Defining small sets from κ-cc families of subsets

Andrés Eduardo Caicedo, John D. Clemens,
Clinton Conley, and Benjamin D. Miller

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Abstract

We show that if X is a set and $\mathcal{A}$ is a non-empty family of n-element subsets of X that does not contain a pairwise disjoint family of cardinality strictly greater than k, then a non-empty subset of X of cardinality strictly less than kn^2 is definable from $\mathcal{A}$. We show that this result is nearly best possible, and investigate analogous questions for families of countable sets. We also study bounds on the size of intersecting families of $\leq n$-element subsets from which no smaller such family can be defined, and classify all families of this form for $n = 3$.

Contents

1 Introduction

Let X be a set. The *power set of* X is the family $\mathcal{P}(X)$ of all subsets of X. Let κ be a (possibly infinite) cardinal number. We say that a family $\mathcal{A} \subseteq \mathcal{P}(X)$ is an *antichain* iff

$$\forall A, B \in \mathcal{A}\ (A \neq B \Rightarrow A \cap B = \emptyset),$$

and we say that $\mathcal{A}$ satisfies the *κ-chain condition*, or simply $\mathcal{A}$ is *κ-cc*, iff it does not contain an antichain of cardinality κ. If $\kappa = 2$, the set $\mathcal{A}$ is also called an *intersecting family*; if $\kappa = \aleph_0 = |\mathbb{N}|$, we say that $\mathcal{A}$ is *finite-cc*; if $\kappa = \aleph_1$, we say that $\mathcal{A}$ is *ccc*. Notice that our notion of antichain is more restrictive than the usual condition that elements of $\mathcal{A}$ be $\subseteq$-incomparable; we call such families *weak antichains* and only encounter them briefly in §6.

Several results in diverse fields of mathematics depend on purely combinatorial principles. To name just two examples, Ramsey's theorem was established to prove a decidability result in mathematical logic (and was rediscovered by Erdős and Szekeres to prove a result in geometry), and the study of automorphism groups of strongly regular graphs and Steiner triple systems played a role in the classification of the finite simple groups. It is often the case that the combinatorial principles behind results in other fields turn out to be interesting in their own right, and their study provides us with further applications of these principles.

A recent result of Clemens-Conley-Miller [2] in the theory of definable equivalence relations relies on the ability to define finite sets in a canonical way from (possibly infinite) intersecting families of finite sets. Our aim in this paper is to investigate the purely combinatorial issues underlying the proof of this result and in particular to study the minimum size of sets definable from such intersecting families. Extremal set theory and in particular the size of intersecting families is a well known area of study within combinatorics; the twist we add is the consideration of definability conditions. This provides us with its own advantages, such as access to techniques from mathematical logic, and with a few additional difficulties, such as the fact that probability arguments, Ramsey arguments, and non-constructive proofs do not yet seem applicable within our framework.

To keep the paper reasonably self-contained, we formalize in §2 the concept of *definability* that our results deal with. We wish to emphasize that these results are combinatorial in nature, and a reader with only an intuitive understanding of definability will be able to follow our arguments without difficulties.

Definition 1.1. We denote by $\xi : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ the map that assigns to each pair (k, n) the least l such that every non-empty $(k+1)$-cc family $\mathcal{A}$ of n-element sets can be used to define a non-empty subset of $\bigcup \mathcal{A}$ of cardinality at most l.

In §3, we show that ξ is well defined, i.e., that for all positive integers k and n there is such an l, and provide bounds on its value. For this, we say that $\mathcal{A}$ is a *low* family iff no proper subset of $\bigcup \mathcal{A}$ or of $\mathcal{A}$ can be defined from $\mathcal{A}$; our bounds come from the study of low families.

In §4, we introduce the concept of *minimal* intersecting families $\mathcal{A} \subseteq [X]^{\leq n}$. This is analogous to the concept of low but now rather than asking that no proper subset of X or $\mathcal{A}$ is definable, we ask that no smaller intersecting family of $\leq n$-element subsets of X (which need not even be a subfamily of $\mathcal{A}$) is definable. Let $\psi(n)$ be the size of the largest minimal family of $\leq n$-element sets. The arguments of §3 provide a super-exponential upper bound for ψ. We also find a super-exponential lower bound. In §4.3, we classify all minimal intersecting families of ≤ 3-element sets. In particular, we show that if $\mathcal{A}$ is such a family, then $|\mathcal{A}| \leq 10$, and show that there are exactly two non-isomorphic families for which this bound is attained. The classification also implies that $\xi(1,3) = 7$.

In §5, we study low intersecting families. We prove a lifting theorem that allows us to build larger low families from smaller ones. We also adapt an argument from §4.3 to show that if $\mathcal{A}$ is a low family of 3-element sets, then $|\mathcal{A}| \leq 10$.

In §6, we show that one cannot in general hope to define non-empty sets of small cardinality from intersecting families of (countably) infinite sets. We also give some special cases in which it is possible to define such sets.

We close in §7 with suggestions for further research.

We violate our goal of being self-contained in two short subsections, §1.1 and §6.1. In §1.1, we briefly discuss the result from Clemens-Conley-Miller [2] leading to the combinatorial result that motivated this paper. In §6.1, we announce some recent applications of the results of §6 to the study of ends of graphs in the descriptive set theoretic context. Both subsections require a more thorough understanding of descriptive set theory than discussed in this paper, and we refer to Clemens-Conley-Miller [2] and references within for additional details.

1.1 A Glimm-Effros-style dichotomy

A topological space X is said to be *Polish* if it is second countable and admits a compatible complete metric. An equivalence relation E on such a space is said to be *countable* if each of its equivalence classes $[x]_E$ is countable. A *reduction* from an equivalence relation E on X to another equivalence relation F on Y is a function $\pi : X \to Y$ with the property that $x_1 E x_2 \iff \pi(x_1) F \pi(x_2)$, for all $x_1, x_2 \in X$. The study of the Borel reducibility quasi-order ($\leq_B$) on the class of countable Borel equivalence relations on Polish spaces has played an important role in descriptive set theory over the last two decades.

A *partial transversal* of a countable Borel equivalence relation E is a set which intersects every E-class in at most one point. We denote by $\mathcal{I}_E$ the σ-ideal consisting of those sets contained in the union of countably many Borel partial transversals of E. We say that E is *smooth* if $\mathcal{I}_E$ trivializes, i.e., if $X \in \mathcal{I}_E$. By well-known results of classical descriptive set theory, such equivalence relations are $\leq_B$-minimal among all countable Borel equivalence relations on uncountable Polish spaces.

A remarkable result, which goes back to work of Glimm-Effros in operator algebras from the 1960s, is that there is a $\leq_B$-minimal non-smooth countable Borel equivalence relation on a Polish space. An example of such an equivalence relation is E_0 on $\{0,1\}^{\mathbb{N}}$, which is given by

$$xE_0y \iff \exists n \in \mathbb{N}\, \forall m \geq n\ (x(m) = y(m)).$$

Speaking very roughly, this result is proved as follows. Given a countable Borel equivalence relation E, there is a natural attempt at recursively building a continuous injective reduction of E_0 into E, which essentially entails trying to build up copies of level-by-level approximations to E_0 within E. If this attempt fails to produce the desired reduction, then it necessarily provides a countable family of Borel partial transversals whose union is X, thus E is smooth.

