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WEAKLY POINTED TREES AND PARTIAL INJECTIONS

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Abstract. We define the notion of a weakly pointed tree, and characterize the amount of genericity necessary to prevent a uniformly branching tree being weakly pointed. We use these ideas to show there is no topological analogue of a measure-theoretic selection theorem of Graf and Mauldin.

We consider some topological and recursion-theoretic questions motivated by the following measure-theoretic result of Graf and Mauldin (Theorem 4.4 of [1]):

THEOREM. *Let X and Y be analytic spaces, λ a probability measure on X, μ a probability measure on Y, and $R \subseteq X \times Y$ a Borel set such that R_x is uncountable for λ-a.e. $x \in X$ and R^y is uncountable for μ-a.e. $y \in Y$. Then there exists a Borel set $A \subseteq X$ with $\lambda(A) = 1$, a Borel set $B \subseteq Y$ with $\mu(B) = 1$, and a Borel isomorphism f from A onto B whose graph is contained in R.*

This says that a sufficiently thick plane set admits a selector defined almost everywhere which is injective. We can consider the following topological analogue of this result:

QUESTION. *Suppose X and Y are Polish spaces, and $R \subseteq X \times Y$ is a Borel set such that R_x is uncountable for a comeager set of $x \in X$ and R^y is uncountable for a comeager set of $y \in Y$. Mush there exist a comeager Borel set $A \subseteq X$, a comeager Borel set $B \subseteq Y$, and a Borel isomorphism f from A to B whose graph is contained in R?*

This turns out to be false, as we will show in Section 2. A counterexample is the set $\{(x, y) \in 3^\omega \times 2^\omega : \forall n\ (x(n) \neq 2 \Rightarrow y(n) = x(n))\}$. In the next section we introduce the notion of a weakly pointed tree, and draw a connection between this set and such trees. We then show that a generic tree is not weakly pointed, and characterize the precise amount of genericity necessary to prevent weak pointedness. A suitable generalization will then provide the counterexample to the above question.

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§1. Weakly pointed trees. Recall that a tree T is *pruned* if every node in T has a proper extension in T, and T is *perfect* if every node in T has two incompatible proper extensions in T. We introduce a coding of uniformly branching pruned trees by elements of 3^ω.

DEFINITION 1. Let $x \in 3^\omega$. Then x encodes the uniformly branching tree

$$T_x = \{s \in 2^{<\omega} : \forall n < |s| \ (x(n) \neq 2 \Rightarrow s(n) = x(n))\}.$$

Note that $T_x \equiv_T x$.

DEFINITION 2. For $x \in 3^\omega$ and $y \in 2^\omega$ we say y *is consistent with* x if

$$\forall n \in \omega \ (x(n) \neq 2 \Rightarrow x(n) = y(n)).$$

We use the same notation for finite strings, imposing requirements only for n less than the length of the shorter string. Note that y is consistent with x if and only if $y \in [T_x]$.

DEFINITION 3. A pruned tree T is *pointed* if $T \leq_T y$ for every branch $y \in [T]$. We say that T is *weakly pointed* if there is some branch $y \in [T]$ such that $T \leq_T y$. For a uniformly branching tree encoded by $x \in 3^\omega$, this is equivalent to saying that there is a y consistent with x such that $x \leq_T y$. Note that every pointed tree is weakly pointed.

DEFINITION 4. Let $B_x = \{n \in \omega : x(n) = 2\}$ be the branching levels of T_x. Note that $B_x \leq_T x$, and for $y \in [T_x]$ we have $x \leq_T y$ if and only if $B_x \leq_T y$. Hence T_x is weakly pointed iff there is a $y \in [T_x]$ with $B_x \leq_T y$.

DEFINITION 5. An element $x \in 3^\omega$ is *n-generic* if for every Σ^0_n set $A \subseteq 3^{<\omega}$ there is a string $\sigma \sqsubseteq x$ such that either (a) $\sigma \in A$, or (b) $\forall \tau \sqsupseteq \sigma$ $(\tau \notin A)$. We say that x is *weakly n-generic* if for every dense Σ^0_n set $A \subseteq 3^{<\omega}$ there is $\sigma \sqsubseteq x$ such that $\sigma \in A$. Note that n-generic implies weakly n-generic, and weakly $(n+1)$-generic implies n-generic.

