$$A \oplus B = \mathbb{Z} \Longleftrightarrow \forall k \in \mathbb{Z} \exists! a \in A \exists! b \in B \ (k = a+b).$$

For a given set $A \subseteq \mathbb{Z}$ or $A \subseteq \mathbb{N}$, we say A is *complemented* if there is a set B with $A \oplus B = \mathbb{Z}$. Complemented sequences of natural numbers were first studied by DeBruijn in [1].

There is a useful fact about direct sums which we will exploit.

Definition 3. For a set $A \subseteq \mathbb{N}$ or $A \subseteq \mathbb{Z}$, the *difference set of* A is

$$D(A) = \{a_2 - a_1 : a_1, a_2 \in A \text{ and } a_2 \geq a_1\}.$$

Similarly, the *sum set of* A is

$$S(A) = \{a_1 + a_2 : a_1, a_2 \in A\}.$$

We then get the following easy characterization of when A and B have a direct sum.

Lemma 4. *We have that* $A+B = A \oplus B$ *if and only if* $D(A) \cap D(B) = \{0\}$.

Proof: This is an immediate consequence of the fact that $a_1 + b_1 = a_2 + b_2$ if and only if $a_1 - a_2 = b_2 - b_1$. □

We call a set B a *difference set* (from $\mathbb{N}$, resp. $\mathbb{Z}$) if $B = D(A)$ for some $A \subseteq \mathbb{N}$ (resp. $A \subseteq \mathbb{Z}$). We will say more about these sets in Section 2. There are two further observations we will use. First, if we define the *shift* of a set A by an integer n to be $A+n = \{a+n : a \in A\}$, then we see that a shift of a set has the same difference set as the original set: $D(A+n) = D(A)$. Thus, shifts of A and B have a direct sum if and only if A and B do. Second, if $A \oplus B = \mathbb{Z}$, then also $(A+n) \oplus B = \mathbb{Z}$.

We say that an increasing sequence $H = \langle \ldots h_{-1}, h_0, h_1, h_2, \ldots \rangle$ is a *hitting sequence* if for each finite consecutive subsequence $h_{-n}, \ldots, h_0, \ldots h_n$, this subsequence occurs shifted to both the right and left in H. That is, there are positive numbers k_l, k_r, n_l and n_r in $\mathbb{N}$ such that

$$h_{i+k_r} = h_i + n_r \text{ and } h_{i-k_l} = h_i - n_l \text{ for } -n \leq i \leq n.$$

We say that such an H has the *shift-repeat property.*

We can now give the characterizations of the four collections of sequences which we will use for our classification. This characterization is due to Eigen and Hajian ([3]), extending work of Kamae ([7]):

Theorem (Eigen and Hajian, Kamae). *For Ω in $[\mathbb{N}]^{\mathbb{N}}$:*

1. $\Omega \in \mathcal{WW} \iff$ *there is $H \subseteq \mathbb{Z}$ such that $H + \Omega = H \oplus \Omega$.*
2. $\Omega \in \mathcal{WW}_0 \iff$ *there is a hitting sequence $H \subseteq \mathbb{Z}$ such that $H + \Omega = H \oplus \Omega$.*
3. $\Omega \in \mathcal{EWW} \iff$ *there is $H \subseteq \mathbb{Z}$ such that $H \oplus \Omega = \mathbb{Z}$.*
4. $\Omega \in \mathcal{EWW}_0 \iff$ *there is a hitting sequence $H \subseteq \mathbb{Z}$ such that $H \oplus \Omega = \mathbb{Z}$.*

Note that this gives $\mathbf{\Sigma}^1_1$ definitions for each of these sets, so that they are all analytic sets. We shall now show that they are in fact $\mathbf{\Sigma}^1_1$-complete, i.e., they are complete analytic sets. In particular, none of these are Borel sets.

Let $\mathcal{T}$ denote the set of trees on $\mathbb{N}$, i.e. the set of all subsets of $\mathbb{N}^{<\mathbb{N}}$ which are closed under taking initial segments. We write $s \sqsubset t$ to indicate that the sequence s is an initial segment of t. Trees can be naturally coded as elements of $\mathcal{P}(\mathbb{N})$ (which we identify with $2^{\mathbb{N}}$) in the following way. Let $\{s_n : n \in \mathbb{N}\}$ be a recursive enumeration of $\mathbb{N}^{<\mathbb{N}}$ such that if $s_n \sqsubset s_m$ then $n < m$. We can then set:

$$\mathcal{T} = \{T \in \mathcal{P}(\mathbb{N}) : \forall n \forall k \; [(n \in T \;\&\; s_k \sqsubset s_n) \Rightarrow k \in T]\}.$$

This is a closed subset of $2^{\mathbb{N}}$ and can thus be viewed as a Polish space in the relative topology. A tree is *well-founded* if it has no infinite branches. A tree which contains an infinite branch is called *ill-founded*. We let $\mathcal{WF} \subseteq \mathcal{T}$ be the set of well-founded trees. The following is a standard fact (see [8] or [9]):

Theorem. *The set $\mathcal{WF}$ is $\mathbf{\Pi}^1_1$-complete as a subset of $\mathcal{T}$. Hence, $\mathcal{T} \setminus \mathcal{WF}$ is $\mathbf{\Sigma}^1_1$-complete.*

We will now show that the four sets $\mathcal{WW}$, $\mathcal{WW}_0$, $\mathcal{EWW}$, and $\mathcal{EWW}_0$ are all $\mathbf{\Sigma}^1_1$-complete by showing that $\mathcal{T} \setminus \mathcal{WF}$ is the continuous inverse image of each of them. We in fact will prove a more general fact. The following is the main technical lemma of this paper:

Main Lemma. *There is a continuous function $f : \mathcal{T} \to [\mathbb{N}]^{\mathbb{N}}$ sending T to Ω_T, and a continuous function $g : \mathcal{T} \to [\mathbb{N}]^{\mathbb{N}}$ sending T to β_T, such that:*

1. *If T is ill-founded, then there is a hitting sequence H such that $H \oplus \Omega_T = \mathbb{Z}$.*
2. *If T is well-founded then $D(\Omega_T) \setminus \{0\}$ meets every infinite difference set (from $\mathbb{N}$ or from $\mathbb{Z}$).*
3. *If T is ill-founded then there is an infinite set H such that $D(H) \subseteq \beta_T$.*

We will give the proof of the Main Lemma in Section 3. Here we use it to prove the descriptive classification of the four sets of weakly wandering sequences.

Theorem 5. *The four sets $\mathcal{WW}$, $\mathcal{WW}_0$, $\mathcal{EWW}$, and $\mathcal{EWW}_0$ are all $\mathbf{\Sigma}^1_1$-complete. In fact, if X is any set with $\mathcal{EWW}_0 \subseteq X \subseteq \mathcal{WW}$, then X is $\mathbf{\Sigma}^1_1$-hard.*

Proof: We see that with f constructed as in the Main Lemma, we have that $f[\mathcal{T} \setminus \mathcal{WF}] \subseteq \mathcal{EWW}_0$ and $f[\mathcal{WF}] \cap \mathcal{WW} = \emptyset$. Thus, if X is such that $\mathcal{EWW}_0 \subseteq X \subseteq \mathcal{WW}$, we have $\mathcal{T} \setminus \mathcal{WF} = f^{-1}[X]$. □

2 Complemented sets and difference sets

We can also use the Main Lemma to give descriptive classifications of the class of complemented sets and the class of sets which contain an infinite difference set. We recall the two definitions given earlier:

Definition 6. A set A (of integers or natural numbers) is said to be *complemented* (in $\mathbb{Z}$) if there is a set $B \subseteq \mathbb{Z}$ such that $A \oplus B = \mathbb{Z}$.

Definition 7. A set $A \subseteq \mathbb{N}$ is said to be a *difference set* (from $\mathbb{N}$, resp. $\mathbb{Z}$) if there is a set B (of natural numbers, resp. integers) such that $A = D(B)$.

Definition 8. We define the following four classes of sets of natural numbers:

$$\begin{aligned} \mathcal{COMP} &= \{A \subseteq \mathbb{N} : A \text{ is complemented in } \mathbb{Z}\} \\ \mathcal{DF} &= \{A \subseteq \mathbb{N} : \ A \text{ is a difference set from } \mathbb{N}\} \\ \mathcal{DF}_\infty &= \{A \subseteq \mathbb{N} : \ A \text{ is an infinite difference set}\} \\ \mathcal{CDF} &= \{A \subseteq \mathbb{N} : \ A \text{ contains an infinite difference set}\}. \end{aligned}$$

All four sets are $\mathbf{\Sigma}^1_1$. Note that the set $\mathcal{COMP}$ is the same as $\mathcal{EWW}$, so the Main Lemma immediately gives the following.

Corollary 9. *The set $\mathcal{COMP}$ is $\mathbf{\Sigma}^1_1$-complete.*

There are several results characterizing when certain types of sets are complemented; see for instance [4]. This corollary, though, shows that in a descriptive context no simpler classification for arbitrary sets is possible than the original definition.