Given both the central nature of the σ-ideal $\mathcal{I}_E$ in this proof, as well as its alternative characterization as the σ-ideal generated by the Borel sets on which E is $\leq_B$-minimal, it is natural to ask for the extent to which $\mathcal{I}_E$ determines the $\leq_B$-class of E. An answer to this question has been provided by Clemens-Conley-Miller [2], where it is showed that the existence of a Borel homomorphism from $\mathcal{I}_E$ to $\mathcal{I}_F$ is equivalent to the existence of a smooth-to-one Borel homomorphism from E to F (we refer the reader to Clemens-Conley-Miller [2] for the exact definitions). While the latter notion is strictly weaker than Borel reducibility, it does agree with Borel reducibility when restricted to several of the most important classes of countable Borel equivalence relations.

Lying at the heart of the Clemens-Conley-Miller [2] result is the fact that if E is a countable Borel equivalence relation on X, F is a countable Borel equivalence relation on Y, and $\phi : X \to Y$ is Borel, then either there is a Borel perturbation of ϕ which is a homomorphism from $\mathcal{I}_E$ to $\mathcal{I}_F$, or else there is a continuous injective reduction π of E_0 into E with the property that the points of the form $\phi \circ \pi(x)$, for $x \in 2^{\mathbb{N}}$, are pairwise F-inequivalent. The proof resembles that of the Glimm-Effros theorem. Again, there is a natural attempt at recursively building the desired reduction of E_0 to E. This attempt fails if X is in the σ-ideal $\mathcal{I}_\phi$ generated by the Borel sets $B \subseteq X$ which have the property that for some finite set Γ of Borel automorphisms whose graphs are contained in E, the collection of sets of the form $\{[\phi(\gamma \cdot x)]_F : \gamma \in \Gamma\}$, for $x \in B$, is an intersecting family. In this case, one then obtains the desired Borel perturbation of ϕ by appealing to the fact that there is a Borel way of defining finite sets from intersecting families, which easily follows from the purely combinatorial result which we study in this paper.

2 Preliminaries on definability

Here we state precisely the notion of definability that we have in mind. We work in the language $\{\in, A\}$ of set theory with one unary predicate symbol. An *atomic formula* is a formula of the form $x = y$, $x \in y$, or Ax. Let $\mathcal{L}$ denote the smallest class of formulae which includes the atomic formulae and satisfies the following:

1. If ϕ is in $\mathcal{L}$, then $\neg\phi$ is in $\mathcal{L}$.
2. If ϕ and ψ are in $\mathcal{L}$, then $(\phi \vee \psi)$ is in $\mathcal{L}$.
3. If ϕ is in $\mathcal{L}$, then $\forall x\ (\phi)$ is in $\mathcal{L}$.

Then $\mathcal{L}$ consists of the *first-order formulae* (again, in the language $\{\in, A\}$). We associate to each set X and family $\mathcal{A} \subseteq \mathcal{P}(X)$ the structure

$$\hat{\mathcal{A}}_X := (\mathcal{P}(X) \sqcup X, \mathcal{A}, \hat{\in}),$$

where $\sqcup$ denotes *disjoint* union, and

$$\alpha \mathrel{\hat{\in}} \beta \ \text{ iff } \ \alpha \in X, \beta \in \mathcal{P}(X), \ \text{and } \alpha \in \beta.$$

We interpret the $\mathcal{L}$-formulae in the usual fashion, with $=$, $\in$, $\neg$, $\vee$, and $\forall$ serving as shorthand for *equality*, *in* (i.e., $\hat{\in}$), *not*, *or*, and *for all*, respectively, and where the quantifier ranges over elements or subsets of X. We interpret Ax as $x \in \mathcal{A}$ (in particular, Ax implies that $x \in \mathcal{P}(X)$). Notice that other standard connectives and quantifiers can be interpreted as well: $(\phi \wedge \psi)$, ϕ *and* ψ, can be stated as $\neg(\neg\phi \vee \neg\psi)$; $\exists x\,(\phi)$, *there exists an* x *such that* ϕ, can be stated as $\neg\forall x\,(\neg\phi)$; $(\phi \Rightarrow \psi)$, ϕ *implies* ψ, can be stated as $(\neg\phi \vee \psi)$; etc.

The sole purpose of taking disjoint unions and using $\hat{\in}$ rather than $\in$ is to stop the internal set structure of X from introducing unexpected definability; for example, if $X = \{\emptyset, \{\emptyset\}\}$ and $\mathcal{A} = \emptyset$, we could define both elements of X if in $\hat{\mathcal{A}}_X$ we were to use $\in$ rather than $\hat{\in}$, or if we were to take the usual union of X and $\mathcal{P}(X)$, while it is our intention that X should be thought of as a 2-element set with no distinguishing features between its elements. We will gloss over this technicality in what follows.

If we want to emphasize that a formula ϕ is true when interpreted in $\hat{\mathcal{A}}_X$, we write $\hat{\mathcal{A}}_X \models \phi$ ($\hat{\mathcal{A}}_X$ *satisfies* ϕ).

A set Y is *(first-order) definable from* $\mathcal{A}$ iff there is a first-order formula $\phi(z)$ (where this notation indicates that any variable occurring in ϕ other than z is bound by a quantifier) with

$$Y = \{Z : \hat{\mathcal{A}}_X \models \phi(Z)\}.$$

There is a similar notion of a set being *definable with parameters*; here, ϕ is allowed to have additional free variables $\vec{x}$ and there is a tuple $\vec{A}$ of elements or subsets of X such that Y is the collection of Z for which $\phi(Z, \vec{A})$ holds. Of course, any finite set is definable with parameters. We do not include parameters in our formulas since, except for §6, all of our structures are finite. An exception is made in §6 in the context of infinite structures, see Theorem 6.5.

What we have just described is the standard notion of definability in logic (see Cori-Lascar [3] for an introduction), and is the one we use throughout the paper. It provides us with several technical advantages, see for example Lemma 2.12 and Theorem 4.18. It must be pointed out, however, that different alternatives are possible and perhaps some are even more natural. For example, one could argue that rather than first-order definability, the natural notion of definability to consider is that of being *definable in set theory*. This notion is a bit more technical, so we opted for the simpler alternative; in a general context it is the appropriate notion, but for finite families, the simpler setting suffices. Our definitions tend to focus on parameters involving the size of certain sets. In set theory, one can define these parameters directly. In first-order logic, no such definitions are possible in general. However, for any fixed $n \in \mathbb{N}$, we can in first-order logic state that a (definable) set *has precisely n elements*. Since the sizes we consider are finite, we can then overcome the obstacle by a detour: Rather than requiring, for example, that a point x has maximal degree in a graph (this notion not being first-order definable), we can instead require that for a given fixed number k, the point x has degree k. If k happens to be the maximal degree in a specific instance, then this gives (indirectly) the required definition in that instance.

Notice that the class of subsets of $\mathcal{P}(X) \cup X$ definable from $\mathcal{A}$ forms a Boolean algebra (it is closed under complements, set theoretic differences, finite intersections and finite unions), and contains X, $\mathcal{A}$, and $\mathcal{P}(X)$.

It will be convenient to have at our disposal a condition which ensures non-definability. Let S_X denote the *symmetric group* of all permutations of X.

Note that the natural action of S_X on X induces an action of S_X on $\mathcal{P}(X)$ given by $\sigma \cdot A = \sigma[A] := \{\sigma(a) : a \in A\}$, as well as an action of S_X on $\mathcal{P}(\mathcal{P}(X))$ given by $\sigma \cdot \mathcal{A} = \sigma[\mathcal{A}] = \{\sigma \cdot A : A \in \mathcal{A}\}$.

