Isomorphism of subshifts is a universal countable Borel equivalence relation

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Abstract

We use the theory of Borel equivalence relations to analyze the equivalence relation of isomorphism among one-dimensional subshifts. We show that this equivalence relation is a universal countable Borel equivalence relation, so that it admits no definable complete invariants fundamentally simpler than the equivalence classes. We also see that the classification of higher dimensional subshifts up to isomorphism has the same complexity as for the one-dimensional case.

The problem of classifying one-dimensional and higher dimensional subshifts has been well studied, with the aim of finding invariants for isomorphism. One can consider this equivalence relation from the standpoint of descriptive set theory, and consider its complexity among Borel equivalence relations under the relation of Borel reducibility, $\leq_B$.

Definition 1. Let A be a finite set of symbols. A *one-dimensional subshift on* A is a closed subset of $A^{\mathbb{Z}}$ which is invariant under the shift operator S, where $S(x)(n) = x(n+1)$. Two subshifts X on A and Y on B are *isomorphic* if there is a homeomorphism $\varphi : X \to Y$ which commutes with S.

A subshift may be defined by a set of *forbidden words* $W \subseteq A^{<\mathbb{N}}$ (where $A^{<\mathbb{N}}$ is the set of finite sequences from A), where W determines the subshift

$$X_W = \{x \in A^{\mathbb{Z}} : \forall w \in W \ (w \not\sqsubseteq x)\}$$

(and $w \sqsubseteq x$ means that w occurs as a subword of x, that is, there is a $k \in \mathbb{Z}$ such that $x(k+i) = w(i)$ for all i with $0 \leq i < |w|$). A subshift is said to be of *finite type* if it is determined by a finite set W. While these have been more thoroughly studied, we will consider arbitrary subshifts in this paper.

A basic result is the following (see [4]):

Theorem (Curtis-Hedlund-Lyndon). *Suppose X and Y are subshifts on A and B, respectively, and $\varphi : X \to Y$ is a morphism of subshifts, i.e., a map commuting with the shift. Then φ is given by a block code, i.e., there is a natural number r and a function $\pi : A^{\{-r,\dots,r\}} \to B$ such that $\varphi(x)(n) = \pi(S^n(x) \restriction \{-r, \dots, r\})$. In particular, this applies when φ is an isomorphism of subshifts.*

In the next section we briefly review the notion of Borel reducibility on definable equivalence relations. In Section 2 we set out our representation of subshifts and the isomorphism relation. Section 3 presents the main result that isomorphism of subshifts is a universal countable Borel equivalence relation. In Section 4 we consider higher dimensional subshifts, and in Section 5 we list some open questions.

1 Borel reducibility

A *Borel equivalence relation* is an equivalence relation E on a Polish space X such that E is Borel as a subset of X^2. A standard method of comparing Borel equivalence relations is via the Borel reducibility relation. We say that an equivalence relation E on a space X is *Borel reducible* to an equivalence relation F on a space Y, $E \leq_B F$, if there is a Borel-measurable function $f : X \to Y$ such that for all $x_1, x_2 \in X$ we have $x_1 \; E \; x_2$ if and only if $f(x_1) \; F \; f(x_2)$. We write $E \sqsubseteq_B F$ when we can find such an f which is injective. Note that having $E \leq_B F$ amounts to having a definable injection from the quotient space X/E into Y/F, so that when $E \leq_B F$ we may view the relation F as being at least as complicated as E.

We say that an equivalence relation E is *countable* if every equivalence class of E is countable. We say E is a *universal countable Borel equivalence relation* if E is a countable Borel equivalence relation and for every countable Borel equivalence relation F we have $F \leq_B E$. Several such examples are known; one which we will use here is the *(left) shift relation* of the free group on two generators $\mathbb{F}_2$ on the space $2^{\mathbb{F}_2}$, denoted $E(\mathbb{F}_2, 2)$. A good introduction to the theory of countable Borel equivalence relations is [2].

