Isometry of Polish Metric Spaces

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June 19, 2007

Abstract

We consider the equivalence relation of isometry of separable, complete metric spaces, and show that any equivalence relation induced by a Borel action of a Polish group on a Polish space is Borel reducible to this isometry relation. We also consider the isometry relation restricted to various classes of metric spaces, and produce lower bounds for the complexity in terms of the Borel reducibility hierarchy.

In this article we consider the equivalence relation of isometry of Polish metric spaces and ask how complicated it is. By a *Polish metric space* we mean a Polish space X together with a complete metric d on X. Two spaces are *isometric* if there is a bijection between them which preserves the metric. We wish to characterize the complexity of this isometry relation. We do this by considering it as an equivalence relation on an appropriately defined Polish space, and using the theory of Borel reducibility of equivalence relations.

Definition 1. Let E and F be equivalence relations on the Polish spaces X and Y. We say that E is *Borel reducible* to Y, $E \leq_B Y$, if there is a Borel function $f : X \to Y$ such that for all $x_1, x_2 \in X$ we have $x_1 \, E \, x_2$ iff $f(x_1) \, F \, f(x_2)$.

When E is reducible to F, we may view the classification of E as at most as complicated as that of F. We will develop techniques for reducing equivalence relations induced by Borel actions of Polish groups to the isometry relation. We will show that any such equivalence relation is reducible to the isometry relation, a result obtained independently by Gao and Kechris in [8]. We will also use these techniques to find bounds on the complexity

[0]2000 Mathematical Subject Classification. Primary 03E15; Secondary 54H05.

of the isometry relation restricted to certain classes of Polish metric spaces. This paper is in many respects a companion to [8]. The techniques here focus heavily on the distance sets of the metric spaces constructed, whereas Gao and Kechris focus on the isometry groups. Many of these results were announced in [5].

In the next section we formalize the isometry relation, and in Section 2 we present the proof that any orbit equivalence relation induced by a Borel action of a Polish group on a Polish space is reducible to the isometry relation. Section 3 discusses some particular classes of metric spaces for which we can get immediate lower bounds on the complexity of the isometry relation using these techniques. This discussion is continued in Section 4 where we consider Polish metric spaces which are Polish groups with invariant metrics, and in Section 5 where we consider ultra-homogeneous spaces.

1 The isometry relation

We begin by formalizing the isometry relation as an equivalence relation on the space of codes for Polish metric spaces. Since we are dealing with separable, complete metric spaces, a space is completely determined by its metric restricted to a countable dense subset. More precisely, we may fix a countable dense set $\{x_i : i \in \mathbb{N}\}$ in a space, and then code it by an array $\langle d_{i,j}\rangle_{i,j\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$, where $d_{i,j}$ is the distance between the points x_i and x_j. The array will satisfy the conditions of a metric on the set $\{x_i : i \in \mathbb{N}\}$. We will use such arrays as codes for Polish metric spaces, although we will permit two points to be at distance 0 to allow for the coding of finite spaces (in which case we identify the two points in the resulting space). Such an array then codes the Polish metric space which is the completion of the given countable space (which will be dense in the resulting space).

Definition 2. Let the space $\mathcal{M}$ of *codes for Polish metric spaces* be:

$$\begin{aligned}\mathcal{M} = \{\langle d_{i,j}\rangle_{i,j\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}} \text{ such that:} & \\ & \forall i \forall j \ (d_{i,j} \geq 0) \wedge \\ & \forall i \ (d_{i,i} = 0) \wedge \\ & \forall i \forall j \ (d_{i,j} = d_{j,i}) \wedge \\ & \forall i \forall j \forall k \ (d_{i,k} \leq d_{i,j} + d_{j,k}) \ \}.\end{aligned}$$

This is a closed subspace of $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ and is hence a Polish space in the relative topology. We now define the isometry relation, $\cong_i$, on this space.

Definition 3. For $\langle d_{i,j}\rangle$ and $\langle \widetilde{d}_{i,j}\rangle$ in $\mathcal{M}$, we set $\langle d_{i,j}\rangle \cong_i \langle \widetilde{d}_{i,j}\rangle$ if and only if the metric spaces coded by the two arrays are isometric.

Lemma 4. *The isometry relation $\cong_i$ is a $\mathbf{\Sigma}^1_1$ (analytic) equivalence relation on $\mathcal{M}$.*

Proof: We have the following calculation:

$$\begin{aligned}\langle d_{i,j}\rangle \cong_i \langle \widetilde{d}_{i,j}\rangle \iff \exists f : \mathbb{N} \to \mathbb{N}^{\mathbb{N}} \text{ sending } n \mapsto f_n \text{ such that:}\\ \forall n \forall k \exists N \forall i, j \geq N \ \left(\widetilde{d}_{f_n(i),f_n(j)} < \frac{1}{k}\right) \ \wedge\\ \forall n \forall m \forall k \exists N \forall i \geq N \ \left(\left|d_{n,m} - \widetilde{d}_{f_n(i),f_m(i)}\right| < \frac{1}{k}\right) \ \wedge\\ \forall n \forall k \exists m \exists N \forall i \geq N \ \left(\widetilde{d}_{n,f_m(i)} < \frac{1}{k}\right).\end{aligned}$$

That is, there is a function f which induces a map sending each point x_i to a $\hat{d}$-Cauchy sequence (by the first condition); the second condition guarantees that this map is an isometric embedding, and the third condition guarantees that it is surjective. The calculation shows that $\cong_i$ is the projection of a Borel subset of the Polish space $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}\times\mathbb{N}} \times \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$, and hence is $\mathbf{\Sigma}^1_1$. □

The reducibility results below will show that $\cong_i$ is in fact a $\mathbf{\Sigma}^1_1$-complete equivalence relation. We note that this is not the only possible way of coding Polish metric spaces; for instance, they may also be coded as closed subsets of Urysohn's universal metric space $\mathbb{U}$, as is done in [8].

In attempting to classify Polish metric spaces (or subclasses thereof) up to isometry, one can characterize the difficulty of the classification in terms of the complexity of the corresponding equivalence relation using the notion of Borel reducibility, $\leq_B$. This gives a measure of how complicated a set of complete invariants must be. The simplest equivalence relations are those which are *concretely classifiable*, i.e., those which admit real numbers (or elements of some other Polish spaces) as complete invariants. In the case of compact metric spaces, the isometry relation is quite simple:

Theorem (Gromov [9]). *Isometry of compact metric spaces is concretely classifiable.*

Gromov's proof involves introducing a metric, the *Gromov-Hausdorff metric*, on the space of compact metric spaces. An alternate proof of this result is obtained by observing that two compact metric spaces are isometric if and only if they have the same sets of n-point distance configurations, Spec_n, for all $n \geq 2$. The set of such distance configurations for a given compact space will be compact, and so the sequence of Spec_n's can be coded by a real.

The general problem of classifying Polish metric spaces up to isometry, on the other hand, is quite difficult:

Theorem (Gao-Kechris [8]). *Isometry of Polish metric spaces, $\cong_i$, is Borel bi-reducible with the universal equivalence relation of a Borel action of a Polish group on a Polish space.*

In the next section we give an independently discovered proof of one direction of this theorem, that any orbit equivalence relation induced by a Borel action of a Polish group on a Polish space is reducible to the isometry relation. The method presented here produces metric spaces which maintain certain properties of the group action being reduced, and will allow us to derive lower bounds on the complexity of the isometry relation restricted to various classes of metric spaces.

Let us note one consequence of the fact that the isometry relation is reducible to the orbit equivalence relation of a Polish group. In the article [2] we give a construction which produces a Polish metric space whose set of distances is an arbitrary given analytic set of non-negative reals (which includes 0, and has 0 as a limit point if uncountable). The result of Gao and Kechris shows that we can not produce such a metric space uniformly (in any definable way), even for $\mathbf{\Sigma}^0_2$ distance sets. The reason for this is that the equivalence relation E_1, defined on $2^{\mathbb{N}\times\mathbb{N}}$ by

$$\langle x_n\rangle_{n\in\mathbb{N}} E_1 \langle y_n\rangle_{n\in\mathbb{N}} \iff \forall^\infty n \ (x_n = y_n),$$

is known not to be reducible to the orbit equivalence relation of any Polish group (Kechris and Louveau, [12]). Since this is a $\mathbf{\Sigma}^0_2$ equivalence relation, it is easily reducible to the relation of equality of (codes for) $\mathbf{\Sigma}^0_2$ sets, which then can not be reducible to the isometry relation. In a future article ([4] we will consider the difficulty of classifying up to isometry all Polish metric spaces with a fixed set of distances.

2 Reducing Polish group actions

In this section we show how to reduce the orbit equivalence relation induced by a Borel action of a Polish group on a Polish space to the isometry relation $\cong_i$. Let G be a Polish group, and let E^X_G be a *Borel G-space*, i.e. the equivalence relation on a Polish space X induced by a Borel action of G on X. When the action is continuous, we call E_G^X a *Polish G-space*. We will show that $E_G^X \leq_B \cong_i$.