Definition 2.1. Given $\mathcal{A} \subseteq \mathcal{P}(X)$, by $\mathrm{Aut}(\mathcal{A})$ we mean $\{\sigma \in S_X : \sigma \cdot \mathcal{A} = \mathcal{A}\}$.

This notation is standard, see for example Meyerowitz [7].

Remark 2.2. There is a general notion of the automorphism group of a structure. For $\hat{\mathcal{A}}_X$, this is the group of all bijections of $\mathcal{P}(X) \cup X$ with itself that send $\mathcal{A}$ onto itself and respect $\hat{\in}$. It is easy to see that these conditions imply that any such bijection comes from a $\sigma \in S_X$ in the manner stated above, so we can identify $\mathrm{Aut}(\hat{\mathcal{A}}_X)$ and $\mathrm{Aut}(\mathcal{A})$. Throughout the paper, we use the (slightly imprecise) notation $\mathrm{Aut}(\mathcal{A})$ since X is always clear from context.

Definition 2.3. Given a set X and $\mathcal{A} \subseteq \mathcal{P}(X)$, let $Y \subseteq X \cup \mathcal{P}(X)$ and $\sigma \in \mathrm{Aut}(\mathcal{A})$. We say that Y is *invariant* under σ iff $Y = \sigma \cdot Y$ and we say that Y is *invariant* iff it is invariant under all members of $\mathrm{Aut}(\mathcal{A})$.

Proposition 2.4. *Given a set X and $\mathcal{A} \subseteq \mathcal{P}(X)$, let $Y \subseteq X \cup \mathcal{P}(X)$. If Y is definable from $\mathcal{A}$ then Y is invariant.*

Proof. If $\phi(z)$ is a first-order formula, $\mathcal{A} \subseteq \mathcal{P}(X)$,

$$Y = \{Z \in X \cup \mathcal{P}(X) : \hat{\mathcal{A}}_X \models \phi(Z)\},$$

and $\sigma \in \mathrm{Aut}(\mathcal{A})$, then

$$\begin{aligned}
\sigma \cdot Y &= \{\sigma \cdot Z : \hat{\mathcal{A}}_X \models \phi(Z)\} \\
&= \{Z : \hat{\mathcal{A}}_X \models \phi(\sigma^{-1} \cdot Z)\} \\
&= \{Z : \widehat{\sigma \cdot \mathcal{A}}_X \models \phi(Z)\} \\
&= \{Z : \hat{\mathcal{A}}_X \models \phi(Z)\} \\
&= Y,
\end{aligned}$$

where the third equality is easily established by induction on the complexity (or length) of ϕ. □

It is thus natural to introduce the following notion:

Definition 2.5. Let $\mathcal{A} \subseteq \mathcal{P}(X)$. A set $Y \subseteq X \cup \mathcal{P}(X)$ is *hopelessly undefinable from* $\mathcal{A}$ iff there exists $\sigma \in \mathrm{Aut}(\mathcal{A})$ such that $Y \neq \sigma \cdot Y$.

Notice that this is indeed a very strong obstacle to any reasonable notion of definability. To provide just one example, we can extend first-order logic by allowing the use of *cardinality quantifiers*, with which we can state that a definable set is infinite (or even, say, that it has size precisely κ for some infinite cardinal number κ). Even in this context the argument above goes through, and a hopelessly undefinable set is not definable even if these additional quantifiers are allowed.

Remark 2.6. If X is finite, the collection $\{Y \subseteq X : Y \text{ is invariant}\}$ is definable from $\mathcal{A}$. To see this, let $n = |X \cup \mathcal{P}(X)|$ and notice that if $Y \subseteq X$ then Y is invariant iff $\hat{\mathcal{A}}_X \models \phi(Y)$, where $\phi(y)$ is

$$\begin{aligned}
\forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_n \quad & ((\textstyle\bigwedge_{i<j} x_i \neq x_j \wedge \bigwedge_{i<j} y_i \neq y_j \wedge \bigwedge_i (Ax_i \Leftrightarrow Ay_i) \\
& \wedge \textstyle\bigwedge_{i,j} (x_i \in x_j \Leftrightarrow y_i \in y_j)) \Rightarrow \bigwedge_i (x_i \in y \Leftrightarrow y_i \in y)).
\end{aligned}$$

That is, the formula $\phi(Y)$ states that Y is closed (thus invariant) under any permutation of $X \cup \mathcal{P}(X)$ in $\mathrm{Aut}(\hat{\mathcal{A}}_X) = \mathrm{Aut}(\mathcal{A})$.

Definition 2.7. A set $\mathcal{A} \subseteq \mathcal{P}(X)$ is *low* iff every non-empty, proper subset of X is hopelessly undefinable from $\mathcal{A}$, as is every non-empty, proper subset of $\mathcal{A}$ or, equivalently, iff the group $\mathrm{Aut}(\mathcal{A})$ acts transitively on both X and $\mathcal{A}$.

In Meyerowitz [7], $\mathcal{A}$ is called *transitive* iff $\mathrm{Aut}(\mathcal{A})$ acts transitively on X. This is a weaker notion than being low.

Example 2.8. The following families $\mathcal{A}$ are transitive but not low:

1. $\mathcal{A} = \mathcal{P}(X)$ for any non-empty X.
2. Let X be the set of vertices of a hexagon, labeled clockwise as 1–6, and let $\mathcal{A}$ consist of the rotations of the triangles $\{1, 2, 4\}$ and $\{1, 3, 5\}$.

In the first example, $\mathcal{A}$ is not low since it has sets of different sizes. In the second example this is avoided and $\mathcal{A}$ is intersecting.

We are mainly interested in the case where the family $\mathcal{A}$ is intersecting. The following are typical examples of low intersecting families.

Example 2.9. Let X be the ordered set of size $\aleph_1$ defined by considering a set A of order type the first uncountable ordinal ω_1, and replacing each element of A with a copy of $\mathbb{Q}$. Let $\mathcal{A}$ be the collection of countable non-empty subsets of X closed under predecessors and whose supremum does not exist. Since all countable dense linear orders without endpoints are order isomorphic, it follows that $\mathcal{A}$ is low.

Example 2.10. Recall that an *ultrafilter* on a set X is a maximal filter over the lattice $\mathcal{P}(X)$. Explicitly, $\mathcal{U} \subset \mathcal{P}(X)$ is an ultrafilter on X iff $\emptyset \notin \mathcal{U}$, $\mathcal{U}$ is closed under supersets and (finite) intersections, and given any $A \subseteq X$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$. An ultrafilter $\mathcal{U}$ is *uniform* iff for every $A \in \mathcal{U}$, $|A| = |X|$. Except for the trivial case of a singleton, if a set X admits a uniform ultrafilter, it is infinite, and a trivial application of Zorn's lemma shows that every infinite X admits a uniform ultrafilter.

Let X be an infinite set. Let $\mathcal{U}$ be a uniform ultrafilter on X, and let $\mathcal{A}$ be the collection of sets in $\mathcal{U}$ whose complement also has size $|X|$. Then $\mathcal{A}$ is low.