In contrast to this, consider the case of sets $A \subseteq \mathbb{N}$ which are complemented in $\mathbb{N}$, i.e., where there is a $B \subseteq \mathbb{N}$ such that $A \oplus B = \mathbb{N}$. Let $\mathcal{COMP}_\mathbb{N}$ denote the collection of such sets. These were the types of complements originally studied by DeBruijn. Here the classification is much simpler, descriptively. For a given set A, consider the tree of attempts to build a complement for A by finite approximations. This tree will be finitely branching since the elements of A and B that can produce a given sum k must be bounded between 0 and k. Thus the question of whether a set is complemented in $\mathbb{N}$ amounts to asking whether there is an infinite branch through a finitely branching tree, which is only a $\mathbf{\Pi}^0_2$ question. Hence $\mathcal{COMP}_\mathbb{N}$ is a Borel set. In the case of complements in $\mathbb{Z}$, the difficulty is that potential witnesses for a given sum are unbounded.

As for difference sets, the proof of the Main Lemma produces the following corollary, due originally to Mannsfield.

Corollary 10 (Mannsfield). *The set $\mathcal{CDF}$ is $\mathbf{\Sigma}^1_1$-complete.*

Proof: The function $f : T \mapsto \Omega_T$ constructed in the Main Lemma induces the continuous function $g : T \mapsto \beta_T$. From condition (3) of the lemma we have that if T contains an infinite branch then $D(H) \subseteq \beta_T$ is an infinite difference set. If T is well-founded, then every infinite difference set meets $D(\Omega_T) \setminus \{0\} = \mathbb{N} \setminus \beta_T$, and hence is not contained in β_T. Hence the function g is a reduction of $\mathcal{WF}$ to $\mathcal{CDF}$. $\square$

A related theorem is due to Schmerl:

Theorem (Schmerl [10]). *The set* $\mathcal{DF}$ *is* $\mathbf{\Sigma}^1_1$*-complete.*

From this one easily sees that $\mathcal{DF}_\infty$ is also $\mathbf{\Sigma}^1_1$-complete. There does not seem to be any way to derive one of the two previous theorems from the other. The construction given here, for instance, necessarily produces sets which are not difference sets in the case that a tree T has more than one branch. Schmerl raised the following question in [11]:

Question 11. *Is it the case that every set* X *with* $\mathcal{DF}_\infty \subseteq X \subseteq \mathcal{CDF}$ *is* $\mathbf{\Sigma}^1_1$*-hard?*

If true, this would of course imply both theorems.

We note here one further interesting fact about difference sets. This is not a difficult result, but does not seem to appear in the literature. We need to pretend that sets do not contain 0 to avoid trivialities.

Proposition 12. *The set*

$$\{A \subseteq \mathbb{N} : \ A \setminus \{0\} \text{ meets every infinite difference set}\}$$

is a filter on $\mathbb{N}$.

Proof: The only condition which is non-trivial to check is that if we have $A, B \subseteq \mathbb{N}$ (both containing 0) such that $A \cup B$ contains an infinite difference set, then either A or B contains an infinite difference set. Let C be such that $D(C) \subseteq A \cup B$. Consider the partition of $[C]^2$:

$$P = \{(n, m) \in [C]^2 : |n - m| \in A\}$$

Applying the infinite Ramsey theorem to this partition, we see that there is an infinite set $H \subseteq C$ such that either $[H]^2 \subseteq P$ or $[H]^2 \cap P = \emptyset$. In the first case, $D(H) \subseteq A$, and in the second case $D(H) \subseteq B$. $\square$

3 Complexity of the set of weakly wandering sequences

In this section we prove the Main Lemma, which establishes all of the previously listed complexity results. The techniques of this proof will also be used in the following section to construct particular weakly wandering sequences.

Proof of the Main Lemma: We will define the continuous maps $T \mapsto \Omega_T$ and $T \mapsto \beta_T$. To make things combinatorially simpler, we first replace T with a new tree (which we will also denote by T) where we add the empty sequence (the root) and all sequences of length 1. The point is to make sure that T contains infinitely many nodes. This can be done continuously (in the codes for trees), and will not affect whether or not T contains an infinite branch. So we fix a tree T which we assume has infinitely many nodes. We let $\langle t_i \rangle_{i \in \mathbb{N}}$ enumerate the nodes of T relative to the ordering on $\mathbb{N}^{<\mathbb{N}}$ introduced above. We denote that a node t_i is a *predecessor* of t_j in T by writing $t_i \prec t_j$. Given a node $t_j \in T$, we say that t_i is the *immediate predecessor* of t_j if $t_i \prec t_j$ and there is no node t_k with $t_i \prec t_k \prec t_j$. Every node other than the root has a unique immediate predecessor. Our enumeration of T is such that if $t_i \prec t_j$ then $i < j$; in particular, t_0 is the root of the tree. We say two nodes t_i and t_j are *incomparable*, $t_i \perp t_j$, if neither is a predecessor of the other. We let $|t|$ denote the length of t.

We will build our sequence $\Omega = \Omega_T$ as the union of finite sequences ω_i, and will simultaneously construct potential witnesses H_i which will give a complement to Ω precisely when T contains an infinite branch. We also build the sequence $\beta = \beta_T$ by constructing finite sequences β_n at each step of the construction. We will proceed in stages, defining at stage n the finite sets $\omega_n \subseteq \mathbb{N}$ and $H_n \subseteq \mathbb{Z}$.

A few notational points: We are really building finite *sequences*, rather than subsets, so we will make decisions not only to add some integers to Ω, but also to keep some out. We will thus require, for instance, that ω_{n+1} is an *end-extension* of ω_n, meaning that ω_{n+1} adds no new integers less than the largest integer in ω_n. We will indicate that B end-extends A by writing $B \sqsupseteq A$, and say $B \sqsupset A$ if B properly extends A. For sequences from $\mathbb{Z}$ instead of $\mathbb{N}$, end-extension will mean that the new sequence contains no new elements between the largest and the smallest element of the original sequence. We will use $B \supseteq A$ as usual to mean B contains all elements of A. The largest and smallest elements of a set A are denoted $\max(A)$ and $\min(A)$, respectively. So we can express end-extension as:

$$A \sqsubseteq B \Longleftrightarrow B \cap [\min(A), \max(A)] = A.$$

Also, given a set $A \subseteq \mathbb{Z}$ and an integer n, we let $A+n$ denote the shift of A by n, $A+n=\{a+n : a \in A\}$. If $n>0$ we call this a *shift to the right*, and if $n<0$ we call it a *shift to the left*. We say that one sequence *occurs* in another if some shift of the first sequence forms a consecutive subsequence of the second. We let $A+B$ denote the set $\{a+b : a \in A, b \in B\}$, and we use $A \oplus B$ to indicate that the sum is direct. The set $A-B$ is defined similarly. To distinguish this from the set-theoretic difference of A and B, we will always use $A \setminus B$ to denote $\{n : n \in A \text{ and } n \notin B\}$. We also let $A \pm B = (A+B) \cup (A-B)$.

We will be constructing the sequences $\omega_n \subseteq \mathbb{N}$ and $H_n \subseteq \mathbb{Z}$ at each stage. We let $A_n = D(\omega_n)$, $D_n = D(H_n)$, and $\beta_n = \bigcup_{i \leq n} D_i$. We will inductively construct these sequences to satisfy the following nine conditions at each stage n, for all $i, j \leq n$:

1. If $i<j$ then $\omega_i \sqsubset \omega_j$ and $\beta_i \sqsubset \beta_j$.
2. $A_i \cap \beta_i = \{0\}$.
3. $A_i \cup \beta_i \supseteq [0, \max(\beta_i)]$.
4. If $t_i \prec t_j$ then $H_i \sqsubset H_j$.
5. Given $i<j$, let k be such that t_k is the maximal mutual predecessor of t_i and t_j. Then we require that $D_i \cap D_j \subseteq D_k$.
6. Given $i<j$ with $t_i \perp t_j$, and $a \in D_i \setminus \beta_{i-1}$ and $b \in D_j \setminus \beta_{j-1}$, we require that $|b-a| \in A_j$ (i.e., $|b-a| \notin \beta_j$).
7. Given i, let m be such that t_m is the immediate predecessor of t_i in T. Then we require that shifts of H_m occur in H_i both to the left and the right as consecutive blocks.
8. Let $\langle r_k \rangle_{k \in \mathbb{N}}$ be the enumeration of $\mathbb{Z}$ given by $\langle 0, 1, -1, 2, -2, \ldots \rangle$, and let $k_i = |t_i|$. Then we require that $r_{k_i} \in \omega_i \oplus H_i$.
9. For $i \leq j$ we require that $D(D_i)$ and $S(D_i)$ are disjoint from $\beta_j \setminus D_i$.