We now characterize the amount of genericity necessary to rule out pointedness and weak pointedness. We begin with an easy observation:

LEMMA 6. *If $x \in 3^\omega$ is weakly 1-generic then B_x is infinite, i.e. T_x is perfect.*

PROOF. For $n \in \omega$ let $A_n = \{\sigma : \sigma \text{ contains at least } n \text{ 2's}\}$. Then A_n is a dense r.e. set, and any x meeting all of them must contain infinitely many 2's. ⊣

We first see that pointedness is impossible for 1-generic trees:

THEOREM 7. *If $x \in 3^\omega$ is 1-generic, then T_x is not pointed.*

PROOF. We will show that the leftmost branch of T_x cannot compute T_x. As noted above, a branch y of T_x can compute T_x if and only if y can compute the set of branch points B_x. Let $y = \tilde{x} \in T_x$, where $\tilde{x}(n) = x(n)$ if $x(n) \neq 2$ and $\tilde{x}(n) = 0$ if $x(n) = 2$; this is the leftmost branch of T_x. Suppose there were a partial recursive function φ such that φ^y computed the characteristic function χ_{B_x}; we will show that this yields a contradiction. Define the r.e. set A by

$$A = \{\sigma \in 3^{<\omega} : \exists n < |\sigma| \ (\sigma(n) \neq 2 \ \wedge \ \varphi^{\tilde{\sigma}}(n) \downarrow = 1)\},$$

where $\tilde{\sigma}(n) = \sigma(n)$ if $\sigma(n) \neq 2$ and $\tilde{\sigma}(n) = 0$ if $\sigma(n) = 2$. If there is a $\sigma \sqsubseteq x$ with $\sigma \in A$ then there is an n such that $\sigma(n) \neq 2$ (so $n \notin B_x$) but $\varphi^y(n) = \varphi^{\tilde{\sigma}}(n) = 1$, so $\varphi^y \neq \chi_{B_x}$. Otherwise, there is $\sigma \sqsubseteq x$ such that $\forall \tau \sqsupseteq \sigma\ (\tau \notin A)$. Let τ_k for $k \geq |\sigma|$ be given by $\tau_k(n) = \sigma(n)$ for $n < |\sigma|$ and $\tau_k(n) = y(n)$ for $|\sigma| \leq n < k$, so $\tilde{\tau}_k = y \upharpoonright k$. Taking $\tau = \tau_k$ we then have $\neg\exists n \geq |\sigma|\ (\varphi^{y \upharpoonright k} \downarrow = 1)$, so if φ^y is total then it is finite, and again $\varphi^y \neq \chi_{B_x}$ since B_x must be infinite for x 1-generic. $\dashv$

We can see that at least weak 2-genericity is necessary to prevent weak pointedness, though:

Theorem 8. *There is a 1-generic $x \in 3^\omega$ such that T_x is weakly pointed.*

Proof. We will simultaneously build a sequence $\langle \sigma_i \rangle_{i \in \omega}$ with $\sigma_i \in 3^{<\omega}$ and a sequence $\langle s_i \rangle_{i \in \omega}$ with $s_i \in 2^{<\omega}$, so that letting $x = \sigma_0 \frown \sigma_1 \frown \cdots$ and $y = s_0 \frown s_1 \frown \cdots$ we have that y is a branch in T_x with $x \leq_T y$. Start with $\sigma_0 = \langle 2 \rangle$ and $s_0 = \langle\rangle$. Given σ_n and s_n, we construct σ_{n+1} to force the n^{th} r.e. set, and s_{n+1} to encode σ_{n+1}, as follows. First, we check whether there is a $\sigma = \sigma_0 \frown \cdots \frown \sigma_n \frown \tau$ with $\sigma \in R_n$, where $R_n = \operatorname{range}(\varphi_n)$ is the n^{th} r.e. subset of $3^{<\omega}$ (this can be done uniformly recursively in $0'$). If so, let $\langle i, j \rangle$ be the least pair such that $\varphi_{n,i}(j) \downarrow = \sigma = \sigma_0 \frown \cdots \frown \sigma_n \frown \tau$ for some τ. We let $\sigma_{n+1} = \tau \frown \langle 2 \rangle$ and let $s_{n+1} = \langle 1 \rangle \frown \tilde{\tau}$, where $\tilde{\tau}(n) = \tau(n)$ if $\tau(n) \neq 2$ and $\tilde{\tau}(n) = 0$ if $\tau(n) = 2$. Otherwise, if there is no such σ, we let $\sigma_{n+1} = \langle 2 \rangle$ and we let $s_{n+1} = \langle 0 \rangle$. Note that $s_0 \frown \cdots \frown s_n$ is always one digit shorter than $\sigma_0 \frown \cdots \frown \sigma_n$.