To see this, suppose first that $a, b \in X$ and let $\pi_{a,b} \in S_X$ be the transposition exchanging a and b. For any $A \subseteq X$, the symmetric difference $A \triangle \pi_{a,b} \cdot A$ has size at most two, so $\pi_{a,b} \cdot A \in \mathcal{A}$ iff $A \in \mathcal{A}$ and thus $\pi_{a,b} \in \mathrm{Aut}(\mathcal{A})$.

Suppose now that $A, B \in \mathcal{A}$. Let $\kappa = |X|$. Then $A \cap B \in \mathcal{U}$, so it has size κ and we can partition it into two disjoint sets Γ and Δ, each of size κ. Since $\mathcal{U}$ is an ultrafilter, exactly one of Γ, Δ, say Γ, is in $\mathcal{U}$. Let $C = A \setminus B$ and $D = B \setminus A$. Let $\pi \in S_X$ be an involution such that

- $\pi \restriction X \setminus (A \cup B) = \mathrm{id}$,
- $\pi \restriction \Gamma = \mathrm{id}$,
- $\pi \cdot (C \cup \Delta) = D \cup \Delta$; this is possible since $|C \cup \Delta| = |D \cup \Delta| = \kappa$.

Clearly, $\pi \cdot A = B$. Notice that for any $Y \subseteq X$, $\pi \cdot (X \setminus Y) = X \setminus \pi \cdot Y$. In particular, $|X \setminus Y| = \kappa$ iff $|X \setminus \pi \cdot Y| = \kappa$. Let $Y \in \mathcal{A}$. Then we can partition Y as $(Y \cap \Gamma) \cup Z$ and $\pi \cdot Y \supseteq Y \cap \Gamma \in \mathcal{A}$. It follows that $\pi \cdot \mathcal{U} \subseteq \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, if $\pi \cdot W \in \mathcal{U}$ then $\pi \cdot (X \setminus W) \notin \mathcal{U}$, so $X \setminus W \notin \mathcal{U}$, so $W \in \mathcal{U}$, and it follows that $\pi \cdot \mathcal{U} = \mathcal{U}$ and therefore $\pi \cdot \mathcal{A} = \mathcal{A}$. Thus, $\mathcal{A}$ is low.

Example 2.11. Let X be a set of size 10 and let $\mathcal{A}$ be the collection of all subsets of X of size 6. Then $\mathcal{A}$ is low.

Additional examples can be found throughout the paper; in particular, see Theorems 3.10 and 6.1.

It turns out that for *finite* structures, a set is definable iff it is invariant under automorphisms:

Lemma 2.12. *Assume X is finite and let $B \subseteq \mathcal{P}(X) \cup X$. Then B is definable from $\mathcal{A}$ iff it is invariant under* $\mathrm{Aut}(\mathcal{A})$.

Proof. This is an easy consequence of either Beth's definability theorem or Svenonius's theorem, see Hodges [5, §10.5]. The reader may find it instructive to complete the following sketch: Let $\vec{a} = \langle a_1, \ldots, a_n \rangle$ enumerate $X \sqcup \mathcal{P}(X)$, where $n = |X| + 2^{|X|}$, and let $\phi^{\vec{a}}_{\mathcal{A}}(\vec{x})$ be the formula

$$\bigwedge_{i<j} x_i \neq x_j \wedge \forall y \bigvee_i y = x_i \wedge \bigwedge_{a_i \hat{\in} a_j} x_i \in x_j \wedge \bigwedge_{a_i \hat{\notin} a_j} x_i \notin x_j \wedge \bigwedge_{a_i \in \mathcal{A}} Ax_i \wedge \bigwedge_{a_i \notin \mathcal{A}} \neg Ax_i,$$

where $i, j \in \{1, \ldots, n\}$ in all connectives. Suppose $\hat{\mathcal{B}}_Y \models \exists \vec{y}\ \phi^{\vec{a}}_{\mathcal{A}}(\vec{y})$, as witnessed by $\vec{b}$. Then the map $b_i \mapsto a_i$ is an isomorphism between $\hat{\mathcal{B}}_Y$ and $\hat{\mathcal{A}}_X$.

Assume now that $Y \subseteq \mathcal{P}(X) \cup X$ is invariant under $\mathrm{Aut}(\mathcal{A})$ and let $\varphi(z)$ be the formula $\exists \vec{x}\ \psi(\vec{x}, z)$, where $\psi(\vec{x}, z)$ is the formula

$$\phi^{\vec{a}}_{\mathcal{A}}(\vec{x}) \wedge \bigwedge_{a_i \notin Y} z \neq x_i \wedge \bigvee_{a_i \in Y} z = x_i.$$

Then Y is definable from $\mathcal{A}$ via φ: The tuple $\vec{a}$ witnesses that $Y \subseteq \{a : \varphi(a)\}$, and the invariance of Y under automorphisms guarantees that for any tuple $\vec{b}$, $\{a : \psi(\vec{b}, a)\}$ is either empty (if $\phi^{\vec{a}}_{\mathcal{A}}(\vec{b})$ fails) or Y. □

Corollary 2.13. *If X is finite and $B \subseteq \mathcal{P}(X) \cup X$, then B is not definable from $\mathcal{A}$ iff B is hopelessly undefinable from $\mathcal{A}$.* □

Remark 2.14. In general, the above is not true for infinite structures, so being low is usually strictly stronger than having only trivial definable sets. For example, consider $(\mathbb{N}, <)$. The first-order formulae in this case are defined as before, except that the atomic formulae are of the form $x = y$ or $x < y$. Clearly, every singleton $\{n\}$ and therefore every finite or co-finite subset of $\mathbb{N}$ is definable. One can show that, indeed, these are the only definable subsets of $\mathbb{N}$, so any infinite, co-infinite subset of $\mathbb{N}$ is undefinable. However, since every singleton is definable, the only automorphism of $(\mathbb{N}, <)$ is the identity.

(It is perhaps a bit easier to show that there is *some* undefinable subset, which allows us to draw the same conclusion. For this, simply observe that there are uncountably many subsets of $\mathbb{N}$ but only countably many formulae.)

Example 2.15. For an illustration of the same phenomenon in our context, let $X = A \times B$ where $|A| = \aleph_0$ and $|B| = \aleph_1$ and let $\mathcal{A} = \{A_i : i \in A\} \cup \{B_j : j \in B\}$ where $A_i = \{i\} \times B$ and $B_j = A \times \{j\}$ for $i \in A$ and $j \in B$. Then $\mathcal{A}$ is transitive but not low since not all of its members have the same size. However, only trivial sets are definable from $\mathcal{A}$. First-order logic is not powerful enough to distinguish between different infinite cardinalities.

Since in §6 we deal with infinite sets, we replace there our notion of definability with one more appropriate for that context. First we present results about some particular infinite ccc families of countable sets; the formal setting for definability here is first-order logic with an additional cardinality quantifier. Namely, to clauses 1–3 in the definition of first-order formulae, we add:

4. If ϕ is in $\mathcal{L}$, then $Qx\ (\phi)$ is also in $\mathcal{L}$,

and we interpret Q as "*there are uncountably many*"; the logic so obtained has been extensively studied in the literature and is usually denoted $\mathcal{L}(Q_1)$.