An equivalence relation E on X is *smooth* if it is Borel reducible to the identity relation on some Polish space Y, i.e., there is a Borel-measurable function $f : X \to Y$ such that $x_1 E x_2$ if and only if $f(x_1) = f(x_2)$. Thus, smooth equivalence relations are those which admit single elements of some Polish space as definable complete invariants. A standard example of a non-smooth countable Borel equivalence relation is the relation E_0 of eventual equality on the Cantor space $2^{\mathbb{N}}$. In particular, any universal countable Borel equivalence relation is non-smooth.

The relation of Borel reducibility may be used to gauge the complexity of a classification problem. If we can show that some complicated equivalence relation E is Borel reducible to some notion of equivalence on a suitable class of structures, then there can be no complete invariants for the given relation which are fundamentally simpler than complete invariants for E. For instance, if we can reduce E_0 to some equivalence relation, then this relation does not admit single reals (or even finite sets of reals) as complete invariants.

2 The space of subshifts

We may assume that our alphabet A is $n = \{0, \dots, n-1\}$ for some $n \geq 2$.

Definition 2. For $n \geq 2$, the space $\mathcal{F}_n^S$ is the set of closed, S-invariant subsets of $n^{\mathbb{Z}}$. We view this as a subspace of the compact Polish space $\mathcal{F}(n^{\mathbb{Z}}) = K(n^{\mathbb{Z}})$ in the Vietoris topology (see [3]).

Lemma 3. *$\mathcal{F}_n^S$ is a closed subspace of $K(n^{\mathbb{Z}})$.*

Proof: It suffices to check that the function $\tilde{S} : K(n^{\mathbb{Z}}) \to K(n^{\mathbb{Z}})$ given by $\tilde{S}(X) = S[X]$ is a homeomorphism, since then $\mathcal{F}_n^S = \{X \in K(n^{\mathbb{Z}}) : \tilde{S}(X) = X\}$ is closed. To see this, we note that the Hausdorff metric d_H given by

$$d_H(X, Y) = \sup_{x \in X} \inf_{y \in Y} d(x, y) \quad \text{for } X, Y \neq \emptyset,$$
$$d_H(X, \emptyset) = 1 \quad \text{for } X \neq \emptyset,$$

is a compatible metric for $K(n^{\mathbb{Z}})$, where d is the compatible metric on $n^{\mathbb{Z}}$ given by

$$d(x, y) = 2^{-\min\{|k| : x(k) \neq y(k)\}} \text{ for } x \neq y.$$

Then $d(S(x), S(y)) \leq 2d(x, y)$, so

$$d_H(\tilde{S}(X), \tilde{S}(Y)) = \sup_{x \in \tilde{S}(X)} \inf_{y \in \tilde{S}(Y)} d(x, y) = \sup_{x \in X} \inf_{y \in Y} d(S(x), S(y))$$
$$\leq 2 \sup_{x \in X} \inf_{y \in Y} d(x, y) = 2d_H(X, Y).$$

Hence $\tilde{S}$ is continuous; $\tilde{S}^{-1}$ is similar. □

We next consider the alternate representation of subshifts using forbidden words.

Definition 4. For $n \geq 2$, let Z_n be the set of countable sequences of finite sequences in n, i.e. $Z_n = (n^{<\mathbb{N}})^{\mathbb{N}}$, equipped with the product topology (where $n^{<\mathbb{N}}$ carries the discrete topology). For $W \in Z_n$ we associate the space $X_W \in \mathcal{F}_n^S$ given by

$$X_W = \{x \in n^{\mathbb{Z}} : \forall i \ (W(i) \neq \emptyset \Rightarrow W(i) \not\sqsubseteq x)\}.$$

Then every subshift on n is represented by an element $W \in Z_n$. Several different sets of forbidden words may produce the same subshift; we wish to see that this is not problematic.

Definition 5. For $W_1, W_2 \in Z_n$, we set $W_1 \sim W_2$ if $X_{W_1} = X_{W_2}$.