At the end of the construction we will set:

$$\Omega = \bigcup_i \omega_i$$

$$A = \bigcup_i A_i = D(\Omega)$$

$$\beta = \bigcup_i \beta_i.$$

Note that we will have $A \cap \beta = \{0\}$ and $A \cup \beta = \mathbb{N}$, i.e., $A \setminus \{0\}$ and $\beta \setminus \{0\}$ partition $\mathbb{N} \setminus \{0\}$.

The purpose of most of these conditions is fairly clear. Conditions (1), (2), and (3) ensure continuity; conditions (2) and (4) ensure that there will be an H having a direct sum with Ω in case T has an infinite branch; condition (7) guarantees that H will be a hitting sequence; condition (8) guarantees that the sum of Ω and H will be all of $\mathbb{Z}$; and conditions (5) and (6) will guarantee that if T is well-founded then $A \setminus \{0\} = D(\Omega) \setminus \{0\}$ meets all infinite difference sets. The last condition is less obvious; this turns out to be the inductive assumption necessary to extend the construction from one stage to the next.

Let us grant that the construction has been carried out, and see that it meets the requirements of the lemma. First, the map $f : T \mapsto \Omega_T$ is continuous, as knowing the first n elements of Ω requires knowing at most the first n nodes of T. The map $g : T \mapsto \beta_T$ is also easily seen to be continuous. So suppose first that T has an infinite branch $\langle t_{m_i} \rangle_{i \in \mathbb{N}}$, i.e. for all i we have $t_{m_i} \prec t_{m_{i+1}}$. Let $H = \bigcup_i H_{m_i}$. Then, since each H_{m_j} end-extends H_{m_i} for $i < j$ (by condition (4)), we will have

$$D(H) = \bigcup_i D(H_{m_i}) = \bigcup_i D_{m_i} \subseteq \beta.$$

Since $A \cap \beta = \{0\}$, we will have $D(\Omega) \cap D(H) = \{0\}$, so that Ω will have a direct sum with H. Condition (7) guarantees that H is a hitting sequence, since any finite subsequence of H must eventually occur as a subsequence of one of the H_{m_i}'s, and thus occurs shifted to the right and left in $H_{m_{i+1}}$ and hence in H. Finally, since each t_{m_i} has length i, each r_i will occur in $\omega_{m_i} \oplus H_{m_i}$, and hence $\Omega \oplus H = \mathbb{Z}$. Thus H gives us the desired witness for Ω_T being in $\mathcal{EWW}_0$.

Conversely, suppose that $D(\Omega_T) \setminus \{0\}$ is disjoint from some infinite difference set; we will show that T has an infinite branch. Any infinite difference set from $\mathbb{Z}$ contains an infinite difference set from $\mathbb{N}$, since if $B = D(C)$ with $C \subseteq \mathbb{Z}$ then one of the two sets:

$$C_+ = \{n : n \in C \text{ and } n \geq 0\}$$
$$C_- = \{-n : n \in C \text{ and } n \leq 0\}$$

will be infinite, and both $D(C_+)$ and $D(C_-)$ are subsets of $D(C)$ and hence subsets of B. Thus, we have some set $C \subseteq \mathbb{N}$ such that $D(C) \cap D(\Omega) = \{0\}$. Since $D(C - n) = D(C)$ for any n, we may shift C so that $0 \in C$. Thus we may suppose $C \subseteq \beta$, since we have $C \subseteq D(C) \subseteq \beta$. Let C_i be the subsequence containing the first i elements of C, and let m_i be the least m such that $C_i \subseteq \beta_m$. Then $m_i \leq m_j$ for $i < j$, and since each β_m is finite we will have that $m_i \to \infty$ as $i \to \infty$.

We claim that $t_{m_i} \preceq t_{m_j}$ for $i < j$. Suppose this fails for some $i < j$, and let m be such that t_m is the maximal mutual predecessor of t_{m_i} and t_{m_j} in T. We thus have $m < m_i$, $m < m_j$, and $m_i < m_j$. By the minimality of m_i and m_j, there are elements $a \in C_i$ and $b \in C_j$, with $a \neq b$, such that:

$$a \in \beta_{m_i} \setminus \beta_{m_i - 1} = D_{m_i} \setminus \beta_{m_i - 1}$$
$$b \in \beta_{m_j} \setminus \beta_{m_j - 1} = D_{m_j} \setminus \beta_{m_j - 1}.$$

Then, by condition (6), we have $|b - a| \in A_{m_j}$, and we know $|b - a| \neq 0$. But $|b - a| \in D(C)$, contradicting that $D(C) \cap A = \{0\}$. Thus, each $t_{m_{i+1}}$ extends t_{m_i} in T, and since these have lengths approaching ∞, we get an infinite branch through T.

Last, we note that when T has an infinite branch $\langle t_{m_i} \rangle_{i \in \mathbb{N}}$, then $H = \bigcup_{i \in \mathbb{N}} H_{m_i}$ will be infinite, so $\beta_T \supseteq D(H)$. Thus, our functions f and g will thus be as desired, once we show that we can build Ω_T as described.

The construction of Ω_T

We begin at stage 0 by setting $\omega_0 = \{0\}$ and $H_0 = \{0\}$. Then $A_0 = D_0 = \beta_0 = \{0\}$. It is easy to see that we have satisfied all the conditions, since $|t_0| = 0$ and $r_0 = 0$. So now suppose the construction has been completed to stage n, with $\omega_0, \ldots, \omega_n$ and $H_0, \ldots, H_n$ defined so as to satisfy the given conditions. We now define ω_{n+1} and H_{n+1} so as to satisfy them at stage $n + 1$.

Let m be such that t_m is the immediate predecessor of t_{n+1} in T. We will proceed in three steps:

Step I We end-extend H_m to $\widetilde{H}_{n+1}$ so as to satisfy condition (7) while not violating any of the other conditions (specifically, we need to ensure that the new differences introduced here to not violate conditions (1), (2), (5), (6), and (9)).

Step II If necessary, we end-extend ω_n to $\widetilde{\omega}_{n+1}$ and extend $\widetilde{H}_{n+1}$ to H_{n+1} in order to satisfy condition (8); otherwise we take $H_{n+1} = \widetilde{H}_{n+1}$ and $\widetilde{\omega}_{n+1} = \omega_n$. Again, we need to preserve conditions (1), (2), (5), (6), and (9).

Step III We end-extend $\widetilde{\omega}_{n+1}$ to ω_{n+1} in order to satisfy condition (3) (and make sure it is a proper extension of ω_n), while not violating condition (2).

Step I. We will choose two numbers Δ_l and Δ_r in $\mathbb{N}$, and define $\widetilde{H}_{n+1}$ to be the union of H_m and two shifts of H_m:

$$\widetilde{H}_{n+1} = (H_m - \Delta_l) \sqcup H_m \sqcup (H_m + \Delta_r).$$

We must choose Δ_l and Δ_r to preserve the necessary conditions. First, to guarantee that we end-extend H_m, we must make sure that the three blocks are disjoint. This is satisfied is we require:

$$\Delta_l, \Delta_r > (\max(H_m) - \min(H_m)) = \max(D_m). \tag{1}$$

To preserve condition (1), we must make sure that any new differences produced are bigger than any of the elements of β_n. The differences we have in D_{n+1} at this point will be in one of the following sets:

$$D_m,\ \Delta_l \pm D_m,\ \Delta_r \pm D_m,\ (\Delta_l + \Delta_r) \pm D_m.$$

New differences, then, are only produced by pairs of elements from different blocks, so it will suffice to ensure

$$\min(H_m) - \max(H_m - \Delta_l) > \max(\beta_n)$$

and

$$\min(H_m + \Delta_r) - \max(H_m) > \max(\beta_n).$$

Both of these are satisfied if we require:

$$\Delta_l, \Delta_r > \max(\beta_n) + \max(D_m). \tag{2}$$

Preserving condition (2) simply requires making sure that all new differences are bigger than the largest element of A_n. This is satisfied if:

$$\Delta_l, \Delta_r > \max(A_n) + \max(D_m). \tag{3}$$

To preserve condition (5), we need to ensure that $D_i \cap D_{n+1} \subseteq D_k$ for $i \leq n$, where t_k is the maximal mutual predecessor of t_i and t_{n+1} in T. Note that t_k is also the maximal mutual predecessor of t_i and t_m in T, so we already know that $D_i \cap D_m \subseteq D_k$. It thus suffices to make sure that the new differences are not in β_n, which is satisfied by our previous condition that

$$\Delta_l, \Delta_r > \max(\beta_n) + \max(D_m).$$

To preserve condition (6), we need to ensure the following: Given $i \leq n$ with $t_i \perp t_{n+1}$, for any $a \in D_i \setminus \beta_{i-1}$ and for any new difference b we need