Afterwards, we work with Polish spaces. The main concepts are recalled in §6, but here it is worth pointing out that the most prominent example of such a space is $\mathbb{R}$ itself. The natural definability setting in this case is that of *second-order definability over the natural numbers*, which is a significant strengthening of first-order logic over $\mathbb{R}$. However, we phrase our results in terms of certain sets being *analytic*. This way, again, we emphasize the combinatorial nature of our arguments. Even a summary explanation of the connection between descriptive set theoretic concepts (like analytic sets) and logic (like definability or second-order arithmetic) would take some space, but fortunately a detailed presentation of these issues can be found in Moschovakis [9], to which we refer for further details.

3 Intersecting families of finite sets

Given a set X and a cardinal κ, let $[X]^\kappa = \{A \subseteq X : |A| = \kappa\}$, $[X]^{<\kappa} = \{A \subseteq X : |A| < \kappa\}$, and $[X]^{\leq\kappa} = \{A \subseteq X : |A| \leq \kappa\}$. If $\kappa = \aleph_0$, we write $[X]^{\mathbb{N}}$, $[X]^{<\mathbb{N}}$, and $[X]^{\leq\mathbb{N}}$, respectively . For each family $\mathcal{A} \subseteq \mathcal{P}(X)$ and for each $D \subseteq X$, let $\deg_{\mathcal{A}}(D)$ denote $|\{A \in \mathcal{A} : D \subseteq A\}|$. We often write $\deg_{\mathcal{A}}(x)$ instead of $\deg_{\mathcal{A}}(\{x\})$. An $\mathcal{A}$-*extension* of D (or simply an *extension* of D) is any $A \in \mathcal{A}$ with $D \subseteq A$.

For m a positive integer and cardinal l, let

$$\mathcal{A}^{(m,l)} = \{D \in [X]^m : \deg_{\mathcal{A}}(D) > l\}.$$

Notice that if l is finite (as in this section), then $\mathcal{A}^{(m,l)}$ is definable from $\mathcal{A}$ and if $l = \aleph_0$ (as in §6), then $\mathcal{A}^{(m,\aleph_0)}$ is $\mathcal{L}(Q_1)$-definable from $\mathcal{A}$.

Example 3.1. Suppose that $A \cap B = \emptyset$ and $|A| \geq |B| = 3$. Let $\mathcal{A}$ denote the family of sets of the form $\{a\} \cup (B \setminus \{b\})$, for $a \in A$ and $b \in B$. Then $\mathcal{A}$ is an intersecting family of 3-element sets and $\mathcal{A}^{(2,2)}$ is an intersecting family of 2-element sets.

Fact 3.2. *Suppose $\mathcal{A} \subseteq [X]^{<\mathbb{N}}$ is finite-cc. Then $\mathcal{A}$ is $(k+1)$-cc for some $k \in \mathbb{Z}^+$.*

Proof. Let $\mathcal{B} \subseteq \mathcal{A}$ be a maximal antichain, and let $k = |\bigcup \mathcal{B}|$. Let $\mathcal{C} \in [\mathcal{A}]^{k+2}$. Every non-empty $A \in \mathcal{C}$ contains an element of $\bigcup \mathcal{B}$ by maximality of $\mathcal{B}$, so at least two members of $\mathcal{C}$ have an element in common, and $\mathcal{C}$ is not an antichain. Of course, $k+2$ can be replaced with $k+1$ if $\emptyset \notin \mathcal{A}$. □

Remark 3.3. Our assumption that the members of $\mathcal{A}$ are finite cannot be relaxed in Fact 3.2. To see this, for each $k \in \mathbb{Z}^+$ let $A_{k1}, \ldots, A_{kk}$ be a partition of $\mathbb{N}$ into k infinite sets such that if $k < l$ then $A_{ki} \cap A_{lj}$ is infinite for each i, j. Then $\mathcal{A} = \{A_{ki} : k \in \mathbb{Z}^+, 1 \leq i \leq k\}$ is finite-cc but not $(k+1)$-cc for any $k \in \mathbb{Z}^+$.

For each non-empty $(k+1)$-cc $\mathcal{A} \subseteq [X]^n$ and for each $m \leq n$, let $d_m = \sup_{D \in [X]^m} \deg_{\mathcal{A}}(D)$. In particular, $d_n = 1$. Let

$$\mathcal{A}_m = \mathcal{A}^{(m,(kn)d_{m+1})} = \{D \in [X]^m : \deg_{\mathcal{A}}(D) > (kn)d_{m+1}\}.$$

Lemma 3.4. *Suppose that $\mathcal{A} \subseteq [X]^n$ is $(k+1)$-cc. Then for each $m < n$, $\mathcal{A}_m$ is $(k+1)$-cc (but possibly empty). Furthermore, if $\mathcal{A}_m \neq \emptyset$ and $\mathcal{A}_k = \emptyset$ for all $k > m$, then $\mathcal{A}_m$ is definable from $\mathcal{A}$.*

Proof. Suppose, towards a contradiction, that there exist pairwise disjoint

$$D_0, \ldots, D_k \in \mathcal{A}_m.$$

We inductively construct $A_0, \ldots, A_k \in \mathcal{A}$ such that for each $i \leq k$:

1. A_i is an extension of D_i,
2. for all $j > i$, $D_j \cap A_i = \emptyset$, and
3. for all $j < i$, $A_j \cap A_i = \emptyset$.

Suppose we have found $A_0, \ldots, A_{i-1}$ satisfying the above conditions. Notice that for any $x \in X \setminus D_i$, at most d_{m+1} extensions of D_i contain x, or else $\deg_{\mathcal{A}}(D_i \cup \{x\}) > d_{m+1}$. Consequently, no more than $(kn)d_{m+1}$ extensions of D_i can meet $\bigcup_{j<i} A_j \cup \bigcup_{j>i} D_j$, so there exists an extension A_i of D_i disjoint from $\bigcup_{j<i} A_j \cup \bigcup_{j>i} D_j$, completing this step of the construction. Of course, $\{A_0, \ldots, A_k\}$ is an antichain of size $k+1$, contradicting the fact that $\mathcal{A}$ is $(k+1)$-cc.

It remains to show that if $m < n$ is largest such that $\mathcal{A}_m \neq \emptyset$, then $\mathcal{A}_m$ is definable from $\mathcal{A}$. For this, argue by induction on $n-m$ that each d_i, $m < i \leq n$, is finite. In particular, d_{m+1} is finite, and the result follows. □

Proposition 3.5. *Suppose that $\mathcal{A} \subseteq [X]^n$ is $(k+1)$-cc. Then there exists a non-empty $(k+1)$-cc family $\mathcal{A}' \subseteq [X]^{\leq n}$ definable from $\mathcal{A}$ with $|\mathcal{A}'| \leq (kn)^n - kn + k$.*

Proof. The result is clear when $n = 1$ (with $\mathcal{A}' = \mathcal{A}$). We proceed by induction on n. If for some $m < n$, $\mathcal{A}_m$ is non-empty, definable and $(k+1)$-cc, then the conclusion follows from the inductive hypothesis. Thus, by Lemma 3.4, we may assume that for all $m < n$, $\mathcal{A}_m$ is empty. That is, for all $m < n$, $d_m \leq (kn)d_{m+1}$ and, in particular, $d_1 \leq (kn)^{n-1}$. Fix a maximal antichain $\{A_0, \ldots, A_{k'-1}\}$ in $\mathcal{A}$, where $k' \leq k$. Every point in $\bigcup_{i<k'} A_i$ is contained in at most $(kn)^{n-1} - 1$ additional sets in $\mathcal{A}$. Since every set in $\mathcal{A}$ intersects this maximal antichain, we have $|\mathcal{A}| \leq (k'n)((kn)^{n-1} - 1) + k'$, thus $|\mathcal{A}| \leq (kn)^n - kn + k$. □

In particular, Proposition 3.5 implies the (crude) bound

$$\xi(k, n) \leq n((kn)^n - kn + k),$$

where ξ is as in Definition 1.1. We now give an improved bound as a function of k, n, and d_1.