Theorem 5. *Let G be a Polish group and E_G^X a Borel G-space. Then $E_G^X \leq_B \cong_i$.*

The rest of this section is devoted to the proof of this theorem. We begin by restricting the group actions we need to consider. The following theorem will simplify matters:

Theorem (Becker and Kechris, [1]). *For any Borel G-space E_G^X, there is a Polish G-space E_G^Y, with Y a compact Polish space, such that $E_G^X \leq_B E_G^Y$.*

So it will suffice to reduce equivalence relations of the form E_G^Y to the isometry relation, where G acts continuously on a compact space Y. We will define, for each $z \in Y$, a Polish metric space (X_z, d_z) such that for all $z_1, z_2 \in Y$ we have

$$z_1 E_G^Y z_2 \iff (X_{z_1}, d_{z_1}) \cong_i (X_{z_2}, d_{z_2}).$$

Let d_Y be a complete metric on Y compatible with the topology, with $d_Y \leq 1$, and let d_G be a compatible left-invariant metric on G with $d_G \leq 1$. Recall that a metric d_G on a topological group is left-invariant if

$$(\forall g_1, g_2 \in G)(\forall h \in G)\ [d_G(hg_1, hg_2) = d_G(g_1, g_2)].$$

Any Polish group G admits a compatible left-invariant metric, although it does not necessarily admit a complete left-invariant metric. Also, given any metric d, we can form the new metric d' given by

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

This new metric will have values less than 1, and is induces the same topology as d. It will preserve most of the properties of the original metric; in particular, it will continue to be left-invariant when d is.

Before we begin, let us give the naive idea behind the construction. Suppose for the moment that G a has a complete left-invariant metric, that Y is the unit interval $[0, 1]$ with the usual metric, and that the action of G on Y is well-behaved in the sense that for all y in Y and all g_1 and g_2 in G we have

$$\left|g_1^{-1} \cdot y - g_2^{-1} \cdot y\right| \leq 2d_G(g_1, g_2).$$

In this case, we can define a map $z \mapsto (X_z, d_z)$ to directly encode the orbit of z into the set of distances in X_z. Let the underlying set X_z be

$$X_z = \{x^*\} \cup \{x_g : g \in G\},$$

and set

$$\begin{aligned} d_z(x_{g_1}, x_{g_2}) &= d_G(g_1, g_2) && \text{for } g_1, g_2 \in G \\ d_z(x^*, x_g) &= \frac{3}{2} + \frac{1}{2} g^{-1} \cdot z && \text{for } g \in G. \end{aligned}$$

Thus, X_z consists of an isometric copy of G together with a distinguished point x^*, and we use the distances from x^* to the points in the copy of G to directly encode the orbit of z. Checking the triangle inequality only requires noting that, by the assumption above,

$$\begin{aligned} |d_z(x^*, x_{g_1}) - d_z(x^*, x_{g_2})| &= \frac{1}{2} \left| g_1^{-1} \cdot y - g_2^{-1} \cdot y \right| \\ &\leq d_G(g_1, g_2) \\ &= d_z(x_{g_1}, x_{g_2}). \end{aligned}$$

It is immediate that if we have two points z_1 and z_2 in distinct G-orbits, then the set of distances from x^*, $\{d_z(x^*, x_g) : g \in G\}$, will be different in X_{z_1} than it is in X_{z_2}. Hence $(X_{z_1}, d_{z_1}) \not\cong_i (X_{z_2}, d_{z_2})$, since we can see that any isometry between the two spaces would have to send the distinguished point x^* of the first space to that of the second space. We could also note that if f is an isometry of (X_{z_1}, d_{z_1}) and (X_{z_2}, d_{z_2}) which sends x_{1_G} to x_h, then we must have $z_1 = h^{-1} \cdot z_2$.

Conversely, if z_1 and z_2 are in the same orbit, say $z_2 = h \cdot z_1$, then we can define the map f by:

$$\begin{aligned} f(x^*) &= x^* \\ f(x_g) &= x_{hg} \text{ for } g \in G. \end{aligned}$$

The left-invariance of d_G ensures this is an isometry on the copy of G, and we check:

$$\begin{aligned} d_{z_2}(f(x^*), f(x_g)) &= d_{z_2}(x^*, x_{hg}) \\ &= \frac{3}{2} + \frac{1}{2}(hg)^{-1} \cdot z_2 \\ &= \frac{3}{2} + \frac{1}{2} g^{-1} \cdot (h^{-1} \cdot z_2) \\ &= \frac{3}{2} + \frac{1}{2} g^{-1} \cdot z_1 \\ &= d_{z_1}(x^*, x_g). \end{aligned}$$

Thus, we can think of the two spaces as encoding the orbit relative to the parameters z_1 and z_2, respectively, and the isometry as simply changing the parameterization.

There are, of course, three problems with implementing this strategy in general. First, G may not have a complete left-invariant metric. We will resolve this problem by instead using copies of the Polish space $\widehat{G}$, which will be the completion of G with respect to a left-invariant metric. G will be dense in this space, and this will turn out to be sufficient to distinguish orbits. The second problem is that our space Y will not in general be the unit interval (or even continuously embeddable in the unit interval), so we can not literally encode the points in an orbit using distances from a single distinguished point. Instead, we will fix a countable dense subset of Y and encode the distances from a given point to each element of the dense set, which will be elements of $[0, 1]$. Two points with identical distances from all points in a dense set must in fact be the same. This will require that we have countably many distinguished points with which to do the encoding, rather than being able to use a single point as we did before. Finally, the action will not generally satisfy the distance condition we assumed. We can fix this by modifying the metric on G to make this condition hold. We could now carry out the construction using a copy of $\widehat{G}$ along with a countable set of distinguished points, but we will give a somewhat more complicated construction, using countably many copies of $\widehat{G}$ instead, which allows for easier generalization in the following sections.

We will first define a new metric, d'_G, on G which has better behavior with respect to the action of G on Y.

Definition 6. Let d_G be a metric on G with $d_G \leq 1$, and let d_Y be a metric on the compact space Y with $d_Y \leq 1$. For g_1 and g_2 in G, set

$$d'_G(g_1, g_2) = \frac{1}{2} d_G(g_1, g_2) + \frac{1}{2} \sup\{d_Y(g_1^{-1} \cdot y, g_2^{-1} \cdot y) : y \in Y\}.$$

Lemma 7. *The metric d'_G is a compatible left-invariant metric on G with $d'_G \leq 1$, and d'_G is complete if d_G is.*

Proof: Clearly $d'_G \leq 1$. Left-invariance is also easy to check:

$$\begin{aligned}
d'_G(hg_1, hg_2) &= \frac{1}{2}d_G(hg_1, hg_2) + \frac{1}{2}\sup\{d_Y((hg_1)^{-1}\cdot y, (hg_2)^{-1}\cdot y) : y \in Y\} \\
&= \frac{1}{2}d_G(g_1, g_2) + \frac{1}{2}\sup\{d_Y(g_1^{-1}\cdot(h^{-1}\cdot y), g_2^{-1}\cdot(h^{-1}\cdot y)) : y \in Y\} \\
&= \frac{1}{2}d_G(g_1, g_2) + \frac{1}{2}\sup\{d_Y(g_1^{-1}\cdot y, g_2^{-1}\cdot y) : y \in Y\} \\
&= d_G(g_1, g_2).
\end{aligned}$$

We now show that a sequence $\langle g_n \rangle_{n\in\mathbb{N}}$ converges with respect to d'_G if and only if it converges with respect to d_G, which shows that the identity map on G is a homeomorphism. This will give that d'_G is compatible with the original topology of G, and also show that d'_G is complete if and only if d_G is complete. Since $d'_G \geq \frac{1}{2}d_G$, any d'_G-convergent sequence is d_G-convergent and so we need only check the converse. By left-invariance, we only need to check this for sequences converging to 1_G.

Let $\langle g_n \rangle_{n\in\mathbb{N}} \to 1_G$ with respect to d_G. Since the action of G on Y is continuous from $G \times Y \to Y$, we have

$$\forall y_0 \forall \epsilon \exists \delta \forall g \forall y \ \left[d_G \times d_Y((g, y), (1_G, y_0)) < \delta \Longrightarrow d_Y(g\cdot y, y_0) < \epsilon\right],$$

where $d_G \times d_Y$ is the usual product metric on $G \times Y$. But Y is compact, so in fact we get

$$\forall \epsilon \exists \delta \forall g \ \left[d_G(g, 1_G) < \delta \Longrightarrow \forall y \ (d_Y(g\cdot y, y) < \epsilon)\right].$$

Thus, as $\langle g_n \rangle \to 1_G$ in d_G, we have $\sup\{d_Y(g_n^{-1}\cdot y, y) : y \in Y\} \to 0$, which shows that the sequence will converge with respect to d'_G. □

Note that d'_G now has the property that for any $z \in Y$ and $g_1, g_2 \in G$, we have

$$d'_G(g_1, g_2) \geq \frac{1}{2}d_Y(g_1^{-1}\cdot z, g_2^{-1}\cdot z). \tag{*}$$

Now fix a countable dense subset of Y, enumerated as $\{y_n\}_{n\in\mathbb{Z}}$. Let $\widehat{G}$ be the completion of G in d'_G. Note that $\widehat{G}$ is no longer a Polish group if d_G is not complete, but it is a Polish space. Thus, since G is a Polish subspace of $\widehat{G}$ in the relative topology, we have that G is a dense G_δ subset of $\widehat{G}$, and hence comeager in $\widehat{G}$.