Lemma 6. *The function $F : Z_n \to \mathcal{F}_n^S$ given by $F(W) = X_W$ is Borel-measurable.*

Proof: A subbasis for the Vietoris topology on $K(n^{\mathbb{Z}})$ is given by sets of the form $C_U = \{K : K \subseteq U\}$ and $M_U = \{K : K \cap U \neq \emptyset\}$ for $U \subseteq n^{\mathbb{Z}}$ open. It suffices to show that $F^{-1}[C_U]$ and $F^{-1}[M_U]$ are all Borel. We calculate:

$$\begin{aligned} W \in F^{-1}[C_U] &\Leftrightarrow F(W) \in C_U \Leftrightarrow X_W \subseteq U \\ &\Leftrightarrow \neg \exists x \in n^{\mathbb{Z}}\ (x \in X_W \setminus U). \end{aligned}$$

Since $X_W \setminus U$ is closed, and hence compact, the projection is compact and so $F^{-1}[C_U]$ is open. Secondly:

$$\begin{aligned} W \in F^{-1}[M_U] &\Leftrightarrow F(W) \in M_U \Leftrightarrow X_W \cap U \neq \emptyset \\ &\Leftrightarrow \exists x \in n^{\mathbb{Z}}\ (x \in X_W \cap U). \end{aligned}$$

Since $X_W \cap U$ is K_σ the projection is also K_σ and hence Borel. Thus F is Borel-measurable (Baire class 1 in fact). □

Since $W_1 \sim W_2$ if and only if $F(W_1) = F(W_2)$, we thus have:

Corollary 7. *The relation $\sim$ is a smooth Borel equivalence relation on Z_n.*

3 Classification of the isomorphism relation

We now consider the isomorphism relation on the spaces $\mathcal{F}_n^S$.

Definition 8. For $X, Y \in \mathcal{F}_n^S$, we set $X\ E\ Y$ if $(X, S) \cong (Y, S)$, i.e., there is a homeomorphism of X and Y which commutes with S.

Lemma 9. *E is a countable Borel equivalence relation on each $\mathcal{F}_n^S$.*

Proof: Each equivalence class of E is clearly countable since $X\,E\,Y$ is witnessed by a block code π; there are only countable many choices for π and each determines Y uniquely from X. To see that E is Borel, let f_π denote the (continuous) function given by a block code $\pi : n^{\{-r,\dots,r\}} \to n$. We then have:

$$\begin{aligned} X\ E\ Y \Leftrightarrow &\exists \pi \exists \tilde{\pi} \forall x \in n^{\mathbb{Z}}\ [(x \in X \Rightarrow f_\pi(x) \in Y) \wedge (x \in Y \Rightarrow f_{\tilde{\pi}} \in X) \wedge \\ &\qquad (x \in X \Rightarrow f_{\tilde{\pi}} \circ f_\pi(x) = f_\pi \circ f_{\tilde{\pi}}(x) = x)] \\ \Leftrightarrow &\exists \pi \exists \tilde{\pi} \neg \exists x \in n^{\mathbb{Z}}\ [(x \in X \setminus f_\pi^{-1}[Y]) \vee (x \in Y \setminus f_{\tilde{\pi}}^{-1}[X]) \vee \\ &\qquad (x \in X \wedge (f_{\tilde{\pi}}(f_\pi(x)) \neq x \vee f_\pi(f_{\tilde{\pi}}(x)) \neq x))]. \end{aligned}$$

The matrix is K_σ so the projection is also K_σ; as π and $\tilde{\pi}$ range over a countable set this relation is Σ_3^0 and hence Borel. □

We will show that this relation is already of maximal complexity for the smallest non-trivial alphabet $n = 2$. Let $E(\mathbb{F}_2, 2)$ be the equivalence relation induced by the (left) shift action of the free group on two generators, $\mathbb{F}_2$, on the space $2^{\mathbb{F}_2}$ given by $g \cdot x(h) = x(g^{-1}h)$. This equivalence relation is a universal countable Borel equivalence relation (see [1]).