Proposition 3.6. *Suppose that* $\mathcal{A} \subseteq [X]^n$ *is* $(k+1)$*-cc with* d_1 *finite, and let* $Y = \{x \in X : \deg_{\mathcal{A}}(x) = d_1\}$*. Then*

$$|Y| \leq k(n^2 - (n/d_1)(n-1)).$$

Proof. Simply observe that

$$\begin{aligned} \frac{d_1|Y|}{n} &\leq \frac{1}{n}\sum_{x \in X} \deg_{\mathcal{A}}(x) \\ &= |\mathcal{A}| \\ &\leq k(1 + n(d_1 - 1)), \end{aligned}$$

thus $|Y| \leq k(n^2 - (n/d_1)(n-1))$. □

As a corollary, we see that $\xi(k,n) \leq kn^2 - 1$, which is a quantitative improvement of the observation of Clemens-Conley-Miller [2] that gave rise to this paper:

Theorem 3.7. *Suppose that* k *and* n *are positive integers.*

1. *Suppose that* $\mathcal{A} \subseteq \mathcal{P}(X)$ *is finite-cc,* $\mathcal{A} \cap [X]^n \neq \emptyset$*, and* $\mathcal{A} \cap [X]^n$ *admits a maximal antichain of size* k*. Then a non-empty subset of* X *of cardinality at most* $kn^3 - 1$ *is definable from* $\mathcal{A}$*.*

2. *If* $\mathcal{A} \subseteq [X]^n$ *is* $(k+1)$*-cc, then a non-empty subset of* X *of cardinality at most* $kn^2 - 1$ *is definable from* $\mathcal{A}$*.*

Proof. We prove (1); the argument for (2) is identical. By replacing $\mathcal{A}$ with $\mathcal{A} \cap [X]^n$, we can assume that $\mathcal{A} \subseteq [X]^n$. By Fact 3.2, $\mathcal{A}$ is $(kn+1)$-cc. By Proposition 3.5, we can assume that d_1 is finite, and Proposition 3.6 then gives the desired result. □

Corollary 3.8. *Suppose that* $\mathcal{A} \subseteq [X]^{<\mathbb{N}}$ *and that there exist non-empty (not necessarily definable) sets* $\mathcal{A}_0, \mathcal{A}_1$ *such that* $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$ *and for all* $A_0 \in \mathcal{A}_0$ *and* $A_1 \in \mathcal{A}_1$*,* $A_0 \cap A_1 \neq \emptyset$*. Then there is a non-empty, finite subset of* X *definable from* $\mathcal{A}$*.*

Proof. Notice that the assumption ensures that $\emptyset \notin \mathcal{A}$. In light of Theorem 3.7.1, it suffices to show that $\mathcal{A}$ is finite-cc. For this, simply note that any antichain $\mathcal{C}$ is contained in $\mathcal{A}_i$ for some $i \in \{0,1\}$. Since every element of $\mathcal{C}$ intersects every $A \in \mathcal{A}_{1-i}$, we see that $|\mathcal{C}| \leq |A|$ for any such A. □

Remark 3.9. The proof of Corollary 3.8 in fact shows that we can find a definable set of size at most

$$(\max_{i<2} \min_{A \in \mathcal{A}_i} |A|)(\min_{A \in \mathcal{A}} |A|)^2 - 1.$$

We now show that the bound of $kn^2 - 1$ from Theorem 3.7.2 is best possible, in the sense that we cannot replace it with ϵkn^2 for any $0 < \epsilon < 1$, using some well-known examples from the study of combinatorial designs.

Recall that the notion of low was introduced in Definition 2.7. Notice that $\bigcup \mathcal{A}$ is definable from $\mathcal{A}$, so if $\mathcal{A}$ is low, then $X = \bigcup \mathcal{A}$. We say that $\mathcal{A}$ has *degree* k iff $\forall x \in X$ $(\deg_{\mathcal{A}}(x) = k)$. If the degree of $\mathcal{A}$ is defined (for example, if $\mathcal{A}$ is low), then $\mathcal{A}$ is called *regular*.

Theorem 3.10. *Suppose that k and n are positive integers.*

1. *There is a set X of cardinality $k(n^2/2 + n/2)$ and a low $(k+1)$-cc family $\mathcal{A} \subseteq [X]^n$ of degree 2.*
2. *If $n-1$ is a power of a prime, then there is a set X of cardinality $k(n^2 - n + 1)$ and a low $(k+1)$-cc family $\mathcal{A} \subseteq [X]^n$ of degree n.*

Proof. We will handle only the case that $k = 1$, as the general case follows by taking the disjoint union of k copies of the examples we describe. To see (1), let $I = \{0, 1, \ldots, n\}$ and $X = [I]^2$. Then

$$|X| = n^2/2 + n/2.$$

For each $i \in I$, define

$$A_i = \{x \in X : i \in x\},$$

so $|A_i| = n$ and

$$\mathcal{A} = \{A_i : i \in I\} \subseteq [X]^n$$

is an intersecting family of degree 2 since if $x = \{i, j\} \in X$, then $A_i \cap A_j = \{x\}$ and $x \in A_k \iff k \in x$. To see that $\mathcal{A}$ is low, define

$$G = \{g \in S_X : \exists \sigma \in S_I \, \forall \{i, j\} \in X \ (g \cdot \{i, j\} = \{\sigma \cdot i, \sigma \cdot j\})\},$$

and observe that $G \leq \mathrm{Aut}(\mathcal{A})$ and G acts transitively on X and $\mathcal{A}$.

To see (2), set $q = p^k = n - 1$. Let $\mathbb{F}_q$ be the field of size q, and let V be a vector space of dimension 3 over $\mathbb{F}_q$.

Let

$$X = \{W \leq V : \dim W = 1\},$$

so

$$|X| = (q^3 - 1)/(q - 1) = q^2 + q + 1 = n^2 - n + 1,$$

since two non-zero vectors $\mathbf{x}$ and $\mathbf{y}$ on V determine the same line W iff there is $k \in \mathbb{F}_q \setminus \{0\}$ such that $\mathbf{x} = k\mathbf{y}$.