$|b-a| \notin \beta_{n+1}$. If we require any new differences to be bigger than $2 \cdot \max(\beta_n)$, we will have that $|b-a| = b-a > \max(\beta_n)$ and so $|b-a|$ is not in β_n. We thus first require:

$$\Delta_l, \Delta_r > \max(D_m) + 2 \cdot \max(\beta_n). \tag{4}$$

We must still ensure that we do not have $b-a \in \beta_{n+1}$, which will only happen if $b-a \in D_{n+1}$, i.e., if we add two new differences b_1 and b_2 such that there is an element a in $D_i \setminus \beta_{i-1}$ with $b_2 - a = b_1$, which amounts to saying that $b_2 - b_1 \in D_i \setminus \beta_{i-1}$. To prevent this, it suffices to make sure that $D(D_{n+1})$ is disjoint from $D_i \setminus \beta_{i-1}$ whenever $t_i \perp t_{n+1}$. Note that if we satisfy condition (9) at stage $n+1$, then (taking $i = j = n+1$), we will have that $D(D_{n+1})$ is disjoint from $\beta_{n+1} \setminus D_{n+1} = \beta_n \setminus D_{n+1}$. Thus we will have:

$$D(D_{n+1}) \cap D_i \subseteq D(D_{n+1}) \cap \beta_n \subseteq D_{n+1}.$$

Then, since $t_i \perp t_{n+1}$ by assumption, we have $D_i \cap D_{n+1} \subseteq D_k$ where $k < i$ is the maximal mutual predecessor of t_i and t_{n+1} in T. But then $D_k \subseteq \beta_{i-1}$, so $D(D_{n+1})$ will be disjoint from $D_i \setminus \beta_{i-1}$. So we will preserve condition (6) if we preserve condition (9).

We lastly check that we can choose Δ_l and Δ_r so as to preserve condition (9). This will amount to ensuring two things at this stage. First, for $i \leq n$, we need $D(D_i)$ and $S(D_i)$ to be disjoint from $\beta_{n+1} \setminus D_i$. Second, we need $D(D_{n+1})$ and $S(D_{n+1})$ to be disjoint from $\beta_{n+1} \setminus D_{n+1}$.

For the first requirement, note that we already know by the inductive assumption that $D(D_i)$ and $S(D_i)$ are disjoint from $\beta_n \setminus D_i$, so we need only ensure that new differences in D_{n+1} are disjoint from $D(D_i)$ and $S(D_i)$ for all $i \leq n$. This will hold if we require:

$$\Delta_l, \Delta_r > \max(D_m) + \max\left(\bigcup_{i \leq n} (D(D_i) \cup S(D_i))\right). \tag{5}$$

We now consider the second requirement. Note that $\beta_{n+1} \setminus D_{n+1} = \beta_n \setminus D_{n+1}$, and we know by assumption that $D(D_m)$ and $S(D_m)$ are disjoint from $\beta_n \setminus D_m$. Thus, we will have that $D(D_m)$ and $S(D_m)$ are disjoint from $\beta_{n+1} \setminus D_{n+1}$. To simplify computations, we will require:

$$\begin{aligned} \Delta_l &> \Delta_r + 2 \cdot \max(D_m) \qquad (6) \\ \Delta_r &> 2 \cdot \max(D_m). \end{aligned}$$

Considering the elements of D_{n+1} added so far, we then see that elements of $D(D_{n+1})$ at this step will be in one of the following sets:

$$D(D_m),\ S(D_m),\ \Delta_l \pm D(D_m),\ \Delta_r \pm D(D_m),\ (\Delta_l - \Delta_r) \pm D(D_m),$$
$$(\Delta_l + \Delta_r) \pm D(D_m),\ \Delta_l \pm S(D_m),\ \Delta_r \pm S(D_m),\ (\Delta_l - \Delta_r) \pm S(D_m),$$
$$(\Delta_l + \Delta_r) + S(D_m).$$

Similarly, elements of $S(D_{n+1})$ so far will be in one of the sets:

$$S(D_m),\ \text{or}\ \left\{ \begin{array}{llll} \Delta_l, & 2\Delta_l, & 2\Delta_l + \Delta_r, & \Delta_l + \Delta_r, \\ \Delta_r, & 2\Delta_r, & \Delta_l + 2\Delta_r, & 2\Delta_l + 2\Delta_r \end{array} \right\} \pm \left\{ \begin{array}{l} D(D_m) \\ S(D_m) \end{array} \right\}$$

(with the exception of $\Delta_l - S(D_m)$ and $\Delta_r - S(D_m)$). By this we mean that we take one of the numbers in the first set and either add or subtract the elements of one of the sets $D(D_m)$ or $S(D_m)$. We already saw that $D(D_m)$ and $S(D_m)$ would not cause problems, and all of the other elements can be kept out of β_n (and hence out of $\beta_{n+1} \setminus D_{n+1}$) if we require

$$\Delta_l, \Delta_r, (\Delta_l - \Delta_r) > \max(\beta_n) + \max(S(D_m)). \tag{7}$$

Thus, if we pick Δ_l and Δ_r to satisfy all the inequalities in equations 1–7, we will be able to meet condition (7) without violating any of the other conditions at this step. So we can fix some enumeration of $\mathbb{N}^2$ and define:

$$(\Delta_l, \Delta_r) = \text{ the least pair satisfying inequalities 1–7.}$$

We then set:

$$\widetilde{H}_{n+1} = (H_m - \Delta_l) \sqcup H_m \sqcup (H_m + \Delta_r).$$

This finishes step I.

Step II. Now we must meet condition (8). So let $r = r_{|t_{n+1}|}$; we will ensure $r \in \omega_{n+1} \oplus H_{n+1}$. Let $\widetilde{H}_{n+1}$ be the set produced in step I, and set:

$$\widetilde{D}_{n+1} = D(\widetilde{H}_{n+1})$$
$$\widetilde{\beta}_{n+1} = \beta_n \cup \widetilde{D}_{n+1}.$$

If r is already in $\omega_n \oplus \widetilde{H}_{n+1}$ we need do nothing at this step; set $\widetilde{\omega}_{n+1} = \omega_n$ and $H_{n+1} = \widetilde{H}_{n+1}$. Otherwise, we need to add r to the sum. We will add an element $h < 0$ to $\widetilde{H}_{n+1}$ and add $w = r - h$ to ω_n. We will set:

$$H_{n+1} = \widetilde{H}_{n+1} \cup \{h\}$$
$$\widetilde{\omega}_{n+1} = \omega_n \cup \{w\}.$$

We need to see that we can choose an h so as not to violate any of the other conditions. First, we require

$$h < \min(\widetilde{H}_{n+1}) \tag{8}$$

in order to make sure condition (4) holds. To satisfy the first part of condition (1), we must make sure that w is bigger than $\max(\omega_n)$. This can be achieved by requiring:

$$-h > \max(\omega_n) + r \tag{9}$$

(recall that h is to be negative). For the second part of (1), we require:

$$-h > \max(\widetilde{\beta}_{n+1}) - \min(\widetilde{H}_{n+1}). \tag{10}$$

Now we consider the new differences added, both to β_n and to A_n. Let:

$$\widehat{D} = \{h' - h : h' \in \widetilde{H}_{n+1}\}$$
$$\widehat{A} = \{w - w' : w' \in \omega_n\}.$$

We will then have:

$$D_{n+1} = \widetilde{D}_{n+1} \cup \widehat{D}$$
$$\beta_{n+1} = \widetilde{\beta}_{n+1} \cup \widehat{D}$$
$$\widetilde{A}_{n+1} = A_n \cup \widehat{A}.$$

To meet condition (2), we must guarantee that the three sets

$$\widetilde{\beta}_{n+1} \cap \widehat{A},\ A_n \cap \widehat{D},\ \text{and } \widehat{D} \cap \widehat{A}$$

are all empty (since we already guaranteed that $\widetilde{\beta}_{n+1} \cap A_n$ was empty in step I). The first of these will be empty if we make sure that $\min(\widehat{A}) > \max(\widetilde{\beta}_{n+1})$, so we require:

$$-h > \max(\widetilde{\beta}_{n+1}) + \max(\omega_n) - r. \tag{11}$$

The second set will be empty if we ensure that $\min(\widehat{D}) > \max(A_n)$, so we require:

$$-h > \max(A_n) - \min(\widetilde{H}_{n+1}). \tag{12}$$

For the third set, we need to be sure that we do not have $h' - h = w - w'$ for some $h' \in \widetilde{H}_{n+1}$ and $w' \in \omega_n$. But this would mean that $w' + h' = w + h = r$, so we would already have had $r \in \omega_n \oplus \widetilde{H}_{n+1}$, contrary to our assumption.

To satisfy condition (5), it suffices that $\widehat{D} \cap \beta_n \subseteq D_m$. We have already ensured this by making $\min(\widehat{D}) > \max(\widetilde{\beta}_{n+1})$. Preserving condition (6) at this step amounts to showing that if $a \in D_i \setminus \beta_{i-1}$ for some $i \leq n$ with $t_i \perp t_{n+1}$, and $b \in \widehat{D}$, then $b - a \notin \beta_{n+1}$. As in step I, this reduces to ensuring that $D(\widehat{D})$ is disjoint from $D_i \setminus \beta_{i-1}$ for such i. But here

$$\begin{aligned} D(\widehat{D}) &= \{(h_1' - h) - (h_2' - h) : h_1', h_2' \in \widetilde{H}_{n+1} \text{ and } h_1' \geq h_2'\} \\ &= \{h_1' - h_2' : h_1', h_2' \in \widetilde{H}_{n+1} \text{ and } h_1' \geq h_2'\} = D(\widetilde{D}_{n+1}). \end{aligned}$$

We already ensured in step I that $D(\widetilde{D}_{n+1}) \cap D_i \subseteq \beta_{i-1}$ for these i, so this is satisfied.