Lemma 10. *There is an $\mathbb{F}_2$-invariant Borel set $A \subseteq 2^{\mathbb{F}_2}$ such that:*

1. *$E(\mathbb{F}_2, 2) \sqsubseteq_B E(\mathbb{F}_2, 2) \restriction A$, so $E(\mathbb{F}_2, 2) \restriction A$ is a universal countable Borel equivalence relation.*

2. *For $x, y \in A$ with $\neg x E(\mathbb{F}_2, 2) y$ we have that there are infinitely many $g \in \mathbb{F}_2$ with $x(g) \neq y(g)$.*

Proof: Let $\mathbb{F}_2$ be generated by a and b. Let $H \subseteq \mathbb{F}_2$ be the subgroup generated by a^2 and b^2, and $\rho : \mathbb{F}_2 \to H$ the isomorphism induced by $\rho(a) = a^2$ and $\rho(b) = b^2$. For $x \in 2^{\mathbb{F}_2}$ define $f(x) \in 2^{\mathbb{F}_2}$ by

$$f(x)(w) = \begin{cases} x(u) & \text{if } w = \rho(u) \text{ for some } u \text{ or} \\ & \quad w = \rho(u)abv \text{ for some } u \text{ and some reduced word } v \\ & \quad\quad \text{whose leftmost symbol is not } b^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is an injection, so $f[2^{\mathbb{F}_2}]$ is Borel. Let A be the $\mathbb{F}_2$-saturation of this, $A = \mathbb{F}_2 \cdot f[2^{\mathbb{F}_2}]$. Then A is Borel, and we claim it has the desired properties.

We first check that f witnesses $E(\mathbb{F}_2, 2) \sqsubseteq_B E(\mathbb{F}_2, 2) \restriction A$. If $y = g \cdot x$, then easily $f(y) = \rho(g) \cdot f(x)$. Conversely, suppose there is $h \in \mathbb{F}_2$ such that $f(y) = h \cdot f(x)$. Then we must have $h = \rho(g)$ for some g, so that $f(y) = \rho(g) \cdot f(x) = f(g \cdot x)$ so $y = g \cdot x$.

Finally, let $x, y \in A$ with $\neg x \, E(\mathbb{F}_2, 2) \, y$ and suppose that $x(g) = y(g)$ for all but finitely many $g \in \mathbb{F}_2$. Then there are $g_x, g_y \in \mathbb{F}_2$ and $x_0, y_0 \in 2^{\mathbb{F}_2}$ such that $x = g_x \cdot f(x_0)$ and $y = g_y \cdot f(y_0)$. We then get that $f(y_0)(g) = g_y^{-1} g_x \cdot f(x_0)(g)$ for all but finitely many g. From this we must have that $g_y^{-1} g_x = \rho(w)$ for some w, so $f(y_0)(g) = f(w \cdot x_0)(g)$ for all but finitely many g. This requires that $y_0 = w \cdot x_0$, so that $x \, E(\mathbb{F}_2, 2) \, y$ contrary to our initial assumption. □

Theorem 11. *The isomorphism relation on one-dimensional subshifts is a universal countable Borel equivalence relation.*

Proof: We will show that $E(\mathbb{F}_2, 2) \restriction A \leq_B E$, where A is the set from the previous lemma and E is the isomorphism relation on subshifts on the alphabet 2. Given a point $x \in 2^{\mathbb{F}_2}$, we produce a subshift $\varphi(x) = X_x$ as follows.

Let $s \frown t$ denote the concatenation of two finite sequences s and t, and $|s|$ the length of a sequence. Let $\mathbb{F}_2$ be generated by a and b. Define a map $\rho : \mathbb{F}_2 \to 2^{<\mathbb{N}}$ by letting $\rho(e) = 11000011$, $\rho(a) = 11100011$, $\rho(a^{-1}) = 11010011$, $\rho(b) = 11001011$, $\rho(b^{-1}) = 11000111$, and

$$\rho(w) = \rho(w_0) \frown \cdots \frown \rho(w_n) \quad \text{for } w = w_0 \frown \cdots \frown w_n \neq e \text{ a reduced word}$$
$$\text{with each } w_i \in \{a, a^{-1}, b, b^{-1}\} \text{ and } w_{i+1} \neq w_i^{-1}.$$

Note that for each $w \in \mathbb{F}_2$, $|\rho(w)| = 8k$ for some $k > 0$.