In the end we let $x = \sigma_0 \frown \sigma_1 \frown \cdots$ and $y = s_0 \frown s_1 \frown \cdots$. It is clear that $y \in T_x$ since $y(n) = x(n)$ whenever $x(n) \neq 2$. We see that x is 1-generic since for each r.e. set $R_n \subseteq 3^{<\omega}$ we either have that $\sigma_0 \frown \cdots \frown \sigma_{n+1} \in R_n$, or for any $\tau \sqsupseteq \sigma_0 \frown \cdots \frown \sigma_n$ we have $\tau \notin R_n$. It remains to check that $x \leq_T y$. We will reconstruct the sequence $\langle \sigma_i \rangle_{i \in \omega}$ recursively in y. Let $\sigma_0 = \langle 2 \rangle$ and $i_0 = y(0)$. Given σ_n and i_n, we find σ_{n+1} and i_{n+1} as follows. If $i_n = 1$ then we know that there was a $\sigma \sqsupseteq \sigma_0 \frown \cdots \frown \sigma_n$ with $\sigma \in R_n$, so we can calculate the least pair $\langle i, j \rangle$ such that $\varphi_{n,i}(j) \downarrow = \sigma \sqsupseteq \sigma_0 \frown \cdots \frown \sigma_n$ and let $\sigma_{n+1} = \sigma \frown \langle 2 \rangle$. Otherwise, if $i_n = 0$ there was no such σ and we let $\sigma_{n+1} = \langle 2 \rangle$. We then let $l = |\sigma_0 \frown \cdots \frown \sigma_{n+1}|$ and let $i_{n+1} = y(l-1)$. $\dashv$

Note that $x <_T 0'$. By adding a second 2 after each stage, we can actually construct x such that T_x contains a pointed subtree. We do not know whether we can find an x such that T_x has a branch y with $x \equiv_T y$.

Finally, we show that weak 2-genericity is the precise level of genericity needed to rule out weak pointedness:

Theorem 9. *If $x \in 3^\omega$ is weakly 2-generic, then T_x is not weakly pointed.*

Proof. Let $x \in 3^\omega$ be weakly 2-generic, and let $y \in [T_x]$. We will show that there is no partial recursive function φ such that $\varphi^y = \chi_{B_x}$. Given φ, define the Σ^0_2 set

$$\begin{aligned} A = \{\sigma \in 3^{<\omega} : \forall s \in 2^{|\sigma|} &\text{ consistent with } \sigma \\ &[\exists n < |\sigma|\ (\sigma(n) \neq 2 \ \wedge\ \varphi^s(n) \downarrow = 1) \ \vee \\ &\neg\exists t \sqsupseteq s \exists n\ (|s| \leq n < |t| \ \wedge\ \varphi^t(n) \downarrow = 1)]\}. \end{aligned}$$

Suppose there is $\sigma \sqsubseteq x$ such that $\sigma \in A$. Let $s = y \restriction |\sigma|$, so s is consistent with σ. If there is $n < |\sigma|$ such that $\sigma(n) = x(n) \neq 2$ and $\varphi^s(n) = \varphi^y(n) \downarrow= 1$, then immediately $\varphi^y \neq \chi_{B_x}$. Otherwise, $\neg\exists t \sqsupseteq s \exists n\ (|s| \leq n < |t| \ \wedge\ \varphi^t(n) \downarrow= 1)$, so $\neg\exists n \geq |\sigma|\ (\varphi^y(n) \downarrow= 1)$. Hence if φ^y is total then it must compute a finite set, and since B_x is infinite we again have $\varphi^y \neq \chi_{B_x}$.

It remains to check that A is dense. Suppose not, so there is a σ such that $\forall \tau \sqsupseteq \sigma\ (\tau \notin A)$. We will show that this yields a contradiction. We have:

$$\begin{aligned}
&\forall \tau \sqsupseteq \sigma \exists s \in 2^{|\tau|} \text{ consistent with } \tau \\
&\qquad [\neg\exists n < |\tau|\ (\tau(n) \neq 2 \ \wedge\ \varphi^s(n) \downarrow= 1) \ \wedge \\
&\qquad \exists t \sqsupseteq s \exists n\ (|s| \leq n < |t| \ \wedge\ \varphi^t(n) \downarrow= 1)].
\end{aligned}$$

Specializing this to $\tau = \sigma \frown t$ where $t \in 2^{<\omega}$ we then have:

$$\begin{aligned}
&\forall t \in 2^{<\omega} \exists s \in 2^{|\sigma|}\ [s \text{ is consistent with } \sigma \ \wedge \\
&\qquad \neg\exists n\ (|\sigma| \leq n < |\sigma \frown t| \ \wedge\ \varphi^{s \frown t}(n) \downarrow= 1) \ \wedge \\
&\qquad \exists t' \sqsupseteq t \exists n\ (|\sigma \frown t| \leq n < |\sigma \frown t'| \ \wedge\ \varphi^{s \frown t'}(n) \downarrow= 1)].
\end{aligned}$$