We must finally preserve condition (9). Again we have two cases to check. For $i \leq n$ we need $\beta_{n+1} \setminus D_i$ disjoint from $D(D_i)$ and $S(D_i)$, and we need $D(D_{n+1})$ and $S(D_{n+1})$ disjoint from $\beta_{n+1} \setminus D_{n+1}$. We already know from step I that $\widetilde{\beta}_{n+1} \setminus D_i$ is disjoint from $D(D_i)$ and $S(D_i)$, and that $D(\widetilde{D}_{n+1})$ and $S(\widetilde{D}_{n+1})$ are disjoint from $\widetilde{\beta}_{n+1} \setminus \widetilde{D}_{n+1}$.

So for the first case, we just need to make sure that $\widehat{D}$ is disjoint from $D(D_i)$ and $S(D_i)$ for $i \leq n$. This is achieved if

$$\min(\widehat{D}) > \max\left(\bigcup_{i \leq n} (D(D_i) \cup S(D_i))\right),$$

which is satisfied if we require:

$$-h > 2 \cdot \max(\beta_n) - \min(\widetilde{H}_{n+1}). \tag{13}$$

For the second case, note that $\beta_{n+1} \setminus D_{n-1} = \widetilde{\beta}_{n+1} \setminus \widetilde{D}_{n+1}$. We first consider $D(D_{n+1})$. We have:

$$D(D_{n+1}) = D(\widetilde{D}_{n+1} \cup \widehat{D}) = D(\widetilde{D}_{n+1}) \cup (\widehat{D} - \widetilde{D}_{n+1}) \cup D(\widehat{D}).$$

So we need only ensure that $\widehat{D} - \widetilde{D}_{n+1}$ and $D(\widehat{D})$ are disjoint from $\widetilde{\beta}_{n+1} \setminus \widetilde{D}_{n+1}$. The first set will be disjoint from $\widetilde{\beta}_{n+1} \setminus \widetilde{D}_{n+1}$ if we have $\min(\widehat{D}) > \max(\widetilde{D}_{n+1}) + \max(\widetilde{\beta}_{n+1})$. For this we require:

$$-h > \max(\widetilde{D}_{n+1}) + \max(\widetilde{\beta}_{n+1}) - \min(\widetilde{H}_{n+1}). \tag{14}$$

For the second set, we observe that $D(\widehat{D}) = D(\widetilde{H}_{n+1}) = \widetilde{D}_{n+1}$, and hence this set is trivially disjoint from $\widetilde{\beta}_{n+1} \setminus \widetilde{D}_{n+1}$.

We next consider $S(D_{n+1})$. Here we have:

$$S(D_{n+1}) = S(\widetilde{D}_{n+1} \cup \widehat{D}) = S(\widetilde{D}_{n+1}) \cup (\widetilde{D}_{n+1} + \widehat{D}) \cup S(\widehat{D}).$$

We have already ensured that the first of these sets is disjoint from $\widetilde{\beta}_{n+1}$. The second and third sets will also be disjoint from $\widetilde{\beta}_{n+1}$ by the previously imposed conditions.

Thus we can take h to be any number satisfying the inequalities in equations 8–14. So we set:

$$h = \text{ the greatest integer satisfying inequalities 8–14}$$

and take H_{n+1} and $\widetilde{\omega}_{n+1}$ to be as defined above. This completes step II.

Step III. We must now add elements to $\widetilde{\omega}_{n+1}$ in order to satisfy condition (3) while not violating condition (2). Let $a_0, \ldots, a_{l-1}$ enumerate $\max(\beta_{n+1}) \setminus (\beta_{n+1} \cup D(\widetilde{\omega}_{n+1}))$. We will successively pick pairs (w_i, w_i') for $i < l$ such that $w_i' - w_i = a_i$. We will then let

$$\omega_{n+1} = \widetilde{\omega}_{n+1} \cup \{w_i, w_i' : i < l\}.$$

We need to ensure that any new differences introduced are not in β_{n+1}. We can do this by making:

$$w_0 > \max(\beta_{n+1}) + \max(\widetilde{\omega}_{n+1}) \tag{15}$$

and

$$w_i > \max(\beta_{n+1}) + w_{i-1}' \text{ for } 1 \leq i \leq l. \tag{16}$$

We thus successively define w_i to be the least number satisfying these conditions, and let $w_i' = w_i + a_i$.

This completes step III, and hence stage $n+1$ of the construction. We thus see that the construction can be continued from one stage to the next, and the lemma is established. □

4 Constructing particular sequences

We can use the techniques developed in the proof of the Main Lemma to show that all of the obvious inclusions among the sets $\mathcal{WW}$, $\mathcal{WW}_0$, $\mathcal{EWW}$, and $\mathcal{EWW}_0$ are proper inclusions. We start with a case of particular interest. In their paper [3], Eigen and Hajian ask (essentially): If Ω is an exhaustive weakly wandering sequence for some transformation T, must Ω be an exhaustive weakly wandering sequence for some *ergodic* transformation T'? The answer is no:

Theorem 13. *There is a sequence $\Omega_1 \in [\mathbb{N}]^{\mathbb{N}}$ such that $\Omega_1 \in \mathcal{EWW}$ but $\Omega_1 \notin \mathcal{WW}_0$ (so Ω_1 is exhaustive weakly wandering for some transformation, but is not weakly wandering for any ergodic transformation).*

Proof: This amounts to showing that we can construct a sequence Ω for which there is a sequence H with $\Omega \oplus H = \mathbb{Z}$, but for which there is no *hitting* sequence H' with $\Omega + H' = \Omega \oplus H'$. The trick will be to build Ω so that $(\mathbb{N} \setminus D(\Omega)) \cup \{0\}$ contains no arithmetic progressions of length 3 (as a subset, not necessarily as a subsequence). We claim that for such an Ω and for any hitting sequence H' we have $D(\Omega) \cap D(H') \neq \{0\}$, so that H' does not have a direct sum with Ω. To see this, it suffices to show that for any hitting sequence H' (or even a sequence with the shift-repeat property in one direction), $D(H')$ contains an arithmetic progression of length 3.

Suppose H' has the shift-repeat property. By shifting if necessary, we may assume that $0 \in H'$. Let $h_1 > 0$ be the first positive element of H' (we can handle the case where all elements of H are negative in essentially the same way). Then, by the shift-repeat property, there are h_n and h_{n+1} in H' such that $h_{n+1} - h_n = h_1 - 0$. But then $D(H')$ contains the elements $h_n - h_1$, $h_n - 0$, and $h_{n+1} = h_n + h_1$, which form an arithmetic progression of length 3 with common difference h_1.

So we will build Ω and H such that $\Omega \oplus H = \mathbb{Z}$ but $(\mathbb{N} \setminus D(\Omega)) \cup \{0\}$ contains no arithmetic progressions of length 3, and we will make sure that H extends infinitely in both directions. We will again build Ω and H in stages, where we construct at stage n the finite sequences ω_n and H_n. At the end we set:

$$\Omega = \bigcup_n \omega_n$$
$$H = \bigcup_n H_n.$$

We set $A_n = D(\omega_n)$ and $D_n = D(H_n)$ as before. Then, at stage n we require the following for $i, j \leq n$:

1. If $i < j$ then $\omega_i \sqsubset \omega_j$, $H_i \sqsubset H_j$, and $D_i \sqsubset D_j$ (where, for the H_i's, we require that we extend properly in both directions).

2. $A_i \cap D_i = \{0\}$.

3. $A_i \cup D_i \supseteq [0, \max(D_i)]$.

4. With $\langle r_k \rangle_{k \in \mathbb{N}}$ enumerating $\mathbb{Z}$ (with $r_0 = 0$), we require that $r_i \in \omega_i \oplus H_i$.

5. D_i does not contain any arithmetic progression of length 3.

From the remarks above, it is evident that if we have carried out the construction then $\Omega_1 = \Omega$ satisfies the conclusions of the theorem. So we proceed with the construction.

Let $\omega_0 = \{0\}$ and $H_0 = \{0\}$. The conditions are clearly satisfied at stage 0, so assume that ω_n and H_n have been defined so that the conditions hold at stage n. We construct ω_{n+1} and H_{n+1} to continue to satisfy them. We proceed in three steps:

Step I We add an element $h_- < 0$ to H_n and (if necessary) an element w to ω_n such that $w + h_- = r_{n+1}$ to satisfy condition (4), while preserving conditions (1), (2), and (5).