For $Y \leq V$ of dimension 2, let $\bar{Y} = \{W \leq Y : \dim W = 1\}$ and set

$$\mathcal{A} = \{\bar{Y} : Y \leq V, \dim Y = 2\}.$$

Notice that

$$|\bar{Y}| = \frac{q^2 - 1}{q - 1} = q + 1 = n.$$

We can endow V with a formal inner product by fixing an arbitrary basis B of V over $\mathbb{F}_q$ and setting $\mathbf{x} \cdot \mathbf{y} = \sum_i x_i y_i$ where $\langle x_1, x_2, x_3 \rangle$ and $\langle y_1, y_2, y_3 \rangle$ are the coordinates of $\mathbf{x}$ and $\mathbf{y}$, respectively, with respect to the basis B. The map that sends W to its orthogonal complement is a one-to-one correspondence between spaces W of dimension 1 and spaces Y of dimension 2, so

$$|\mathcal{A}| = |X|.$$

Clearly, $\mathcal{A}$ is intersecting. Given W of dimension 1 and Y of dimension 2, let $Y_\perp$ be the orthogonal complement of Y and let $W^\perp$ be the orthogonal complement of W. Then $W \in \bar{Y}$ iff $Y = (W^\perp)_\perp$, so

$$\deg_{\mathcal{A}}(W) = |\overline{W^\perp}| = n.$$

(For $k = 1$, what we are describing should remind the reader of a well known Steiner $S(2, p-1, p^2+p+1)$ system, see van Lint-Wilson [10].) To see that $\mathcal{A}$ is low, define

$$G = \{g \in S_X : \exists T \in \mathrm{GL}_3(\mathbb{F}_q)\, \forall W \in X \ (g \cdot W = TW)\},$$

and observe that $G \leq \mathrm{Aut}(\mathcal{A})$ and G acts transitively on X and $\mathcal{A}$. □

Example 3.11. When $n = 3$ and $k = 1$, Theorem 3.10.2 provides us with a low intersecting family of triples with $|X| = 7$, the *Fano plane*. This can also be described by letting $X = \mathbb{Z}/7\mathbb{Z}$ and taking as $\mathcal{A}$ the family of translations of $\{0, 1, 3\}$:

$$\mathcal{A} = \{013, 124, 235, 346, 450, 561, 602\}.$$

This shows that $\xi(1, 3) \geq 7$. We in fact have equality, see Proposition 4.41.

4 Minimal intersecting families

Until Section 6, we confine our attention to intersecting families. In the previous section we found bounds for the smallest size of a subset of X definable from $\mathcal{A}$, where $\mathcal{A} \subseteq [X]^n$ is intersecting. This was done in two stages; in the first, we found a finite intersecting $\mathcal{A}' \subseteq [X]^{\leq n}$ definable from $\mathcal{A}$. In the second, we started from such a finite $\mathcal{A}'$ and found bounds on the size of subsets of X definable from $\mathcal{A}'$ (and therefore, from $\mathcal{A}$). Here we address the question of how small we can find $\mathcal{A}'$ itself.

Definition 4.1. An intersecting family $\mathcal{A} \subseteq [X]^{\leq n}$ is *n-minimal* (or, simply, *minimal* if n is clear from context) iff there is no intersecting $\mathcal{B} \subseteq [X]^{\leq n}$ definable from $\mathcal{A}$ with $|\mathcal{B}| < |\mathcal{A}|$.

Remark 4.2. Given a set X and intersecting families $\mathcal{A}_0, \mathcal{A}_1 \subseteq \mathcal{P}(X)$, the relation "$\mathcal{A}_0$ is definable from $\mathcal{A}_1$" is a quasi-order. It is not a partial order since, for example, if $X = \{0, 1, 2\}$, then $[X]^2$ and $[X]^3$ can both be defined from each other.

If $\mathcal{A}$ is minimal, we may assume $\bigcup \mathcal{A} = X$, so we adopt this convention (often without comment) in what follows; this is a minor technicality intended to avoid situations like the following: We could have a set X of size at least 10, let $Y \subset X$ have size 5, and let $\mathcal{A} = [Y]^3$. Then $\mathcal{A}$ is 3-minimal and of size 10. We have that $\bigcup \mathcal{A} = Y$ and $[X \setminus Y]^3$ is definable from $\mathcal{A}$; moreover, if $|X| = 10$, then $[X \setminus Y]^3$ is intersecting and of size 10 as well. Obviously, $X \setminus Y$ has nothing to do with the combinatorics of $\mathcal{A}$, so we prefer to discard it rather than having to pay attention to its size.

Notice that if $\mathcal{A} \subseteq [X]^{\leq n}$ is minimal, then there is $m \leq n$ such that $\mathcal{A} \subseteq [X]^m$.

Definition 4.3. Let $\psi(n)$ be the largest possible size of an n-minimal intersecting family of sets. A family $\mathcal{A} \subseteq [X]^{\leq n}$ is said to be *n-large* (or, simply, *large* if n is clear from context) if it is intersecting, minimal, and has size $\psi(n)$.

Notice that $\psi(n)$ is finite for all n. In fact, $\psi(n) \leq n^n - n + 1$, by Proposition 3.5.

4.1 Examples

Example 4.4. Let $X = \{1, \dots, 2n-1\}$ and $\mathcal{A} = [X]^n$. Then $\mathcal{A}$ is intersecting, low and minimal. Hence

$$\binom{2n-1}{n} \leq \psi(n).$$

It follows that $\psi(2) \geq 3$ and $\psi(3) \geq 10$. Moreover, it is easy to see that $\psi(2) = 3$.

Example 4.5. Besides the family of 3-element subsets of a set of size 5, there is another example of a minimal family of triples of size 10: Let $X = \{a, b, c, d, e, f\}$ and consider the family

$$\mathcal{A} = \{abc, abd, ace, adf, aef, bcf, bde, bef, cde, cdf\}.$$

To see that $\mathcal{A}$ is minimal, first we show that it is low. For this, it suffices to consider the subgroup of $\mathrm{Aut}(\mathcal{A})$ generated by the permutations π_1 and π_2, where

- $\pi_1(a) = b$, $\pi_1(b) = d$, $\pi_1(c) = a$, $\pi_1(d) = e$, $\pi_1(e) = c$ and $\pi_1(f) = f$, and
- $\pi_2(a) = f$, $\pi_2(b) = b$, $\pi_2(c) = e$, $\pi_2(d) = c$, $\pi_2(e) = a$ and $\pi_2(f) = d$.

But then it follows that $\mathcal{A}$ is minimal: Suppose that $\mathcal{B} \subseteq [X]^{\leq 3}$ is intersecting and definable from $\mathcal{A}$. If $\mathcal{B} \cap [X]^{\leq 2} \neq \emptyset$, then from $\mathcal{B} \cap [X]^{\leq 2}$ (and, therefore, from $\mathcal{A}$) we can define a non-empty subset of X of size at most 3, contradicting that $\mathcal{A}$ is low. Thus, $\mathcal{B} \subseteq [X]^3$. If $\mathcal{B} \cap \mathcal{A} \neq \emptyset$ then $\mathcal{B} \cap \mathcal{A} = \mathcal{A}$, since $\mathcal{A}$ is low, but then $|\mathcal{B}| \geq |\mathcal{A}|$. So we may assume that $\mathcal{B} \cap \mathcal{A} = \emptyset$. But then, again considering $\langle \pi_1, \pi_2 \rangle$, $\mathcal{B} \supseteq [X]^3 \setminus \mathcal{A}$, so $|\mathcal{B}| = 10 = |\mathcal{A}|$. (Notice that $[X]^3 \setminus \mathcal{A}$ is intersecting and definable from $\mathcal{A}$, so this last possibility occurs.)

Notice that under the action of S_X on $\mathcal{P}(\mathcal{P}(X))$, the permutation π given by

$$\pi(a) = d, \pi(b) = f, \pi(c) = e, \pi(d) = b, \pi(e) = a, \pi(f) = c$$

exchanges $\mathcal{A}$ and $\mathcal{B}$, showing that $\hat{\mathcal{A}}_X$ and $\hat{\mathcal{B}}_X$ are isomorphic.