Define also the map $d : 2 \to 2^{<\mathbb{N}}$ by $d(0) = 101$ and $d(1) = 111$; note that both of these sequences have length 3. Fix an enumeration of $\mathbb{F}_2$, $\mathbb{F}_2 = \{w_i :$

$i \in \mathbb{N}\}$. For a finite sequence $s \in 2^{<\mathbb{N}}$, let $\bar{0} \frown s \frown \bar{0}$ denote the element of $2^{\mathbb{Z}}$ whose negative coordinates are all 0, followed by s, followed by all 0's. Let $\bar{s}$ be the periodic element of $2^{\mathbb{Z}}$ with period $|s|$ induced by s (starting at coordinate 0). Finally, let $p(i)$ denote the i^{th} prime.

Now for an element $x \in 2^{\mathbb{F}^2}$ define the countable set A_x to be:

$$A_x = \{\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\} \cup \\ \{\overline{d(0) \frown 0^{p(2n+2)-3}} : n \in \mathbb{N}\} \cup \{\overline{d(1) \frown 0^{p(2n+3)-3}} : n \in \mathbb{N}\},$$

and let $\varphi(x) = X_x$ where

$$X_x = \overline{\bigcup_{n\in\mathbb{Z}} S^n[A_x]}$$

is the smallest subshift containing A_x.

Then ϕ is a Borel-measurable map from $2^{\mathbb{F}_2}$ to $\mathcal{F}_2^s$. To see this, note that the intermediate map $x \mapsto W_x$ given by

$$s \in W_x \Leftrightarrow \neg\exists i \exists k \exists n \; (s \sqsubseteq 0^m \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown 0^m) \wedge \\ \neg\exists n \exists m \; (s \sqsubseteq (d(0) \frown 0^{p(2n+2)-3})^m) \wedge \\ \neg\exists n \exists m \; (s \sqsubseteq (d(1) \frown 0^{p(2n+3)-3})^m)$$

is Borel, and φ is this map followed by the Borel map sending W_x to X_x.

It remains to check that φ witnesses that $E(\mathbb{F}_2, 2) \upharpoonright A \leq_B E$. Let us note some properties of the subshifts X_x which will be important for this. First, for any x, the periodic points of X_x are (shifts of) the following:

$\bar{0}$	period 1
$\overline{d(0) \frown 0^{p(2n+2)-3}}$	period $p(2n+2)$
$\overline{d(1) \frown 0^{p(2n+3)-3}}$	period $p(2n+3)$
$\overline{\rho(w)}$ for $w \in \mathbb{F}_2$	period $8 \cdot |w|$ (with $|e| = 1$)

For every x, X_x also includes the points $\bar{0} \frown d(0) \frown \bar{0}$, $\bar{0} \frown d(1) \frown \bar{0}$, and $\bar{0} \frown \rho(w) \frown \bar{0}$ for $w \in \mathbb{F}_2$. It also contains all left-, right-, or bi-infinite sequences of $\rho(w)$'s (with other coordinates 0). The only other limit points will be among $\bar{0} \frown d(i) \frown 0^{10+k^2} \frown \rho[w^*]$ (where w^* is a right-infinite word); the inclusion of these points for various i, k, and w^* depends on the orbit of x.

Suppose first that $x, y \in 2^{\mathbb{F}_2}$ with $xE(\mathbb{F}_2, 2)y$; we show that $(X_x, S) \cong (Y_x, S)$ (this does not require $x, y \in A$). Since a and b generate $\mathbb{F}_2$, it will be sufficient to show this when $y = a \cdot x$ or $y = b \cdot x$. Consider the case $y = a \cdot x$ (the case of $y = b \cdot x$ is identical). Let P be the set $\{\overline{d(0) \frown 0^{p(2n+2)-3}} : n \in$

$\mathbb{N}\} \cup \{\overline{d(1) \frown 0^{p(2n+3)-3}} : n \in \mathbb{N}\}$; then we have:

$$\begin{aligned} A_x &= P \cup \{\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\} \\ A_y &= P \cup \{\bar{0} \frown d(w_i \cdot y(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\} \\ &= P \cup \{\bar{0} \frown d(w_i a \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0} : i, k \in \mathbb{N}\} \\ &= P \cup \{\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i a^{-1}) \frown \bar{0} : i, k \in \mathbb{N}\}. \end{aligned}$$