Defining (X_z, d_z)

Fix $z \in Y$. We will now define the Polish metric space (X_z, d_z). We will let X_z have the set $\widehat{G} \times \mathbb{Z} \times \mathbb{Z}_2$ as its underlying set, where $\mathbb{Z}_2$ is the cyclic group $\{0, 1\}$. We will define d_z on the dense subset $G \times \mathbb{Z} \times \mathbb{Z}_2$. If we were to fix a countable dense set $G_0 \subseteq G$ then in fact $G_0 \times \mathbb{Z} \times \mathbb{Z}_2$ would be a countable dense subset of X_z, but it will be simpler to define d_z on the given dense subset. So let

$$X_z = \{x_{\hat{g}}^{i,n}\}_{\hat{g} \in \widehat{G}, n \in \mathbb{Z}, i \in \{0,1\}}.$$

Let $\pi : \mathbb{Z} \leftrightarrow \mathbb{N}$ be the bijection given by

$$\pi(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -1 - 2n & \text{if } n < 0. \end{cases}$$

Definition 8. For $g_0, g_1 \in G$, $n_0, n_1 \in \mathbb{Z}$, and $i_0, i_1 \in \{0, 1\}$, let

$$d_z(x_{g_0}^{i_0,n_0}, x_{g_1}^{i_1,n_1}) =$$
$$\begin{cases} d'_G(g_0, g_1) & \text{if } i_0 = i_1 \text{ and } n_0 = n_1 \\ \frac{3}{2} + 4^{-|n_0 - n_1|}\,[1 + d'_G(g_0, g_1)] & \text{if } i_0 = i_1 \text{ and } n_0 \neq n_1 \\ 1 + 4^{-1-\pi(n_0 - n_1)}\left[1 + d_Y(y_{(n_0 - n_1)}, g_1^{-1} \cdot z)\right] & \text{if } i_0 = 0 \text{ and } i_1 = 1 \\ 1 + 4^{-1-\pi(n_1 - n_0)}\left[1 + d_Y(y_{(n_1 - n_0)}, g_0^{-1} \cdot z)\right] & \text{if } i_0 = 1 \text{ and } i_1 = 0. \end{cases}$$

Our space thus consists of two $\mathbb{Z}$-chains of isometric copies of $\widehat{G}$. Distances between two points in distinct copies of $\widehat{G}$ are bounded away from 0, and the metric is set up to establish some rigidity in the two chains. Distances between points in different chains are then used to encode the orbit of z under the G-action on Y.

Lemma 9. *The above definition of d_z gives a metric on $G \times \mathbb{Z} \times \mathbb{Z}_2$ whose completion is a metric on X_z.*

Proof: The definition is symmetric, so we need only check the triangle inequality. If we have three points which are all in the same G-block (i.e., the same copy of G), then d_z behaves like d'_G, so we are fine. If all three points are in distinct blocks, then each distance is in the interval $[1, 2]$, and any such triple satisfies the triangle inequality. So we need only consider the case where two points are in the same G-block, and the third is in a distinct block. Let the three points be $x_g^{i,n}$, $x_{h_1}^{j,m}$, and $x_{h_2}^{j,m}$, where $(i, n) \neq (j, m)$.

Set:

$$\delta_0 = d_z(x^{j,m}_{h_1}, x^{j,m}_{h_2})$$
$$\delta_1 = d_z(x^{i,n}_{g}, x^{j,m}_{h_1})$$
$$\delta_2 = d_z(x^{i,n}_{g}, x^{j,m}_{h_2}).$$

Note that $\delta_0 = d'_G(h_1, h_2)$ is necessarily the shortest of these distances, so it will suffice to show that $|\delta_1 - \delta_2| \leq \delta_0$. We have three cases:

1. If $i = j$, then $n \neq m$, so δ_1 and δ_2 are both defined by the second clause of the definition of d_z, and we have:

$$\begin{aligned} |\delta_1 - \delta_2| &= 4^{-|n-m|} \left| d'_G(g, h_1) - d'_G(g, h_2) \right| \\ &\leq \frac{1}{4} d'_G(h_1, h_2) \quad \text{(since } d'_G \text{ is a metric)} \\ &\leq d'_G(h_1, h_2) = \delta_0. \end{aligned}$$

2. If $i = 0$ and $j = 1$ then δ_1 and δ_2 are both defined by the third clause, and we get:

$$\begin{aligned} |\delta_1 - \delta_2| &= 4^{-1-\pi(n-m)} \left| d_Y(y_{(n-m)}, h_1^{-1} \cdot z) - d_Y(y_{(n-m)}, h_2^{-1} \cdot z) \right| \\ &\leq 4^{-1-\pi(n-m)} d_Y(h_1^{-1} \cdot z, h_2^{-1} \cdot z) \\ &\leq \frac{1}{4} d_Y(h_1^{-1} \cdot z, h_2^{-1} \cdot z) \\ &\leq \frac{1}{2} d'_G(h_1, h_2) \quad \text{(by } (*)) \\ &\leq d'_G(h_1, h_2) = \delta_0. \end{aligned}$$

3. If $i = 1$ and $j = 0$ then δ_1 and δ_2 are defined by the fourth clause and we get:

$$\delta_1 = 4^{-1-\pi(m-n)} \left[1 + d_Y(y_{(m-n)}, g^{-1} \cdot z) \right] = \delta_2,$$

   so this case is fine as well.

Finally, we note that any d_z-Cauchy sequence must eventually be contained entirely within a single G-block, and is hence Cauchy with respect to d'_G. Since the completion of G under d'_G is $\widehat{G}$, we get that the completion of $G \times \mathbb{Z} \times \mathbb{Z}_2$ under d_z is $\widehat{G} \times \mathbb{Z} \times \mathbb{Z}_2 = X_z$. □

Now we check that the map $z \mapsto (X_z, d_z)$ is a reduction of E^Y_G to $\cong_i$.

Lemma 10. *If* $z_1 \, E_G^Y \, z_2$ *then* $(X_{z_1}, d_{z_1}) \cong_i (X_{z_2}, d_{z_2})$.

Proof: Let $h \in G$ witness that $z_1 \, E_G^Y \, z_2$, i.e., $z_2 = h \cdot z_1$. Consider the map $m_h : g \mapsto hg$. This is an isometry of (G, d'_G) since d'_G is left-invariant. Moreover, since G is dense in $\widehat{G}$ this map extends uniquely to an isometry of $(\widehat{G}, d'_G)$. We now define $f : X_{z_1} \to X_{z_2}$. We will use $x_g^{i,n}$ to denote a point is X_{z_1} and $y_g^{i,n}$ to denote a point in X_{z_2}. It will suffice to define f on $G \times \mathbb{Z} \times \mathbb{Z}_2$ since this is dense and our function will be continuous. So let

$$\begin{aligned} f(x_g^{0,n}) &= y_g^{0,n} \\ f(x_g^{1,n}) &= y_{hg}^{1,n}. \end{aligned}$$

Since f restricted to a given G-block is either the identity or the map m_h, this is a continuous bijection. We check that it is an isometry. Let $x_1 = x_{g_1}^{i_1,n_1}$ and $x_2 = x_{g_2}^{i_2,n_2}$. Note that the first two cases in the definition of d_z are independent of z, so if $i_1 = i_2$ then isometry will be guaranteed by the left-invariance of d'_G. So we may restrict attention to the case where $i_1 \neq i_2$; without loss of generality we may assume that $i_1 = 0$ and $i_2 = 1$. Then let $y_1 = f(x_1) = y_{g_1}^{0,n_1}$ and $y_2 = f(x_2) = y_{hg_2}^{1,n_2}$. We compute:

$$\begin{aligned} d_{z_2}(y_1, y_2) &= d_{z_2}(y_{g_1}^{0,n_1}, y_{hg_2}^{1,n_1}) \\ &= 1 + 4^{-1-\pi(n_1-n_2)} \left[1 + d_Y(y_{(n_1-n_2)}, (hg_2)^{-1} \cdot z_2)\right] \\ &= 1 + 4^{-1-\pi(n_1-n_2)} \left[1 + d_Y(y_{(n_1-n_2)}, g_2^{-1} \cdot h^{-1} \cdot z_2)\right] \\ &= 1 + 4^{-1-\pi(n_1-n_2)} \left[1 + d_Y(y_{(n_1-n_2)}, g_2^{-1} \cdot z_1)\right] \\ &= d_{z_1}(x_{g_1}^{0,n_1}, x_{g_2}^{1,n_2}) = d_{z_1}(x_1, x_2). \end{aligned}$$

Thus f is the required isometry from (X_{z_1}, d_{z_1}) to (X_{z_2}, d_{z_2}). □

Lemma 11. *If* $(X_{z_1}, d_{z_1}) \cong_i (X_{z_2}, d_{z_2})$ *then* $z_1 \, E_G^Y \, z_2$.