Step II We add an element $h_+ > 0$ to H_n, again preserving conditions (1), (2), and (5).

Step III We add elements to ω_n in order to satisfy condition (3), while preserving conditions (1) and (2).

Step I. If it is already the case that $r_{n+1} \in \omega_n \oplus H_n$, then we will only add an element h_- to H_n at this step, making sure that h_- is to the left of all previous elements. Otherwise, we will also add the element $w = r_{n+1} - h_-$ to ω_n, which we want to be to the right of previous elements. Let $r = r_{n+1}$. If we are not adding an element w, we ignore any of the requirements below related to w. The new differences we introduce will then be:

$$\widehat{D} = \{h' - h_- : h' \in H_n\}$$
$$\widehat{A} = \{w - w' : w' \in \omega_n\}.$$

Condition (1) is satisfied if we make $w > \max(\omega_n)$, $h_- < \min(H_n)$, and $\min(\widehat{D}) > \max(D_n)$. So we require:

$$\begin{aligned} -h_- &> \max(\omega_n) - r \\ h_- &< \min(H_n) \\ -h_- &> \max(D_n) - \min(H_n). \end{aligned} \tag{17}$$

Preserving condition (2) requires that

$$(D_n \cup \widehat{D}) \cap (A_n \cup \widehat{A}) = \{0\}.$$

We know that $D_n \cap A_n = \{0\}$, so we will make sure that the sets $\widehat{D} \cap A_n$, $D_n \cap \widehat{A}$, and $\widehat{D} \cap \widehat{A}$ are all empty. For the first set, it suffices that $\min(\widehat{D}) > \max(A_n)$, so we require:

$$-h_- > \max(A_n) - \min(H_n). \tag{18}$$

For the second set, it suffices that $\min(\widehat{A}) > \max(D_n)$, so we require:

$$-h_- > \max(D_n) + \max(\omega_n) - r. \tag{19}$$

For the third set, we need to ensure that there are not elements $h' \in H_n$ and $w' \in \omega_n$ with $h' - h_- = w - w'$. As in the Main Lemma, this only happens if $h' + w' = r$, in which case we do not add w, so $\widehat{A}$ is empty.

To satisfy condition (5), we need to be sure that $D_n \cup \widehat{D}$ contains no arithmetic progression of length 3. We know that D_n does not, and we know that elements of $\widehat{D}$ are bigger than elements of D_n, so any arithmetic progression must be $\langle d_0, d_1, d_2 \rangle$ with $d_0 < d_1 < d_2$, $d_0 + d_2 = 2d_1$, and $d_2 \in \widehat{D}$. We must have $d_2 = h_2 - h_-$, with $h_2 \in H_n$. We consider three cases, depending on whether d_0 and d_1 are in D_n or in $\widehat{D}$.

1. If $d_0, d_1 \in D_n$, let $d_0 = h_0 - h_0'$ and $d_1 = h_1 - h_1'$, with h_0, h_0', h_1, and h_1' in H_n. We then have:

   $$h_0 - h_0' + h_2 - h_- = 2h_1 - 2h_1',$$

   so that

   $$-h_- = 2h_1 + h_0' - 2h_1' - h_0 - h_2.$$

   We can prevent this from happening by requiring:

   $$-h_- > 3\max(H_n) - 4\min(H_n). \tag{20}$$

2. If $d_0 \in D_n$ and $d_1 \in \widehat{D}$, then $d_0 = h_0 - h_0'$, $d_1 = h_1 - h_-$ where h_0, h_0', and h_1 are in D_n. We then have:

   $$h_0 - h_0' + h_2 - h_- = 2h_1 - 2h_-,$$

   so that

   $$-h_- = h_0 + h_2 - h_0' - 2h_1.$$

   This is prevented if

   $$-h_- > 2\max(H_n) - 3\min(H_n),$$

   which we have already ensured.

3. If d_0 and d_1 are in $\widehat{D}$, let $d_0 = h_0 - h_-$ and $d_1 = h_1 - h_-$, with h_0 and h_1 in H_n. Then

$$h_0 - h_- + h_2 - h_- = 2h_1 - 2h_-,$$

i.e., $h_0 + h_2 = 2h_1$. This would mean that $\langle h_0, h_1, h_2 \rangle$ was an arithmetic progression in H_n, so that $\langle 0, h_1 - h_0, h_2 - h_0 \rangle$ was an arithmetic progression in D_n, contrary to our assumption.

So we can now safely choose h_-:

$h_- =$ the greatest negative number satisfying requirements 17–20.

We then let:

$$\widetilde{H}_{n+1} = H_n \cup \{h_-\}$$

$$\widetilde{\omega}_{n+1} = \begin{cases} \omega_n & \text{if } r_{n+1} \in \omega_n \oplus H_n \\ \omega_n \cup \{w\} & \text{if } r_{n+1} \notin \omega_n \oplus H_n. \end{cases}$$

We also set $\widetilde{A}_{n+1} = A_n \cup \widehat{A}$ and $\widetilde{D}_{n+1} = D_n \cup \widehat{D}$. This finishes step I.

Step II. We now wish to add an element $h_+ > \max(H_n)$. Let:

$$D' = \{h_+ - h : h \in \widetilde{H}_{n+1}\}.$$

For condition (1), we need $\min(D') > \max(\widetilde{D}_{n+1})$, so we require:

$$h_+ > \max(\widetilde{D}_{n+1}) + \max(\widetilde{H}_{n+1}). \tag{21}$$

For condition (2) we need to make sure that $D' \cap \widetilde{A}_{n+1} = \emptyset$, so we require:

$$h_+ > \max(\widetilde{A}_{n+1}) + \max(\widetilde{H}_{n+1}). \tag{22}$$

For condition (5) we have three cases which are essentially the same as in step I. We need only require:

$$h_+ > 4\max(\widetilde{H}_{n+1}) - 3\min(\widetilde{H}_{n+1}). \tag{23}$$

If we now let:

$h_+ =$ the least positive number satisfying requirements 21–23,

then all the conditions will be preserved. We then set:

$$H_{n+1} = \widetilde{H}_{n+1} \cup \{h_+\}.$$

This finishes step II.

Step III. We must now add elements to $\widetilde{\omega}_{n+1}$ in order to satisfy condition (3) while not violating conditions (1) and (2). Let $a_0, \ldots, a_{l-1}$ enumerate $\max(D_{n+1}) \setminus (D_{n+1} \cup D(\widetilde{\omega}_{n+1}))$. As in the Main Lemma, we successively pick pairs (w_i, w_i') for $i < l$ such that $w_i' - w_i = a_1$ and then let

$$\omega_{n+1} = \widetilde{\omega}_{n+1} \cup \left(\bigcup_{i<l} \{w_i, w_i'\} \right).$$

We need to ensure that any new differences introduced are not in D_{n+1}. We can do this by making:

$$\begin{aligned} w_0 &> \max(D_{n+1}) + \max(\widetilde{\omega}_{n+1}) \\ w_i &> \max(D_{n+1}) + w_{i-1}' \text{ for } 1 \leq i \leq l. \end{aligned} \tag{24}$$

We thus successively define w_i to be the least number satisfying these conditions, and let $w_i' = w_i + a_i$. This finishes step III, and the construction. Again, this allows us to proceed to the next stage of the construction and finishes the proof. $\square$

We next show that there is a sequence which is weakly wandering for some ergodic transformation, but is not exhaustive weakly wandering for any transformation.

Theorem 14. *There is a sequence $\Omega_2 \in [\mathbb{N}]^{\mathbb{N}}$ such that $\Omega_2 \in \mathcal{WW}_0 \setminus \mathcal{EWW}$.*

Proof: We build $\Omega = \Omega_2$ and a hitting sequence H such that $\Omega + H = \Omega \oplus H$ to guarantee that $\Omega \in \mathcal{WW}_0$. We will prevent Ω being in $\mathcal{EWW}$ by requiring that for any $w \in \Omega$ we have $w + 1 \in D(\Omega)$. To see that this suffices, suppose there were an H' with $\Omega \oplus H' = \mathbb{Z}$. By shifting, we may assume that $0 \in H'$. Then we have some w in Ω and h in H' with $w + h = -1$. But now $-h = w + 1 > 0$ and $0 - h \in D(H')$, contradicting that $D(\Omega) \cap D(H') = \{0\}$.

So we will build Ω and H as before, satisfying the following conditions at stage n for $i, j \leq n$:

1. If $i < j$ then $\omega_i \sqsubset \omega_j$, $H_i \sqsubset H_j$, and $D_i \sqsubset D_j$.
2. $A_i \cap D_i = \{0\}$.
3. $A_i \cup D_i \supseteq [0, \max(D_i)]$.
4. Shifts of H_i occur to the left and right in H_{i+1} as consecutive blocks.
5. For all $w \in \omega_i$, we have $w + 1 \in A_{i+1}$ (so $w \notin D_i$).