We now form sequences $\langle t_i \rangle_{i \in \omega}$ and $\langle s_i \rangle_{i \in \omega}$ of elements of $2^{<\omega}$ such that for each i, $t_i \sqsubseteq t_{i+1}$ and $s_i \in 2^{|\sigma|}$ is consistent with σ. Let $t_0 = \langle\rangle$. Given t_i, by the previous conclusion we can choose $s_i \in 2^{|\sigma|}$ consistent with σ and $t_{i+1} \sqsupseteq t_i$ such that

$$\neg\exists n\ (|\sigma| \leq n < |\sigma \frown t_i| \ \wedge\ \varphi^{s_i \frown t_i}(n) \downarrow= 1)$$

and

$$\exists n\ (|\sigma \frown t_i| \leq n < |\sigma \frown t_{i+1}| \ \wedge\ \varphi^{s_i \frown t_{i+1}}(n) \downarrow= 1).$$

Since there are only finitely many choices for s_i, there must be some fixed s consistent with σ and an infinite sequence $n_0 < n_1 < \cdots$ such that $s_{n_i} = s$ for all i. Let $z = s \frown \bigcup_i t_i$. Then for each i we have

$$\neg\exists n\ (|\sigma| \leq n < |\sigma \frown t_{n_i}| \ \wedge\ \varphi^{s \frown t_{n_i}}(n) \downarrow= 1)$$

so $\neg\exists n \geq |\sigma|\ (\varphi^z(n) \downarrow= 1)$, but for each i we have

$$\exists n\ (|\sigma \frown t_{n_i}| \leq n < |\sigma \frown t_{n_i+1}| \ \wedge\ \varphi^{s \frown t_{n_i+1}}(n) \downarrow= 1)$$

so $\exists^\infty n\ (\varphi^z(n) \downarrow= 1)$, a contradiction. ⊣

Since the set of weakly 2-generics is comeager, we have:

Corollary 10. *The following set is meager:*

$$\{x \in 3^\omega : \exists y \in 2^\omega\ (y \geq_T x \wedge y \text{ is consistent with } x)\}.$$

§2. Partial injections.

The following result has been proved independently by Hjorth. It may be proved using a relativized version of the preceding results applied to ξ-generics, but we present a purely topological proof.

Theorem 11. *There is no Baire-measurable partial injection $F : 3^\omega \to 2^\omega$ with comeager domain such that*

$$\mathit{Graph}(F) \subseteq R = \{(x, y) \in 3^\omega \times 2^\omega : \forall n\ (x(n) \neq 2 \Rightarrow x(n) = y(n))\}.$$

Note that R is Borel, R_x is uncountable for a comeager set of x by Lemma 6, and R^y is uncountable for all y. Hence this gives a negative answer to the initial question.

PROOF. Suppose there were such an F. Then there is a comeager set C such that $F \upharpoonright C$ is continuous (and injective). By removing a meager perfect set from C, we can assume that we have a comeager set C and a Borel bijection $f : 3^\omega \to 2^\omega$ such that $f(x)$ is consistent with x for all $x \in C$. Define the set $A \subseteq \omega \times 2^\omega$ to be

$$A = \{(n, y) : f^{-1}(y)(n) = 2\}.$$

Then A is Borel, and for each $x \in C$ we have $n \in B_x$ if and only if $(n, f(x)) \in A$. We claim that the set

$$D = \{x \in 3^\omega : \exists y \in [T_x] \; \forall n \; (n \in B_x \Leftrightarrow A(n, y))\}$$

is meager, which will yield a contradiction since $C \subseteq D$.

Suppose not; then since D is analytic it has the property of Baire, so there is some $\sigma \in 3^k$ such that D is comeager in N_σ. Since the set of $x \in 3^\omega$ such that B_x is infinite is comeager, we also have that the set

$$D' = \{x \in 3^\omega : B_x \text{ is infinite} \wedge \exists y \in [T_x] \forall n \; (n \in B_x \Leftrightarrow A(n, y))\}$$

is comeager in N_σ.