Proof: Suppose $f : X_{z_1} \to X_{z_2}$ is an isometry. We again use $x_g^{i,n}$ to denote a point in X_{z_1} and $y_g^{i,n}$ to denote a point in X_{z_2} (where g ranges over $\widehat{G}$ here). Note that since distances within $\widehat{G}$-blocks are at most 1, and distances between elements in distinct blocks are greater than 1, the map f must carry each block to another block, i.e., if $f(x_{g_1}^{i,n}) = y_{h_1}^{j_1,m_1}$ and $f(x_{g_2}^{i,n}) = y_{h_2}^{j_2,m_2}$, then $j_1 = j_2$ and $m_1 = m_2$. Moreover, f must respect the ordering of the chains as follows. Note that the cases in the definition produce distances in the intervals: $[0, 1]$, $[\frac{3}{2} + 4^{-|n_0-n_1|}, \frac{3}{2} + 2 \cdot 4^{-|n_0-n_1|}]$, or $[1 + 4^{-1-\pi(n_0-n_1)}, 1 + 2 \cdot 4^{-1-\pi(n_0-n_1)}]$. These intervals are disjoint, so f must send neighboring blocks to neighboring blocks and hence must send each $\mathbb{Z}$-chain in X_{z_1} to one of the two $\mathbb{Z}$-chains in X_{z_2}, although the order may be reversed and the two chains may be interchanged. The chains must also move in the same way, in the sense that if $x_g^{0,n} \mapsto y_h^{i,m}$, then also $x_g^{1,n} \mapsto y_{h'}^{1-i,m}$.

Suppose f maps $\widehat{G} \times \{0\} \times \{0\}$ to $\widehat{G} \times \{j_0\} \times \{m_0\}$ for some $j_0 in \{0,1\}$ and $m_0 \in \mathbb{Z}$, and thus induces an isometry $\hat{f}_0 : \widehat{G} \to \widehat{G}$ given by $f(x_g^{0,0}) = y_{\hat{f}_0(g)}^{j_0,m_0}$. Since G is comeager in $\widehat{G}$, the inverse image $\hat{f}_0^{-1}[G]$ is also comeager in $\widehat{G}$ so we have $G \cap \hat{f}_0^{-1}[G] \neq \emptyset$. Hence there are elements $g_0, h_0 \in G$ such that $\hat{f}_0(g_0) = h_0$. Thus

$$f(x_{g_0}^{0,0}) = y_{h_0}^{j_0,m_0}.$$

By the same argument, we can find $j_1 \in \{0,1\}$, $m_1 \in \mathbb{Z}$, and $g_1, h_1 \in G$ such that

$$f(x_{g_1}^{1,0}) = y_{h_1}^{j_1,m_1}.$$

Notice that we must have $j_1 = 1 - j_0$ and $m_0 = m_1$. Also, we must have either

$$f(x_{g_0}^{0,n}) = y_{h_0}^{j_0,m_0+n} \text{ and } f(x_{g_1}^{1,n}) = y_{h_1}^{1-j_0,m_0+n} \text{ for all } n$$

or

$$f(x_{g_0}^{0,n}) = y_{h_0}^{j_0,m_0-n} \text{ and } f(x_{g_1}^{1,n}) = y_{h_1}^{1-j_0,m_0-n} \text{ for all } n.$$

By considering, say, the blocks $\widehat{G} \times \{0\} \times \{0\}$ and $\widehat{G} \times \{1\} \times \{0\}$, we see that the first case must hold if $j_0 = 0$, and the second case must hold if $j_0 = 1$. That is, if the $\mathbb{Z}$-chains are interchanged, then their ordering is flipped. We can summarize this as follows, where $\oplus$ denotes addition in $\mathbb{Z}_2$:

$$(\forall n \in \mathbb{Z})(\forall i \in \mathbb{Z}_2) \ \left[f(x_{g_i}^{i,n}) = y_{h_i}^{j_0 \oplus i, m_0 + (-1)^{j_0} \cdot n} \right].$$

Note that this suggests the semi-direct product $\mathbb{Z} \ltimes \mathbb{Z}_2$ given by the action $j \cdot n = (-1)^j \cdot n$ of $\mathbb{Z}_2$ on $\mathbb{Z}$; we will use this fact in the next section. Now, for each $n \in \mathbb{Z}$ we must have

$$d_{z_2}(f(x_{g_0}^{0,n}), f(x_{g_1}^{1,0})) = d_{z_1}(x_{g_0}^{0,n}, x_{g_1}^{1,0}).$$

From the definition of X_z we have:

$$d_{z_1}(x_{g_0}^{0,n}, x_{g_1}^{1,0}) = 1 + 4^{-1-\pi(n)} \left[1 + d_Y(y_n, g_1^{-1} \cdot z_1) \right].$$

Because of the order flipping in the case $j_0 = 1$, we also have:

$$d_{z_2}(f(x^{0,n}_{g_0}), f(x^{1,0}_{g_1})) = 1 + 4^{-1-\pi(n)} \left[1 + d_Y(y_n, h^{-1}_{1-j_0} \cdot z_2)\right].$$

But then, setting $h_2 = h_{1-j_0}$, we have

$$(\forall n \in \mathbb{Z}) \; \left[d_Y(y_n, g_1^{-1} \cdot z_1) = d_Y(y_n, h_2^{-1} \cdot z_2)\right].$$

Then, since the y_n's are dense in Y, for any $\epsilon > 0$ we can find an n with $d_Y(y_n, g_1^{-1} \cdot z_1) < \frac{\epsilon}{2}$, so also $d_Y(y_n, h_2^{-1} \cdot z_2) < \frac{\epsilon}{2}$, giving $d_Y(g_1^{-1} \cdot z_1, h_2^{-1} \cdot z_2) < \epsilon$. We thus have

$$g_1^{-1} \cdot z_1 = h_2^{-1} \cdot z_2.$$

So, letting $g_2 = h_2 g_1^{-1}$, we have $z_2 = g_2 \cdot z_1$, so that $z_1 \, E^Y_G \, z_2$, which was our goal. □

Thus, $z \mapsto (X_z, d_z)$ is a reduction as desired. We last check that this map is Borel, i.e., that we can produce codes for these spaces in a Borel way.

Lemma 12. *The map $z \mapsto (X_z, d_z)$ is Borel-measurable as a map from Y to $\mathcal{M}$.*

Proof: Fix a countable dense subset G_0 of G (so that G_0 is also a dense subset of $\widehat{G}$), with

$$G_0 = \{g_k : k \in \mathbb{N}\},$$

and fix a bijection $(i, n, k) \mapsto \langle i, n, k \rangle$ of $\mathbb{N}^3$ with $\mathbb{N}$. We code (X_z, d_z) as an array in $\mathcal{M}$ by sending $z \mapsto \langle d_{i,j} \rangle_{i,j \in \mathbb{N}}$, where

$$d_{\langle i,n.k\rangle,\langle j,m,l\rangle} = d_z(x^{i,n}_{g_k}, x^{j,m}_{g_l}).$$

If we have a convergent sequence $\langle z_i \rangle_{i \in \mathbb{N}} \to z$, then we have that $\langle g_k \cdot z_i \rangle_{i \in \mathbb{N}} \to g_k \cdot z$ for each k by the continuity of the action, and so $\langle d_Y(y_n, g_k \cdot z_i) \rangle_{i \in \mathbb{N}} \to d_Y(y_n, g_k \cdot z)$ for each n and k. Thus the codes for the (X_{z_i}, d_{z_i})'s will approach the code for (X_z, d_z). This shows that the map is in fact a continuous reduction. □

We have thus proved Theorem 5. Notice that for a given Polish G-space E^X_G, all of the metric spaces produced by the embedding are homeomorphic to $\widehat{G} \times \mathbb{Z} \times \mathbb{Z}_2$. Non-isometry was established by producing different distance sets. We needed to use two $\mathbb{Z}$-chains in order to encode a point $z \in Y$ via its distances from the points y_n; if we had an action on a space Y which can be embedded continuously into $\mathbb{R}$, we could have done the construction using only two copies of $\widehat{G}$ by encoding z directly into the metric, rather than encoding z's distances from the y_n's.

3 Isometry restricted to special classes of spaces

We will use the construction from the previous section to get lower bounds on the complexity of isometry restricted to various special classes of metric spaces. Several other special cases are considered in [8]; for an overview of these and other cases see [5]. Two types of lower bounds will be of interest. First, we will be able to show that isometry restricted to certain classes of spaces is not classifiable by countable structures by embedding turbulent group actions An equivalence relation is *classifiable by countable structures* if it is reducible to the isomorphism relation on some collection of countable first-order structures (such as the isomorphism relation among countable graphs). Turbulent actions and classification by countable structures are defined in [10], which shows that any equivalence relation which reduces a turbulent orbit equivalence relation is not classifiable by countable structures.