We can then take $r = 16$ and define a block code $\pi : 2^{\{-r,\dots,r\}} \to 2$ such the induced map f_π is an isomorphism of X_x and X_y. The idea is to ensure that each element $\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i) \frown \bar{0}$ is mapped to $\bar{0} \frown d(w_i \cdot x(w_k)) \frown 0^{10+k^2} \frown \rho(w_i a^{-1}) \frown \bar{0}$ (and likewise for their shifts) and all other sequences are left unchanged. This will provide an S-invariant homeomorphism of A_x with A_y and hence an isomorphism of X_x with X_y.

It will be sufficient to ensure that the following subsequences are mapped as shown:

$$\begin{aligned} 0 \frown \rho(e) \frown 0 &\mapsto 0 \frown \rho(a^{-1}) \frown 0 \\ 0 \frown \rho(a) \frown 0 &\mapsto 0 \frown \rho(e) \frown 0 \\ \rho(a) \frown \rho(a) \frown 0 &\mapsto \rho(a) \frown 0^8 \frown 0 \\ \rho(b) \frown \rho(a) \frown 0 &\mapsto \rho(b) \frown 0^8 \frown 0 \\ \rho(b^{-1}) \frown \rho(a) \frown 0 &\mapsto \rho(b^{-1}) \frown 0^8 \frown 0 \\ \rho(a^{-1}) \frown 0^8 \frown 0 &\mapsto \rho(a^{-1}) \frown \rho(a^{-1}) \frown 0 \\ \rho(b) \frown 0^8 \frown 0 &\mapsto \rho(b) \frown \rho(a^{-1}) \frown 0 \\ \rho(b^{-1}) \frown 0^8 \frown 0 &\mapsto \rho(b^{-1}) \frown \rho(a^{-1}) \frown 0 \end{aligned}$$

and all other sequences should be left unchanged. None of the listed sequences overlaps a shift of another, so this can be ensured by choosing the block code π appropriately; taking $r = 16$ is sufficient for each image digit to "see enough" of the input to tell if and wherein it is part of one of the listed sequences.

For the other direction, suppose $x, y \in A$ are such that $X_x \cong X_y$ via the function f_π induced by the block code $\pi : 2^{\{-r,\dots,r\}} \to 2$. We will show that $x \, E(\mathbb{F}_2, 2) \, y$. The periods of the periodic points must be preserved by f_π, so $f_\pi(\bar{0}) = \bar{0}$ as this is the unique point of period 1; hence $\pi(0^{2r+1}) = 0$. Similarly, considering the points of P for large n we must have that $f_\pi(\bar{0} \frown d(0) \frown \bar{0}) = S^{n(0)}(\bar{0} \frown d(0) \frown \bar{0})$ and $f_\pi(\bar{0} \frown d(1) \frown \bar{0}) = S^{n(1)}(\bar{0} \frown d(1) \frown \bar{0})$ for some $n(0), n(1) \in \{-r, \dots, r\}$. We can also find some sequence s_0 and $m_0 \in \{-r, \dots, r\}$ such that $f_\pi(\bar{0} \frown \rho(e) \frown \bar{0}) = S^{m_0}(\bar{0} \frown s_0 \frown \bar{0})$.

If we then consider the points $\bar{0} \frown d(x(w_k)) \frown 0^{10+k^2} \frown \rho(e) \frown \bar{0}$ in X_x for sufficiently large k, we see that these must map to (shifts of) the points $\bar{0} \frown d(x(w_k)) \frown 0^{10+k^2+n(x(w_k))-m_0} \frown s_0 \frown \bar{0} \in X_y$. Each of these must then be a shift of a point of the form $\bar{0} \frown d(w_{i_k} \cdot y(w_{j_k})) \frown 0^{10+{j_k}^2} \frown \rho(w_{i_k}) \frown \bar{0}$ for some i_k and j_k. Since $\rho(w_{i_k}) = s_0$ for sufficiently large k, there is some fixed

$w \in \mathbb{F}_2$ such that $s_0 = \rho(w)$, and for sufficiently large k we have $w_{i_k} = w$. We must also have $k^2 + n(x(w_k)) - m_0 = j_k^2$, so for $k > r$ we need $j_k = k$ (and hence $n(0) = n(1) = m_0$).