Let $\{\mu_i : i < m\}$ enumerate those $\mu \in 2^k$ such that μ is consistent with σ. We build a sequence $\emptyset = \tau_0 \sqsubseteq \tau_1 \sqsubseteq \cdots \sqsubseteq \tau_m \in 2^{<\omega}$ as follows. Given τ_i with $i < m$, let $\rho_i = \mu_i \frown \tau_i$. We consider two cases. If there are $n > |\rho_i|$ and $\nu \in 2^{<\omega}$ such that $|\rho_i \frown \nu| > n$ and $\{y \sqsupseteq \rho_i \frown \nu : A(n, y)\}$ is comeager in $N_{\rho_i \frown \nu}$, then we let $\tau_{i+1} = \tau_i \frown \nu$. Otherwise we let $\tau_{i+1} = \tau_i$.

We let $\tau = \sigma \frown \tau_m \in 3^{<\omega}$. Note that if $y \in 2^\omega$ is consistent with some $x \in 3^\omega$ such that $\tau \sqsubseteq x$, then $\mu_i \frown \tau_{i+1} \sqsubseteq y$ for some $i < m$. If the first case of the construction held for i, then there is an n such that the set $\{y : A(n, y)\}$ is comeager in N_μ for the $\mu = \mu_i \frown \tau_m \in 2^{<\omega}$ extending the μ_i which is consistent with τ, and $n \notin B_x$ for any $x \sqsupseteq \tau$ since $\tau(n) \in \{0, 1\}$. If the second case held for i then for all $n > |\tau|$ the set $\{y : \neg A(n, y)\}$ is comeager in N_μ for the $\mu = \mu_i \frown \tau_m$ extending the μ_i which is consistent with τ. Hence, the set

$$E = \{y : \{n : A(n, y)\} \text{ is finite} \vee \exists n \; (A(n, y) \wedge \forall x \sqsupseteq \tau \; (n \notin B_x))\}$$

is comeager in the clopen set

$$K = \{y \in 2^\omega : y \text{ is consistent with some } x \in 3^\omega \text{ with } \tau \sqsubseteq x\}.$$

Fix now sets $G_n \subseteq 3^\omega$ for $n \in \omega$ which are open dense in N_τ such that $\bigcap_{n \in \omega} G_n \subseteq D'$, and fix sets $H_n \subseteq 2^\omega$ for $n \in \omega$ which are open dense in K such that $\bigcap_{n \in \omega} H_n \subseteq E$. We build $x \sqsupseteq \tau$ as follows.

Let $\tau_0 = \tau$. Given τ_i, let $\sigma_i \sqsupseteq \tau_i$ with $|\sigma_i| > |\tau_i|$ and $N_{\sigma_i} \subseteq G_i$. Let $\{\mu_j : j < m_i\}$ enumerate those $\mu \in 2^{|\sigma_i|}$ such that μ is consistent with σ_i. Build $\emptyset = \nu_0 \sqsubseteq \nu_1 \sqsubseteq \cdots \sqsubseteq \nu_{m_i}$ such that for each $j < m_i$ we have $N_{\mu_j \frown \nu_{j+1}} \subseteq H_i$. Let $\tau_{i+i} = \sigma_i \frown \nu_{m_j}$. At the end set $x = \bigcup_{i \in \omega} \tau_i \in 3^\omega$.

First, $x \in D'$ since x is in each G_i, so B_x is infinite and there is a $y \in [T_x]$ such that $\forall n \; (n \in B_x \Leftrightarrow A(n, y))$. We must then have $y \in H_i$ for each i, so $y \in E$.

But then either $\{n : A(n, y)\}$ is finite, in which case B_x is finite, or there is an n with $A(n, y)$ and $n \notin B_x$. In either case we have reached a contradiction. $\dashv$

§3. Questions. We used an analysis of pointed trees to rule out a topological analogue of the theorem of Graf and Mauldin stated at the beginning. We could conversely ask how this theorem applies to weakly pointed trees. Applying that theorem to the set R, we can conclude that there are countable ordinals α and β such that for every α-random $x \in 3^\omega$ there is a $y \in 2^\omega$ consistent with x such that $x \leq_T y^{(\beta)}$, where $y^{(\beta)}$ is the β-jump of y. We can then ask what the minimum possible values are:

Question 1. *What are the minimal ordinals α and β that satisfy the above conclusion?*

The tree T_x being weakly pointed corresponds to $\beta = 0$, so we can ask:

Question 2. *Is there an α such that T_x is weakly pointed for every α-random x? What about 1-random x?*

We could also ask about three possible technical improvements to our constructions:

Question 3. *Is there a 1-generic x such that there is a branch $y \in [T_x]$ with $x \equiv_T y$?*

Question 4. *If x is weakly 1-generic, can T_x be pointed?*

Question 5. *Must a weakly pointed tree contain a pointed subtree?*

Finally, we have considered only uniformly branching trees here. We could generalize this and ask:

Question 6. *What about general pruned trees encoded by elements of 3^ω?*

REFERENCES

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