We will also show that isometry of some other classes of metric spaces is not concretely classifiable by embedding the equivalence relation E_0. The relation E_0 is defined on the space $2^{\mathbb{N}}$ by setting $x \, E_0 \, y$ iff x and y agree on all but finitely many co-ordinates. This relation is generated by the ideal I_{FIN} of finite subsets of $\mathbb{N}$ acting by symmetric difference Δ on $\mathcal{P}(\mathbb{N})$ (which we identify with $2^{\mathbb{N}}$). This relation is not concretely classifiable, nor is any equivalence relation to which it is reducible (see, e.g., [6]).

In their paper [8], Gao and Kechris show that isometry of discrete Polish metric spaces, ultrametric Polish metric spaces, and zero-dimensional locally compact Polish metric spaces are all bireducible with graph isomorphism. They ask about general zero-dimensional spaces, and homogeneous locally compact spaces. We will be able to show that isometry of zero-dimensional spaces is strictly more complex than graph isomorphism. In a subsequent article ([3]) we will analyze isomorphism of classes of structures with large automorphism groups and use that analysis to show that isometry of homogeneous locally compact spaces is at least as complicated as graph isomorphism.

The main idea used here is that the spaces we constructed above for a given G-action have many of the topological properties of the group G. First, note that in the case that G has a complete left-invariant metric (for instance, if G is locally compact or abelian), then the metric space constructed will have the topology of the Polish group $G \times \mathbb{Z} \times \mathbb{Z}_2$. We thus get:

Theorem 13. *Let E_G^X be a Borel G-space, where G has a complete left-*

invariant metric. Then $E_G^X \leq_B \cong_i$ restricted to Polish group topologies.

Noting that there are abelian Polish groups with turbulent actions (for instance the density ideal (I_d, Δ) discussed below), we get:

Corollary 14. *Isometry of Polish metric spaces with Polish group metrics is not classifiable by countable structures.*

The metrics constructed in this case will not in general be left-invariant (although this group will have a complete left-invariant metric). We will consider invariant metrics below.

Suppose, in addition, that G is a zero-dimensional Polish group. Then the resulting space will also be zero-dimensional, so we have:

Theorem 15. *Let E_G^X be a Borel G-space, where G is zero-dimensional and has a complete left-invariant metric. Then $E_G^X \leq_B \cong_i$ restricted to zero-dimensional Polish group topologies.*

Again taking $G = (I_d, \Delta)$, which is zero-dimensional (see below), we get:

Corollary 16. *Isometry of zero-dimensional Polish metric spaces is not classifiable by countable structures.*

IN the final two sections we will consider metric spaces with rich isometry groups. Recall that a metric space is said to be *homogeneous* if its isometry group acts transitively, i.e., for any two points x and y there is an isometry carrying x to y. A stronger property is the following:

Definition 17. A metric space is *ultra-homogeneous* if any partial isometry between finite subsets of it can be extended to an isometry of the whole space.

This is equivalent to saying that if $\langle x_i \rangle_{i \leq n}$ and $\langle y_i \rangle_{i \leq n}$ are two finite sets of points such that for all $i, j \leq n$ we have $d(x_i, x_j) = d(y_i, y_j)$, then there is an isometry f of the space such that $f(x_i) = y_i$ for all $i \leq n$. Such spaces are determined up to isometry by their n-point distance configurations mentioned earlier.

One way to establish homogeneity is to produce a Polish group with a left-invariant metric, for then left-multiplication by any group element will be an isometry. Thus, given any g_1 and g_2 in the group, left multiplication by $g_2 g_1^{-1}$ will be an isometry sending g_1 to g_2. We will first modify our construction to produce (for certain groups) metric spaces which are in fact Polish groups with left-invariant and two-sided invariant metrics. Producing ultra-homogeneous spaces will require further modification and restrictions on G.

4 Producing invariant metrics

We begin by considering abelian Polish groups. Any abelian Polish group has a complete two-sided invariant metric (in fact, any compatible left-invariant metric will necessarily be two-sided invariant and will be complete). Some of what follows should also be applicable to groups with complete invariant metrics, but we seem to actually require commutativity for part of the argument. Fix a Polish G-space E_G^Y as before, where again we may assume Y is compact, d_G is an invariant metric for G with $d_G \leq 1$, and d_Y is a metric on Y with $d_Y \leq 1$. Let d'_G again be defined as in Definition 6. Fix a point $z \in Y$. We will now define a Polish metric space (X_z, d_z) such that X_z is in fact the Polish group $(G \times \mathbb{Z}) \ltimes \mathbb{Z}_2$, the semi-direct product given by the action of $\mathbb{Z}_2$ on $G \times \mathbb{Z}$ where $1 \cdot (g, n) = (g^{-1}, -n)$, and such that the metric d_z is a complete, compatible left-invariant metric on $(G \times \mathbb{Z}) \ltimes \mathbb{Z}_2$. This will guarantee that the metric space produced is homogeneous.

As before, X_z will have the underlying set

$$x_z = \{x_g^{i,n} : g \in G, i \in \mathbb{Z}_2, n \in \mathbb{Z}\}.$$

Definition 18. For $g_0, g_1 \in G$, $n_0, n_1 \in \mathbb{Z}$, and $i_0, i_1 \in \{0, 1\}$, set:

$$d_z(x_{g_0}^{i_0,n_0}, x_{g_1}^{i_1,n_1}) =$$
$$\begin{cases} d'_G(g_0, g_1) & \text{if } i_0 = i_1 \text{ and } n_0 = n_1 \\ \frac{3}{2} + 4^{-|n_0 - n_1|}\left[1 + d'_G(g_0, g_1)\right] & \text{if } i_0 = i_1 \text{ and } n_0 \neq n_1 \\ 1 + 4^{-1-\pi(n_0 - n_1)}\left[1 + d_Y(y_{(n_0 - n_1)}, (g_0 g_1^{-1}) \cdot z)\right] & \text{if } i_0 = 0 \text{ and } i_1 = 1 \\ 1 + 4^{-1-\pi(n_1 - n_0)}\left[1 + d_Y(y_{(n_1 - n_0)}, (g_1 g_0^{-1}) \cdot z)\right] & \text{if } i_0 = 1 \text{ and } i_1 = 0. \end{cases}$$

Verifying that this is a metric is similar to the proof of Lemma 9, noting that for invariant metrics we have $d_G(g_1^{-1}, g_2^{-1}) = d_G(g_1, g_2)$. Verifying that the map $z \mapsto (X_z, d_z)$ is an embedding of E_G^Y is also the same as before. The difference now is in verifying the left-invariance of the metric. Consider the group $(G \times \mathbb{Z}) \ltimes \mathbb{Z}_2$ where multiplication is given by

$$(g, n, i) \cdot (h, m, j) = (g \cdot h^{(-1)^i}, n + (-1)^i \cdot m, i \oplus j).$$

We will check that d_z is left-invariant under this multiplication.

Lemma 19. *Let (X_z, d_z) be as above. Then for all $h, g_1, g_2 \in G$, for all $m, n_1, n_2 \in \mathbb{Z}$, and for all $j, i_1, i_2 \in \mathbb{Z}_2$ we have*

$$d_z\left(x_{hg_1^{(-1)^j}}^{j\oplus i_1, m+(-1)^j \cdot n_1}, x_{hg_2^{(-1)^j}}^{j\oplus i_2, m+(-1)^j \cdot n_2}\right) = d_z\left(x_{g_1}^{i_1,n_1}, x_{g_2}^{i_2,n_2}\right),$$

i.e., d_z is a left-invariant metric on $X_z = (G \times \mathbb{Z}) \ltimes \mathbb{Z}_2$.

Proof: For simplicity, let

$$(\tilde{g}_k, \tilde{n}_k, \tilde{\imath}_k) = (hg_k^{(-1)^j}, m + (-1)^j \cdot n_k, j \oplus i_k)$$

for $k = 1, 2$. We consider three possible cases in the definition of d_z.

1. If $i_1 = i_2$ and $n_1 = n_2$, then $\tilde{\imath}_1 = \tilde{\imath}_2$ and $\tilde{n}_1 = \tilde{n}_2$ and so

$$\begin{aligned} d_z(x_{\tilde{g}_1}^{\tilde{\imath}_1,\tilde{n}_1}, x_{\tilde{g}_2}^{\tilde{\imath}_2,\tilde{n}_2}) &= d'_G(hg_1^{(-1)^j}, hg_2^{(-1)^j}) \\ &= d'_G(g_1^{(-1)^j}, g_2^{(-1)^j}) \\ &= d'_G(g_1, g_2) = d_z(x_{g_1}^{i_1,n_1}, x_{g_2}^{i_2,n_2}). \end{aligned}$$

2. If $i_1 = i_2$ and $n_1 \neq n_2$, then we have $\tilde{\imath}_1 = \tilde{\imath}_2$ and $|\tilde{n}_1 - \tilde{n}_2| = |n_1 - n_2|$. So we get $d_z(x_{\tilde{g}_1}^{\tilde{\imath}_1,\tilde{n}_1}, x_{\tilde{g}_2}^{\tilde{\imath}_2,\tilde{n}_2}) = d_z(x_{g_1}^{i_1,i_2}, x_{g_2}^{i_2,n_2})$ as in the previous case.