We initially set $\omega_0 = H_0 = \{0\}$, satisfying the conditions at stage 0. We now assume they have been satisfied at stage n and proceed to construct ω_{n+1} and H_{n+1}. We have two steps:

Step I Add shifts of H_n to satisfy condition (4) while respecting conditions (1), (2), and (5).

Step II Add elements to ω_n to satisfy conditions (3) and (5), while respecting conditions (1) and (2).

Step I. We will again pick numbers Δ_l and Δ_r in $\mathbb{N}$ and let:

$$H_{n+1} = (H_n - \Delta_l) \sqcup H_n \sqcup (H_n + \Delta_r).$$

We will then have:

$$D_{n+1} = D_n \cup (\Delta_l \pm D_n) \cup (\Delta_r \pm D_n) \cup ((\Delta_l + \Delta_r) \pm D_n).$$

We know that D_n satisfies the given conditions, so we can satisfy the rest of the conditions by making sure that elements of the remaining sets are bigger than $\max(D_n)$, bigger than $\max(A_n)$, and bigger than $\max(\omega_n) + 1$. The following requirement suffices:

$$\Delta_l, \Delta_r > \max(D_n) + \max(A_n) + 1. \tag{25}$$

Step II. As in the previous constructions, we can now form ω_{n+1} by successively adding pairs to add the necessary differences to A_{n+1} while avoiding D_{n+1}. This will finish the construction. □

We continue by producing a sequence which is weakly wandering for some transformation, but for no ergodic one, and which is not exhaustively weakly wandering for any transformation.

Theorem 15. *There is a sequence $\Omega_3 \in \mathcal{WW} \setminus (\mathcal{WW}_0 \cup \mathcal{EWW})$.*

Proof: We build $\Omega = \Omega_3$ and H in stages such that $\Omega + H = \Omega \oplus H$. We use previously discussed conditions to ensure that there is no such hitting sequence H, and also no H' with $\Omega \oplus H' = \mathbb{Z}$. At stage n we require the following for $i, j \leq n$:

1. If $i < j$ then $\omega_i \sqsubset \omega_j$, $H_i \sqsubset H_j$ (in both directions), and $D_i \sqsubset D_j$.
2. $A_i \cap D_i = \{0\}$.

3. $A_i \cup D_i \supseteq [0, \max(D_i)]$.
4. D_i does not contain any arithmetic progression of length 3.
5. For all $w \in \omega_i$, we have $w \in A_{i+1}$ (so $w \notin D_n$).

We set $\omega_0 = H_0 = \{0\}$. Then we assume the construction is completed to stage n, and construct stage $n+1$. There are three steps:

Step I Add an element $h_- < \min(H_n)$, preserving the other conditions.

Step II Add an element $h_+ > \max(H_n)$.

Step III Extend ω_n to include necessary differences.

Step I. We want to add $h_- < \min(H_n)$. Set:

$$\widehat{D} = \{h - h_- : h \in H_n\}.$$

We want to make $\widehat{D}$ disjoint from A_n and D_n, and bigger than $\omega_n + 1$, so we require:

$$-h_- > \max(A_n) + 1 + \max(D_n) - \min(H_n). \tag{26}$$

We also need to make sure that $D_n \cup \widehat{D}$ contains no arithmetic progressions of length 3. As in the previous construction, the following requirement suffices:

$$-h_- > 3\max(H_n) - 4\min(H_n). \tag{27}$$

We can now take h_- to be the greatest negative number satisfying these two requirements. We let $\widetilde{H}_{n+1} = H_n \cup \{h_-\}$ and $\widetilde{D}_{n+1} = D_n \cup \widehat{D}$.

Step II. We now add an element $h_+ > \max(H_n)$, much like in step I. The following conditions will suffice to preserve the other conditions:

$$\begin{aligned} h_+ &> \max(A_n) + 1 + \max(\widetilde{D}_{n+1}) + \max(\widetilde{H}_{n+1}) \qquad (28) \\ h_+ &> 4\max(\widetilde{H}_{n+1}) - 3\min(\widetilde{H}_{n+1}). \end{aligned}$$

We take h_+ to be the least such number, and let $H_{n+1} = H_n \cup \{h_+\}$.

Step III. We once again add pairs to ω_n to produce the needed differences. This completes the construction. $\square$

The previous three constructions can be viewed as performing the construction in the Main Lemma along a tree with a single infinite branch, with some additional requirements. We now complete the picture of the inclusions among the four sets $\mathcal{WW}$, $\mathcal{WW}_0$, $\mathcal{EWW}$, and $\mathcal{EWW}$ by producing a sequence which is weakly wandering for some ergodic transformation, exhaustive weakly wandering for some other transformation, but exhaustive weakly wandering for no ergodic transformation. This time we will be building two additional sequences as witnesses, and the construction can be viewed as occurring along a tree with two infinite branches.

Theorem 16. *There is a sequence $\Omega_4 \in [\mathbb{N}]^{\mathbb{N}}$ such that $\Omega_4 \in (\mathcal{WW}_0 \cap \mathcal{EWW}) \setminus \mathcal{EWW}_0$.*

Proof: This time we will build $\Omega = \bigcup_i \omega_i$ as well as two other sequences, $H^o = \bigcup H_i^o$ and $H^x = \bigcup H_i^x$. H^o will be a hitting sequence with $\Omega + H^o = \Omega \oplus H^o$ to ensure $\Omega \in \mathcal{WW}_0$, and H^x will be such that $\Omega \oplus H^x = \mathbb{Z}$ to ensure $\Omega \in \mathcal{EWW}$. We will prevent any hitting sequence giving a direct sum of $\mathbb{Z}$. At each stage we set $A_n = D(\omega_n)$, $D_n^o = D(H_n^o)$, $D_n^x = D(H_n^x)$, and $\beta_n = \bigcup_{k \le n}(D_k^o \cup D_k^x)$, and we satisfy the following ten conditions:

1. If $i < j$ then $\omega_i \sqsubset \omega_j$, $H_i^o \sqsubset H_j^o$ and $H_i^x \sqsubset H_j^x$ (in both directions), and $D_i^o \sqsubset D_j^o$ and $D_i^x \sqsubset D_j^x$.
2. $A_i \cap \beta_i = \{0\}$.
3. $D_i^o \cap D_i^x = \{0\}$.
4. $A_i \cup \beta_i \supseteq [0, \max(\beta_i)]$.
5. H_{i+1}^o contains left and right shifts of H_i^o as consecutive blocks.
6. D_i^x contains no arithmetic progression of length 3.
7. With $\langle r_k \rangle_{k \in \mathbb{N}}$ enumerating $\mathbb{Z}$ (with $r_0 = 0$), we have $r_i \in \omega_i \oplus H_i^x$.
8. For all $w \in \omega_i$, we have $w + 1 \notin D_i^o$.
9. If $a \in D_i^o \setminus D_i^x$ and $b \in D_i^x \setminus D_i^o$, then $|b - a| \in A_i$.
10. $D(D_i^o)$ and $S(D_i^0)$ are disjoint from $D_i^x \setminus \{0\}$, and $D(D_i^x)$ and $S(D_i^x)$ are disjoint from $D_i^o \setminus \{0\}$.

It is clear that H^o will be a hitting sequence with $\Omega + H^o = \Omega \oplus H^o$, and that $\Omega \oplus H^x = \mathbb{Z}$. We need to check that the conditions guarantee there is no hitting sequence H with $\Omega \oplus H = \mathbb{Z}$.

Suppose we have such an H, so $D(\Omega) \cap D(H) = \{0\}$. We may assume $0 \in H$ by shifting if necessary. Let $H_- = \{-h : h \in H, h < 0\}$. Then $H_- \subseteq D(H) \subseteq D(H^o) \cup D(H^x)$. We claim that either $H_- \subseteq D(H^o)$ or $H_- \subseteq D(H^x)$. If not, there are a and b in H_- with $a \in D(H^o) \setminus D(H^x)$ and $b \in D(H^x) \setminus D(H^0)$. But then there is an n with $a \in D_n^o \setminus D_n^x$ and $b \in D_n^x \setminus D_n^o$, so that $|b-a| \in A_n$ by condition (9). But $a \neq b$, and $|b-a| \in D(H_-) \subseteq D(H)$, a contradiction.

If $H_- \subseteq D(H^o)$, we claim $\Omega + H \neq \mathbb{Z}$. For if the sum were $\mathbb{Z}$, there would be $w \in \Omega$ and $h \in H$ with $w + h = -1$. We then have that $h < 0$, and so $-h = w + 1 \in H_- \subseteq D(H^o)$. This contradicts condition (8). On the other hand, if $H_- \subseteq D(H^x)$, we claim that H is not hitting. We have $D(H_-) \subseteq D(D(H^x)) = \bigcup_n D(D_n^x)$. We also have that $D(D(H^x))$ is disjoint from $D(H^o) \setminus D(H^x)$ by condition (10) and so $D(D(H^x)) \cap D(H^o) = \{0\}$ by condition (3). Thus $D(H_-) \cap D(H^o) = \{0\}$ and $D(H_-) \cap D(\Omega) = \{0\}$, so that we have $D(H_-) \subseteq D(H^x)$ by condition (4). But now, as in the previous argument, if H is hitting then H_- must contain an arithmetic progression of length 3, which is impossible since $D(H^x)$ does not, by condition (6).