Thus, $w \cdot y(w_k) = x(w_k)$ for all but finitely many k. Since x and y are in A, this requires that $w \cdot y \, E(\mathbb{F}_2, 2)\, x$. This means that $y \, E(\mathbb{F}_2, 2)\, x$ and we are done. □

4 Higher dimensional subshifts

Above we have shown that isomorphism of one-dimensional subshifts is a universal countable Borel equivalence relation. We can see that the same is true for n-dimensional subshifts for any $n \geq 1$. An *n-dimensional subshift* on an alphabet A is a closed subset of $A^{\mathbb{Z}^n}$ which is invariant under the shift maps $S_1, \dots, S_n$, where $S_k(x)(i_1, \dots, i_n) = x(i_1, \dots, i_{k-1}, i_k + 1, i_{k+1}, \dots, i_n)$. Let E_n denote isomorphism relation on the space $\mathcal{F}_2^{S,n}$ of n-dimensional subshifts on the alphabet 2. Then each E_n is again a countable Borel equivalence relation, so each is Borel reducible to isomorphism of one-dimensional subshifts by the above theorem. The reverse is also true:

Theorem 12. *For each $n \geq 1$, we have $E \leq_B E_n$, where E is isomorphism of one-dimensional subshifts on 2. In particular all the relations E_n are universal countable Borel equivalence relations, and hence they are all mutually bi-reducible.*

Proof: Fix $n \geq 1$, and consider the injection $f : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}^n}$ given by

$$f(x)(i_1, \dots, i_n) = x(i_1).$$

This induces the map $\tilde{f} : \mathcal{F}_2^S \to \mathcal{F}_2^{S,n}$ given by $\tilde{f}(X) = f[X]$. We claim that $\tilde{f}$ is a reduction of E to E_n. Let $X, Y \in \mathcal{F}_2^S$. If $g : X \to Y$ is an isomorphism, then $\tilde{g} : \tilde{f}(X) \to \tilde{f}(Y)$ is an isomorphism, where $\tilde{g}(f(x)) = f(g(x))$. Conversely, if $\tilde{\varphi} : \tilde{f}(X) \to \tilde{f}(Y)$ is an isomorphism, then $\varphi : X \to Y$ is an isomorphism, where $\varphi(x) = f^{-1}(\tilde{\varphi}(f(x)))$. □

This result may be somewhat surprising, since two-dimensional subshifts are known to be more complicated than one-dimensional subshifts in a number of ways. The above theorem shows, though, that their classification up to isomorphism is no more difficult than for the one-dimensional case, at least from the standpoint of Borel reducibility.

5 Questions

Here we have considered the two-sided shift on $A^{\mathbb{Z}}$. We could also consider the one-sided shift on $A^{\mathbb{N}}$.

Question 1. *What is the complexity of the isomorphism of one-sided shifts on $2^{\mathbb{N}}$?*

We could also consider larger equivalence relations. For subshifts X and Y, write $X \geq Y$ if there is a shift morphism $f : X \to Y$. We write $X \geq_{\mathrm{inj}} Y$ when f is injective, and $X \geq_{\mathrm{surj}} Y$ when f is surjective (in which case f is a *factor map*). We can then define the following three equivalence relations:

$$
\begin{aligned}
X \, E_{\mathrm{morph}} \, Y &\Leftrightarrow X \geq Y \wedge Y \geq X \\
X \, E_{\mathrm{inj}} \, Y &\Leftrightarrow X \geq_{\mathrm{inj}} Y \wedge Y \geq_{\mathrm{inj}} X \\
X \, E_{\mathrm{surj}} \, Y &\Leftrightarrow X \geq_{\mathrm{surj}} Y \wedge Y \geq_{\mathrm{surj}} X.
\end{aligned}
$$

Question 2. *What are the complexities of the above three equivalence relations?*

Note that all of these contain the isomorphism relation, E_{morph} contains the other two, and these no longer have all equivalence classes countable.

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