3. If $i_1 \neq i_2$ then $\tilde{\imath}_1 \neq \tilde{\imath}_2$. We may assume $i_0 = 0$. There are two sub-cases:

   (a) If $j = 0$ then we have $\tilde{n}_1 - \tilde{n}_2 = n_1 - n_2$, so we get

$$\begin{aligned} d_z(x_{\tilde{g}_1}^{\tilde{\imath}_1,\tilde{n}_1}, x_{\tilde{g}_2}^{\tilde{\imath}_2,\tilde{n}_2}) &= 1 + 4^{-1-\pi(\tilde{n}_1-\tilde{n}_2)} \left[1 + d_Y(y_{\tilde{n}_1-\tilde{n}_2}, \tilde{g}_1\tilde{g}_2^{-1} \cdot z)\right] \\ &= 1 + 4^{-1-\pi(n_1-n_2)} \left[1 + d_Y(y_{n_1-n_2}, hg_1g_2^{-1}h^{-1} \cdot z)\right] \\ &= 1 + 4^{-1-\pi(n_1-n_2)} \left[1 + d_Y(y_{n_1-n_2}, g_1g_2^{-1} \cdot z)\right] \\ &= d_z(x_{g_1}^{i_1,n_1}, x_{g_2}^{i_2,n_2}). \end{aligned}$$

   (b) If $j = 1$ then $\tilde{n}_2 - \tilde{n}_1 = n_1 - n_2$, so we get

$$\begin{aligned} d_z(x_{\tilde{g}_1}^{\tilde{\imath}_1,\tilde{n}_1}, x_{\tilde{g}_2}^{\tilde{\imath}_2,\tilde{n}_2}) &= 1 + 4^{-1-\pi(\tilde{n}_2-\tilde{n}_1)} \left[1 + d_Y(y_{\tilde{n}_2-\tilde{n}_1}, \tilde{g}_2\tilde{g}_1^{-1} \cdot z)\right] \\ &= 1 + 4^{-1-\pi(n_1-n_2)} \left[1 + d_Y(y_{n_1-n_2}, hg_2^{-1}g_1h^{-1} \cdot z)\right] \\ &= 1 + 4^{-1-\pi(n_1-n_2)} \left[1 + d_Y(y_{n_1-n_2}, g_1g_2^{-1} \cdot z)\right] \\ &= d_z(x_{g_1}^{i_1,n_1}, x_{g_2}^{i_2,n_2}). \end{aligned}$$

Thus we get left-invariance in all three cases. □

As before, the embedding is clearly Borel, and so we have:

Theorem 20. *Let G be an abelian Polish group and let E_G^X be a Borel G-space. Then we have $E_G^X \leq_B \cong_i$ restricted to left-invariant metrics for Polish groups.*

Taking $G = (I_d, \Delta)$ again, we get:

Corollary 21. *Isometry of homogeneous Polish metric spaces is not classifiable by countable structures.*

Also, if we take $G = (I_{\mathrm{FIN}}, \Delta)$, which generates the equivalence relation E_0, and note that the spaces produced are discrete (and hence locally compact), we get:

Corollary 22. *Isometry of homogeneous discrete Polish metric spaces (and hence of homogeneous locally compact Polish metric spaces) is not concretely classifiable.*

In the article [3] we will improve this result to show that isometry of homogeneous discrete Polish metric spaces is bireducible with graph isomorphism.

This result should be contrasted with Corollary 5.8 of [8], that the isometry of *pseudo-connected* homogeneous locally compact Polish metric spaces is concretely classifiable. See [8] for the definitions of pseudo-connected spaces and pseudo-connected components of locally compact Polish spaces. We can strengthen this contrast. Let E_0 be represented as the orbit equivalence relation of a $\mathbb{Z}$-action (we can, for instance, essentially use the odometer map on $2^{\mathbb{N}}$), and let $G = \mathbb{Z}$ be given the discrete metric $d_G(n,m) = \frac{1}{2}\left[1 + \frac{|n-m|}{1+|n-m|}\right]$ for $n \neq m$. Then (G, d_G) will be pseudo-connected. If we now let our space (X_z, d_Z) consist of two copies of $\mathbb{Z}$,

$$X_z = \{x_n^i : i \in \{0,1\}, n \in \mathbb{Z}\},$$

and define

$$\begin{aligned} d_z(x_n^i, x_m^i) &= d_G(n,m) \\ d_z(x_n^0, x_m^1) &= 1 + \varphi((n-m) \cdot z), \end{aligned}$$

where $\varphi(y) = \sum_{k \in \mathbb{N}} \frac{y(k)}{2^{k+2}}$, then the resulting space will be homogeneous locally compact with two pseudo-connected components. A similar argument to that above shows that this is a reduction of E_0, so we thus get:

Corollary 23. *The isometry of homogeneous locally compact Polish metric spaces with two pseudo-connected components is not concretely classifiable.*

So Corollary 5.8, and hence Theorem 5.7, of [8] cannot be extended to the case of finitely many pseudo-connected components. The above construction can easily be modified to show that any countable abelian group action can be reduced to isometry of homogeneous discrete spaces with two pseudo-connected components. Louveau (unpublished) has been able to show this for ultra-homogeneous spaces as well:

Theorem (Louveau). *Any orbit equivalence relation of a countable abelian group action is reducible to isometry of ultra-homogeneous locally compact spaces with two pseudo-connected components.*

Here we have only been able to give lower bounds on the complexity in the given cases; the exact classification remains open. So we can restate two of the questions from [8]:

Question 24. *What is the exact complexity of isometry restricted to zero-dimensional Polish metric spaces?*

Based on the distance sets considered in [2], where it is shown that the distance set of a zero-dimensional Polish metric space may be as complicated as that of an arbitrary Polish metric space, it seems plausible to conjecture that isometry of zero-dimensional Polish metric spaces is as complicated as the isometry of arbitrary Polish metric spaces.

Question 25. *What is the exact complexity of the isometry of homogeneous locally compact Polish metric spaces?*

As noted, we will show in [3] that graph isomorphism is a lower bound here, and we suspect that this is also an upper bound.

Towards the solution of Question 24, we can ask how complicated the orbit equivalence relation induced by an action of a zero-dimensional Polish group may be. There are several known examples of universal Polish group actions, but none of these is given by the action of a zero-dimensional group. So we may ask:

Question 26. *Is there a universal Polish group action given by the action of a zero-dimensional Polish group?*

There are two improvements to the above techniques that we we will briefly sketch. First, if G is a group in which every element has order 2, then we can produce two-sided invariant metrics for the group $G \times \mathbb{Z} \times \mathbb{Z}_2$ by replacing "$n_0 - n_1$" with "$|n_0 - n_1|$" throughout the definition of d_z. Second, if Y is a space which embeds continuously in $[0, 1]$, then we can produce metrics for the space $G \times \mathbb{Z}_2$. The representative case here is the action of (I, Δ) on $2^{\mathbb{N}}$, where I is a Polishable ideal. From this we can get:

Corollary 27. *If G is a Polish group in which every element has order 2, then any equivalence relation E_G^X is reducible to isometry of invariant metrics for abelian Polish groups.*

Further slight modifications can be used to apply to other classes of metric spaces which we will not list here. We now turn to the case of ultra-homogeneous spaces.

5 Producing ultra-homogeneous spaces

Recall the definition of ultra-homogeneous spaces given in Section 3. One might suspect that these spaces have a relatively simple classification because of their strong uniformity. In a certain sense they do: Two ultra-homogeneous spaces are isometric if and only if they have the same set of n-point distance configurations for all $n \geq 2$. However, these sets are (potentially) quite complicated analytic sets, so they do not provide nice invariants from a descriptive set-theoretic point of view. We will in fact be able to show that the classification is complicated, in that it can not be classified by countable structures.

An ultra-homogeneous space is clearly homogeneous, so we will attempt to modify the above construction which produced homogeneous spaces. Notice that in that construction two isometric configurations of points must agree on the relative ordering of their G-blocks, so the chief difficulty will lie with G itself. For the spaces produced above to be ultra-homogeneous, G itself must be ultra-homogeneous (with respect to d'_G). So our first task will be to produce an ultra-homogeneous metric for a sufficiently complicated Polish group.