Thus, our construction will suffice once we carry it out. We start with $\omega_0 = H_0^o = H_0^x = \{0\}$. Assume we have finished stage n; we now construct stage $n+1$. We have four steps:

Step I We extend H_n^0 to meet condition (5), while respecting conditions (1), (2), (3), (8), (9), and (10).

Step II We extend H^x to the left and extend ω_n (if necessary) to meet condition (7), respecting conditions (1), (2), (3), (6), (9), and (10).

Step III We extend H^x to the right, preserving the same conditions.

Step IV We extend ω_n to meet condition (4), while preserving conditions (1) and (2).

Step I. As before, we will have

$$H_{n+1}^o = (H_n^o - \Delta_l) \sqcup H_n^o \sqcup (H_n^o + \Delta_r),$$

where we pick Δ_l and Δ_r large enough to make these sets disjoint and to make any new differences bigger than $\max(A_n)+1$, $\max(D_n^x)$, and $\max(D_n^o)$.

The new differences will then be:

$$\widehat{D} = (\Delta_l \pm D_n^o) \cup (\Delta_r \pm D_n^o) \cup ((\Delta_l + \Delta_r) \pm D_n^o).$$

We must still preserve conditions (9) and (10).

For condition (9), we must ensure that for $b \in \widehat{D}$ and $a \in D_n^x \setminus D_n^o$, we have $b - a \notin \beta_n \cup \widehat{D}$. Keeping $b - a$ out of β_n can be done by making Δ_l and Δ_r sufficiently large. For $b - a$ to be in $\widehat{D}$, we would have $D(\widehat{D})$ meeting $D_n^x \setminus D_n^o$. Preserving condition (10) will thus suffice to preserve condition (9).

To preserve condition (10) we must do two things. We must make sure that $\widehat{D}$ is disjoint from $D(D_n^x)$ and $S(D_n^x)$, and we must make sure that $D(D_n^o \cup \widehat{D})$ and $S(D_n^0 \cup \widehat{D})$ are disjoint from $D_n^x \setminus \{0\}$. The first task is achieved by making Δ_l and Δ_r sufficiently large. For the second, we already know that differences and sums from D_n^o are disjoint from $D_n^x \setminus \{0\}$. A difference or sum involving one element of D_n^o and one element of $\widehat{D}$ can be kept out of D_n^x by making Δ_l and Δ_r sufficiently large. As in the proof of the Main Lemma, the inductive assumption then allows us to satisfy the condition by making

$$\Delta_l, \Delta_r, (\Delta_l - \Delta_r) > \max(D_n^x) + \max(S(D_n^o)).$$

Thus, taking Δ_l and Δ_r to be the least pair satisfying the requirements will be sufficient.

Step II. We add an element $h_- < \min(H_n^x)$ to H_n^x. If $r_{n+1} \notin \omega_n \oplus H_n^x$, we also add an element $w > \max(\omega_n)$ to ω_n; otherwise we add nothing to ω_n at this stage. We let the new differences be:

$$\widehat{D} = \{h' - h_- : h' \in H_n^x\}$$
$$\widehat{A} = \{w - w; : w' \in \omega_n\}.$$

The following conditions can be met by making $-h_-$ and w sufficiently large:

- $\widehat{D}$ is disjoint from A_n, D_{n+1}^o, $D(D_{n+1}^o)$ and $S(D_{n+1}^o)$.
- $\widehat{A}$ is disjoint from D_n^x.
- $(D_n^x \cup \widehat{D})$ contains no arithmetic progression of length 3 with at least one element in D_n^x.
- For $b \in \widehat{D}$ and $a \in D_{n+1}^o \setminus \{0\}$, we have $|b - a| \notin (D_n^x \cup D_n^o)$.

- $\widehat{D} - D^x_n$, $\widehat{D} + D^x_n$, and $S(\widehat{D})$ are disjoint from D^o_{n+1}.

The only things left to preserve are the following:

- $\widehat{D}$ is disjoint from $\widehat{A}$.
- $\widehat{D}$ contains no arithmetic progression of length 3.
- $D(\widehat{D})$ is disjoint from $D^o_{n+1} \setminus \{0\}$.

As before, the first of these is guaranteed by the assumption that $r_{n+1} \notin \omega_n \oplus H^x_n$. The second is guaranteed since an arithmetic progression in $\widehat{D}$ would imply one in H^x_n, and the third is guaranteed because $D(\widehat{D}) = D(H^x_n) = D^x_n$ which we know to be disjoint from $D^o_{n+1} \setminus \{0\}$. So we can take h_- to be the greatest negative number satisfying the appropriate conditions and add it to H^x_n. If necessary, we also add $w = r_{n+1} - h_-$ to ω_n. This completes step II.

Step III. This is handled like step II (without adding to ω_{n+1}).

Step IV. We again add pairs to ω_{n+1} to include the necessary differences in A_{n+1}. This will complete the construction at stage $n+1$. □

5 Modifications to the constructions

The four constructions in the previous section show that the sets $\mathcal{WW} \setminus (\mathcal{WW}_0 \cup \mathcal{EWW})$, $\mathcal{EWW} \setminus \mathcal{WW}_0$, $\mathcal{WW}_0 \setminus \mathcal{EWW}$, and $(\mathcal{WW}_0 \cap \mathcal{EWW}) \setminus \mathcal{EWW}_0$ are all non-empty. By combining the construction in the Main Lemma with the following new conditions, we can in fact show that these four sets are all $\mathbf{\Sigma}^1_1$-hard, i.e., any analytic set is a continuous preimage of any of them. We will briefly sketch the modifications necessary.

We will use the original conditions (1)–(6) and condition (9); however, conditions (7) and (8) will sometimes be replaced by alternate conditions. Let us define the two alternatives:

(7′) Each D_i contains no arithmetic progression of length 3.

(8′) If $w \in \omega_i$, then $w + 1 \notin \beta_i$.

Then, to show $\mathcal{EWW} \setminus \mathcal{WW}_0$ is $\mathbf{\Sigma}^1_1$-hard, we would use conditions (7′) and (8), for $\mathcal{WW}_0 \setminus \mathcal{EWW}$ we use (7) and (8′), and for $\mathcal{WW} \setminus (\mathcal{EWW} \cup \mathcal{WW}_0)$ we use (7′) and (8′). For $(\mathcal{EWW} \cap \mathcal{WW}_0) \setminus \mathcal{EWW}_0$ we build H^o_i and H^x_i

and define D_i^o, D_i^x, β_i^o, and β_i^x accordingly. We include the corresponding conditions for D_i^o and D_i^x, and also require $\beta_i^o \cap \beta_i^x = \{0\}$. As in the original argument, we can check that if there is an H which has a direct sum with Ω then there is an infinite branch through T; moreover, H will have the desired properties. We thus obtain:

Theorem 17. *Each of the four sets* $\mathcal{WW} \setminus (\mathcal{WW}_0 \cup \mathcal{EWW})$, $\mathcal{EWW} \setminus \mathcal{WW}_0$, $\mathcal{WW}_0 \setminus \mathcal{EWW}$, *and* $(\mathcal{WW}_0 \cap \mathcal{EWW}) \setminus \mathcal{EWW}_0$ *is* $\mathbf{\Sigma}_1^1$*-hard.*

In all of the above constructions we built the sequence Ω as a subset of $\mathbb{N}$. We can also get the same results for sequences which are subsets of $\mathbb{Z}$ unbounded in both directions. There is no difficulty in adding negative elements to ω_n in the last step of the construction; we simply must make them small enough so that new differences avoid any previously constructed sets. The only modification necessary is that we should replace the condition "$w+1 \notin \beta_i$" by "$|w+1| \notin \beta_i$" for $w \in \omega_n$. This will change our requirements slightly, but causes no difficulty, and will again establish that there is no H with $H \oplus \Omega = \mathbb{Z}$ when necessary.

There is one question we should mention. We have been able to determine the complexity of the set of sequences which are exhaustive weakly wandering for some transformation T and so forth. It is not clear though, what the set of sequences which are, say, weakly wandering for a particular transformation can look like.

Definition 18. For a given transformation T, set:

$$\mathcal{WW}(T) = \{\Omega : \ \Omega \text{ is weakly wandering for } T\}.$$

The set $\mathcal{EWW}(T)$ is defined similarly.

Question 19. *Which subsets of* $[\mathbb{N}]^{\mathbb{N}}$ *can be* $\mathcal{WW}(T)$ *or* $\mathcal{EWW}(T)$ *for some transformation T? For some ergodic T? In particular, what are the possible complexities of these sets?*

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Department of Mathematics
Penn State University
University Park, PA 16802, USA

email: clemens@math.psu.edu