We will focus on the *density ideal* I_d which we have already used above. As the action of I_d on $2^{\mathbb{N}}$ via symmetric difference is turbulent, it will suffice to reduce this action to isometry of ultra-homogeneous Polish metric spaces in order to rule out classifiability by countable structures. This is a free action, which will be crucial to our method. Recall that:

$$I_d = \left\{ x \subseteq \mathbb{N} : \lim_{n\to\infty} \frac{|x \cap n|}{n} = 0 \right\}.$$

This is a *Polishable* ideal, i.e., it can be given a Polish topology compatible with the Borel structure it inherits as a subset of $2^{\mathbb{N}}$ so that it is a Polish group in this topology (with group addition being symmetric difference). We will use a few facts from the theory of Polishable ideals, primarily their representation by *lower semi-continuous submeasures on* $\mathbb{N}$.

Definition 28. A *submeasure on* $\mathbb{N}$ is a map $\varphi : \mathcal{P}(\mathbb{N}) \to [0, \infty]$ such that:

1. $\varphi(\emptyset) = 0$.
2. $0 < \varphi(\{n\}) < \infty$ for all $n \in \mathbb{N}$.
3. $x \subseteq y \implies \varphi(x) \leq \varphi(y)$.
4. $\varphi(x \cup y) \leq \varphi(x) + \varphi(y)$.

A submeasure φ is said to be *lower semi-continuous (l.s.c.)* if

$$\varphi(x) = \lim_{n\to\infty} \varphi(x \cap n) \text{ for all } x.$$

Given an l.s.c. submeasure φ, we define its *exhaustive ideal*,

$$\mathrm{Exh}(\varphi) = \{x : \lim_{n\to\infty} \varphi(x \setminus n) = 0\},$$

where $x \setminus n = x \setminus \{0, \ldots, n-1\}$. A submeasure is said to be *finite* if $\varphi(x) < \infty$ for all x, and it is said to be *exhaustive* if $\mathrm{Exh}(\varphi) = \mathcal{P}(\mathbb{N})$.

We use the following representation from [14]:

Theorem (Solecki). *An ideal I is Polishable if and only if there is a finite l.s.c submeasure φ such that $I = Exh(\varphi)$. In this case, the metric d_φ given by $d_\varphi(x, y) = \varphi(x \Delta y)$ is a complete invariant metric on I compatible with its (unique) Polish topology.*

We will now define such a submeasure for I_d. We will identify subsets of $\mathbb{N}$ with their characteristic functions in $2^{\mathbb{N}}$ when this is convenient. Let $\rho : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be given by:

$$\rho(x)(i) = \begin{cases} 0 & \text{if } i = 0 \\ x(m) & \text{if } i = 2^m(2k+1) \text{ for } k, m \geq 0. \end{cases}$$

That is,

$$\rho(x) = \langle 0, x(0), x(1), x(0), x(2), x(0), x(1), x(0), x(3), \ldots \rangle.$$

The main point is that if $x \neq y$ then $\rho(x) \Delta \rho(y)$ is infinite.

Definition 29. For $x \subseteq \mathbb{N}$, let

$$\varphi(x) = \sup \left\{ \frac{|x \cap 2^n|}{2^{n+1}} : n \in \mathbb{N} \right\} + \sum_{n \in \mathbb{N}} \frac{\rho(x)(n)}{2^{n+1}}.$$

Note that $\varphi(x) \in [0,1]$ since $\rho(x)(0) = 0$.

Lemma 30. *The function φ is a l.s.c. submeasure on $\mathbb{N}$ with $I_d = Exh(\varphi)$.*

Proof: First, note that the map ρ respects Boolean operations, so that the function φ_1 defined by

$$\varphi_1(x) = \sum_{n\in\mathbb{N}} \frac{\rho(x)(n)}{2^{n+1}}$$

is an exhaustive submeasure on $\mathbb{N}$ (in fact, a measure). Thus it will suffice to show that the function φ_0 given by

$$\varphi_0(x) = \sup\left\{\frac{|x \cap 2^n|}{2^{n+1}} : n \in \mathbb{N}\right\}$$

is a l.s.c submeasure on $\mathbb{N}$ such that $I_d = \text{Exh}(\varphi_0)$. It is easily seen to be a submeasure, and it is lower semi-continuous because

$$(\forall\epsilon > 0)(\exists k)\ \left[\frac{|x \cap 2^k|}{2^{k+1}} \geq \sup\left\{\frac{|x \cap 2^n|}{2^{n+1}} : n \in \mathbb{N}\right\} - \epsilon\right].$$

To check chat $I_d = \text{Exh}(\varphi_0)$, we first note that if $x \in I_d$ then $\frac{|x\cap n|}{n} \to 0$, so if we fix an $\epsilon > 0$ then there is an N such that for all $k \geq 2^N$ we have $\frac{|x\cap k|}{2k} < \epsilon$. Consider $x\setminus k$ for $k \geq 2^N$. We have, for $m < N$, that $\frac{|(x\setminus k)\cap 2^m|}{2^{m+1}} = 0$, whereas for $m \geq N$, this quotient is less than ϵ. Taking the supremum we then have $\varphi_0(x \setminus k) < \epsilon$ whenever $k \geq 2^N$, so $x \in \text{Exh}(\varphi_0)$.

Conversely, if $x \notin I_d$ then there is an $\epsilon > 0$ for which there are infinitely many n with $\frac{|x\cap n|}{n} \geq \epsilon$. Fix such an ϵ. Now fix any m and let n_0 be large enough that $n_0 \cdot \epsilon \geq 2m$ and such that $\frac{|x\cap n_0|}{n_0} \geq \epsilon$. Let k be such that $2^{k-1} \leq n_0 < 2^k$. Then:

$$\frac{|(x \setminus m) \cap 2^k|}{2^{k+1}} \geq \frac{|(x \setminus m) \cap n_0|}{2^{k+1}} \geq \frac{n_0 \cdot \epsilon - m}{2^{k+1}} \geq \frac{n_0 \cdot \epsilon}{8n_0} \geq \frac{\epsilon}{8}.$$

Thus $\varphi_0(x \setminus m) \geq \frac{\epsilon}{8}$ for any m, so $x \notin \text{Exh}(\varphi_0)$. Hence $I_d = \text{Exh}(\varphi_0) = \text{Exh}(\varphi)$. □

Note that for $x \in I_d$ we have that $\varphi_0(x)$ defined above achieves its supremum at some value of n. This shows that φ_0 achieves only countably many values on I_d, and since this gives a metric on I_d with only countably many distances we can see that I_d is zero-dimensional.

We now show that the metric induced by φ will make I_d into an ultra-homogeneous space. First we see that φ will limit the number of isometric configurations:

Lemma 31. *If x and y are in I_d and $x \neq y$, then $\varphi(x) \neq \varphi(y)$.*

Proof: As just noted, when $x \in I_d$ we have $\frac{|x \cap 2^n|}{2^{n+1}} \to 0$, so that the supremum in the definition of φ is actually achieved by some value of m. Let $m(x)$ be the least such m, so that we have

$$\varphi(x) = \frac{|x \cap 2^{m(x)}|}{2^{m(x)+1}} + \sum_{n \in \mathbb{N}} \frac{\rho(x)(n)}{2^{n+1}}.$$

Let $\sigma(x) \in 2^{<\mathbb{N}}$ be the "binary expansion" of $\frac{|x \cap 2^{m(x)}|}{2^{m(x)+1}} \leq \frac{1}{2}$, i.e.,

$$\frac{|x \cap 2^{m(x)}|}{2^{m(x)+1}} = \sum_{n < |\sigma(x)|} \frac{\sigma(x)(n)}{2^{n+1}}.$$

Let $\tau(x)$ be defined as the co-ordinate-wise sum with left carry of $\sigma(x)$ and $\rho(x)$ (where we mean by this that the first $|\sigma(x)|$ digits of $\rho(x)$ should be listed, left-to-right, and considered as a binary number, and the same done with $\sigma(x)$, and these added. The rest of $\rho(x)$ is concatenated and left unchanged. So, e.g., $\langle 011 \rangle + \langle 010 \ldots \rangle = \langle 101 \ldots \rangle$). This is well defined since $\rho(x)(0) = 0$ always, and if $\sigma(x)(0) = 1$ then all other digits of $\sigma(x)$ are 0. We thus have:

$$\varphi(x) = \sum_{n \in \mathbb{N}} \frac{\tau(x)(n)}{2^{n+1}}.$$

Now, note that if $x, y \in I_d$ then x and y are not eventually 1, and so $\rho(x)$ and $\rho(y)$ are not eventually 1. Also, if $x \neq y$ then $\rho(x)$ and $\rho(y)$ differ on infinitely many coordinates. Since $\sigma(x)$ and $\sigma(y)$ are finite strings, they affect only finitely-many coordinates in $\tau(x)$ and $\tau(y)$, respectively, so that we will have $\tau(x)$ differing from $\tau(y)$ on infinitely many coordinates as well. In particular $\tau(x) \neq \tau(y)$ when $x \neq y$. This will then ensure (since neither of x and y is eventually 1) that $\varphi(x) \neq \varphi(y)$. □

Basically, we ensure ultra-homogeneity by simply limiting the possible repetitions of n-point configurations.

Lemma 32. *The Polish metric space (I_d, d_φ) is ultra-homogeneous, where $d_\varphi(x, y) = \varphi(x \Delta y)$.*

Proof: Suppose that $x_0, \ldots, x_n$ and $y_0, \ldots, y_n$ are points in I_d such that

$$(\forall i < j \leq n) \ (d_\varphi(x_i, x_j) = d_\varphi(y_i, y_j)).$$

Then $\varphi(x_i \Delta x_j) = \varphi(y_i \Delta y_j)$ for all $i, j \le n$. By the previous lemma, this implies that for all $i, j \le n$ we have $x_i \Delta x_j = y_i \Delta y_j$. Let $w = x_0 \Delta y_0 \in I_d$ and define $f : I_d \to I_d$ by

$$f(x) = x \Delta w.$$

This is an isometry of I_d since the metric d_φ is invariant, and we check:

$$f(x_i) = x_i \Delta w = x_i \Delta (x_o \Delta y_0) = (x_i \Delta x_0) \Delta y_0 = (y_i \Delta y_0) \Delta y_0 = y_i.$$

So we have an isometry carrying each x_i to y_i as required. $\square$

We are now ready to define the reduction. Recall that we are trying to reduce the action of I_d on $2^{\mathbb{N}}$ via symmetric difference. So we fix $z \in I_d$ and define (X_z, d_z). We will let X_z consist of two copies of I_d,

$$X_z = \left\{ x_\alpha^i : i \in \{0,1\}, \alpha \in I_d \right\},$$

and we set:

$$d_z(x_\alpha^i, x_\beta^j) = \begin{cases} \varphi(\alpha \Delta \beta) & \text{if } i = j \\ \frac{3}{2} + \sum_{n \in \mathbb{N}} \frac{\rho(z \Delta \alpha \Delta \beta)(n)}{2^{n+1}} & \text{if } i \ne j. \end{cases}$$

The proof that this defines a metric is much like before. The relationship between φ and φ_0 is that of d'_G to d_G, and the fact that ρ respects Boolean operations ensures that:

$$\begin{aligned} &\left| \sum_{n \in \mathbb{N}} \frac{\rho(z \Delta \alpha \Delta \beta_1)(n)}{2^{n+1}} - \sum_{n \in \mathbb{N}} \frac{\rho(z \Delta \alpha \Delta \beta_2)(n)}{2^{n+1}} \right| \\ &\le \sum_{n \in \mathbb{N}} \frac{|\rho(z \Delta \alpha \Delta \beta_1)(n) - \rho(z \Delta \alpha \Delta \beta_2)(n)|}{2^{n+1}} \\ &= \sum_{n \in \mathbb{N}} \frac{\left(\rho(z \Delta \alpha \Delta \beta_1) \Delta \rho(z \Delta \alpha \Delta \beta_2)\right)(n)}{2^{n+1}} \\ &= \sum_{n \in \mathbb{N}} \frac{\rho(\beta_1 \Delta \beta_2)(n)}{2^{n+1}}. \end{aligned}$$

It is also easy to see that $z_1 \not E_{I_d} z_2 \implies (X_{z_1}, d_{z_1}) \not\cong_i (X_{z_2}, d_{z_2})$ since the orbits are coded more or less directly into the set of distances via ρ. There is one slight difficulty here, namely the case of eventually constant z's, for which the function ρ is not necessarily one-to-one. However, since $I_{\mathrm{FIN}} \subseteq I_d$, these sequences fall into only two orbits which can have overlapping distance sets.

Since there are only a finite number of problematic orbits, we can simply redefine (X, d) in these two cases to be two other distinct ultra-homogeneous spaces.

We can also check that the map f given by

$$f(x^0_\alpha) = x^0_\alpha$$
$$f(x^1_\alpha) = x^1_{\alpha \Delta z_1 \Delta z_2}$$

is an isometry of (X_{z_1}, d_{z_1}) and (X_{z_2}, d_{z_2}) when $z_1 \Delta z_2 \in I_d$. Note that d_z is an invariant metric when we view X_z as the Polish group $(I_d, \Delta) \times \mathbb{Z}_2$, so the space is homogeneous. It remains to check ultra-homogeneity.

Lemma 33. *The space (X_z, d_z) constructed above is ultra-homogeneous.*

Proof: Let $x_0, \dots, x_n$ and $y_0, \dots, y_n$ be points in X_z such that for all $i < j \leq n$ we have $d_z(x_i, x_j) = d_z(y_i, y_j)$. We may assume that $x_0 = y_0$ by homogeneity. If all of the x_i's lie in the same copy of I_d, then all of the y_i's must also lie in this same copy of I_d, and in fact we must have $x_i = y_i$ for all $i \leq n$ as we saw in Lemma 32. In general, the x_i's and y_i's fall into two sets, those in the same copy of I_d as x_0 and those in the other copy. Here again the two sets in the same copy as x_0 must in fact be identical. Let $x_0 = x^j_\alpha$. For two points $x_i = x^{1-j}_{\beta_x}$ and $y_i = x^{1-j}_{\beta_y}$ in the other copy, we must have

$$\frac{3}{2} + \sum_{n \in \mathbb{N}} \frac{\rho(z \Delta \alpha \Delta \beta_x)(n)}{2^{n+1}} = \frac{3}{2} + \sum_{n \in \mathbb{N}} \frac{\rho(z \Delta \alpha \Delta \beta_y)(n)}{2^{n+1}}.$$

This will ensure that $z \Delta \alpha \Delta \beta_x = z \Delta \alpha \Delta \beta_y$ (since if one of these is eventually 1 then so is the other), and thus that $\beta_x = \beta_y$. Thus, both collections of points must be the same. Hence ultra-homogeneity is trivially satisfied. □

We have thus proved:

Theorem 34. *Isometry of ultra-homogeneous Polish metric spaces is not classifiable by countable structures.*

We do not know an upper bound, though.

Question 35. *What is the exact complexity of isometry restricted to ultra-homogeneous Polish metric spaces?*

The technique and modifications presented here doubtless admit further refinements. It would be of interest to see how sharp the lower bounds produced actually are. For instance, it would be very interesting to see if there is any connection between the topological properties of a class of metric spaces and the properties of the Polish groups whose actions can be reduced to isometry of that class of spaces. We have no ideas for what such a relationship might be, if any.

Our technique, for instance, seems inapplicable to the question of the complexity of locally compact Polish metric spaces, which is perhaps the most interesting open question in this area. As it stands, the above technique can only produce locally compact spaces in the case that G is locally compact. Since locally compact groups do not admit turbulent actions, this prevents a non-classifiability result. It also prevents us from using this technique to reduce actions of S_∞, which is a notable weakness in that we know such actions are reducible to isometry of locally compact spaces. It would certainly be reassuring to at least be able to close this gap.

References

[1] H. Becker and A.S. Kechris, **The descriptive set theory of Polish group actions**, London Mathematical Society Lecture Notes Series, Cambridge, 1996.

[2] J.D. Clemens, *The set of distances in a Polish metric space*, preprint.

[3] J.D. Clemens, *Isomorphism of homogeneous spaces*, in preparation.

[4] J.D. Clemens, *Classifying Polish metric spaces with a fixed set of distances*, in preparation.

[5] J.D. Clemens, S. Gao, and A.S. Kechris, *Polish metric spaces: Their classification and isometry groups*, **Bulletin of Symbolic Logic**, vol. 7 (2001), no. 3, pp. 361–375.

[6] R. Dougherty, S. Jackson, and A.S. Kechris, *The structure of hyperfinite Borel equivalence relations*, **Transactions of the American Mathematical Society**, vol. 341 (1994), no. 1, pp. 193–225.

[7] H. Friedman and L. Stanley, *A Borel reducibility theory for classes of countable structures*, **Journal of Symbolic Logic**, vol. 54 (1989), pp. 894–914.

[8] S. Gao and A.S. Kechris, *On the classification of Polish metric spaces up to isometry*, **Memoirs of the American Mathematical Society**, vol. 766, Amer. Math. Soc., 2003.

[9] M. Gromov, **Metric structures for Riemannian and non-Riemannian spaces**, Progress in Mathematics, vol. 152, Birkhäuser, 1999.

[10] G. Hjorth, **Classification and orbit equivalence relations**, Mathematical Surveys and Monographs, vol. 75, American Mathematical Society, 2000.

[11] A.S. Kechris, **Classical descriptive set theory**, Springer-Verlag Graduate Texts in Mathematics, Berlin, 1995.

[12] A.S. Kechris and A. Louveau, *The classification of hypersmooth Borel equivalence relations*, **Journal of the American Mathematical Society**, vol. 10 (1997), no.1, pp. 215–242.

[13] Y.N. Moschovakis, **Descriptive set theory**, North-Holland Press, Amsterdam, 1980.

[14] S. Solecki, *Analytic ideals*, **Bulletin of Symbolic Logic**, vol. 2 (1996), no. 3, pp. 339–348.

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