Example 4.6. The previous example is a particular case of a more general construction of intersecting families, which is worth considering in some detail. Fix n and let $X = \{0, 1, \dots, 2n-1\}$. Let $\mathcal{A} \subset [X]^n$ be such that for any $A \in [X]^n$ either A or $X \setminus A$ is not in $\mathcal{A}$. Then $\mathcal{A}$ is intersecting and

$$|\mathcal{A}| \leq \frac{1}{2}\binom{2n}{n} = \binom{2n-1}{n}.$$

If we want $\mathcal{A}$ to be of maximal size and minimal, our choice as to which one of A and $X \setminus A$ belongs to $\mathcal{A}$ needs to be sufficiently uniform. For example, let $n = 3$. For $A \subseteq X$ let $\sum A = \sum_{x \in A} x$. Since $\sum X = 15$ is odd, we can define an $\mathcal{A}$ of maximal size by letting

$$\mathcal{A} = \{A \in [X]^3 : \sum A \text{ is even}\}.$$

However, this $\mathcal{A}$ is neither minimal nor low, since $\deg_{\mathcal{A}}(0) = 4$ while $\deg_{\mathcal{A}}(1) = 6$.

We can use a similar idea to produce (at least for $n = 3$) families that are actually minimal. Once again, let n be arbitrary and notice that for any $A \subseteq X$, $\sum A + \sum(X \setminus A) = n(2n-1)$, so $\sum A + \sum(X \setminus A) \equiv 0 \pmod{n}$. If n is odd, we can then fix a set S such that for all $i \neq 0$, exactly one of $i, -i$ is in S (mod n). Then we would want to take $\mathcal{A} = \{A \in [X]^n : \sum A \in S \pmod{n}\}$. However, this does not take care of the case when $\sum A \equiv \sum(X \setminus A) \equiv 0 \pmod{n}$, but in that case, since $n(2n-1)$ is an odd multiple of n, exactly one of $\sum A, \sum(X \setminus A)$ is an odd multiple of n. So we just choose one of these options and put the corresponding set into $\mathcal{A}$.

Applying this idea to the case $n = 3$ we obtain the following families:

- $\{A \in [X]^3 : \sum A \equiv 1 \pmod{3} \text{ or } \sum A \equiv 3 \pmod{6}\}$.
- $\{A \in [X]^3 : \sum A \equiv 2 \pmod{3} \text{ or } \sum A \equiv 3 \pmod{6}\}$.
- $\{A \in [X]^3 : \sum A \equiv 1 \pmod{3} \text{ or } \sum A \equiv 0 \pmod{6}\}$.
- $\{A \in [X]^3 : \sum A \equiv 2 \pmod{3} \text{ or } \sum A \equiv 0 \pmod{6}\}$.

It is easy to verify that these four families are all isomorphic to the family from Example 4.5.

Leaving aside issues of minimality, this construction obviously generalizes to larger odd values of n. However, it is not possible to do something like this and obtain a minimal family for all even values of n:

Lemma 4.7. *Let $\mathcal{A} \subseteq [X]^{\leq n}$ be minimal. Assume that $|\mathcal{A}| \geq \binom{2n-1}{n}$ and $|X| \leq 4n-3$. Then $\mathcal{A}$ is low. If $|\mathcal{A}| > \binom{2n-1}{n}$, the same holds for $|X| \leq 4n-1$.*

Proof. Since $\mathcal{A}$ is minimal, by Lemma 2.12, that $\mathcal{A}$ is not low is equivalent to there being a non-empty proper subset of X definable from $\mathcal{A}$. But, if there were such a set A, either A or its complement would have size at most $2n-2$ (or at most $2n-1$ if $|\mathcal{A}| > \binom{2n-1}{n}$). Without loss of generality, assume it is A. Then the family of n-element subsets of A would be intersecting and of size strictly smaller than $\mathcal{A}$, contradicting minimality. □

For example, it follows that for $n = 4$ we would need a family of 35 4-element subsets of a set of size 8 where each element has the same degree d, but this would imply that $8d = 4 \times 35$, a contradiction. (On the other hand, $\binom{2n-1}{n}$ is always even for odd $n > 1$.)

As for the question of whether $\mathcal{A}$ is regular, we can say the following: For definiteness, let $n = 2k-1$ and set $S = \{1, \dots, k-1\}$ and

$$\mathcal{A} = \{A \in [\{0, \dots, 2n-1\}]^n : \sum A \in S \pmod{n} \text{ or } \sum A \equiv n \pmod{2n}\}.$$

The map $i \mapsto i+2 \pmod{2n}$ is in $\mathrm{Aut}(\mathcal{A})$, so even numbers are indistinguishable, and so are odd numbers. The map $i \mapsto n-i$ is an isomorphism between $\hat{A}_X$ and $\widehat{A^c}_X$ that exchanges the roles of even and odd numbers. The map $i \mapsto i+1$ sends $\mathcal{A}$ to $\bar{\mathcal{A}} = \{A : \sum A \in S \pmod{n} \text{ or } \sum A \equiv 0 \pmod{2n}\}$, again exchanging the roles of even and odd numbers. It follows that $\mathcal{A}$ is regular iff $|\{A \in \mathcal{B} : 0 \in A\}| = |\{A \in \mathcal{B} : 1 \in A\}|$, where

$$\mathcal{B} = \{A \in [\{0, \dots, 2n-1\}]^n : \sum A \equiv n \pmod{2n}\}.$$

However, examination of these families for small values of n has failed to provide examples of regular collections $\mathcal{A}$ other than for $n = 3$.

4.2 The size of minimal families

Since minimal intersecting families are necessarily finite, in the current section we restrict our attention to finite intersecting families. Given a finite, intersecting $\mathcal{A} \subseteq [X]^{\leq n}$, recall that

$$d_1 = \max\{\deg_{\mathcal{A}}(x) : x \in X\}$$

and set

$$S_1 := \{x : \deg_{\mathcal{A}}(x) = d_1\}.$$

Question 4.8. *Assume $\mathcal{A} \subseteq [X]^{\leq n}$ is large. Does it follow that $\mathcal{A} \subseteq [X]^n$? Equivalently, is ψ (strictly) increasing?*

Fact 4.9. *Assume $n > 1$ and $\psi(n) < \psi(n+1)$. Let $\mathcal{A} \subseteq [X]^{n+1}$ be a minimal intersecting family with $|\mathcal{A}| > \psi(n)$. Then $\deg_{\mathcal{A}}(x) > 1$ for all $x \in X$.*

Proof. Otherwise, the collection $\mathcal{B}$ of sets $Y \subset X$ of size n such that there is an $x \in X$ with $\deg_{\mathcal{A}}(x) = 1$ and $Y \cup \{x\} \in \mathcal{A}$ is intersecting and definable from $\mathcal{A}$. From it, an intersecting family $\mathcal{C}$ of n-element subsets of X of size at most $\psi(n)$ can be defined, by definition of ψ. But since $\psi(n) < |\mathcal{A}|$, this contradicts minimality of $\mathcal{A}$. □

Note that if $\mathcal{A}$ is minimal, then $m_1 := |A \cap S_1|$ is independent of $A \in \mathcal{A}$.

Lemma 4.10. *Let $\mathcal{A} \subseteq [X]^{\leq n}$ be minimal, $|\mathcal{A}| \geq \binom{2n-1}{n}$. Then $|S_1| \geq 2n-1$, with equality only if $|\mathcal{A}| = \binom{2n-1}{n}$. If $n > 1$ then $m_1 > 1$.*

Proof. If $|S_1| < 2n-1$ then $[S_1]^n$ is intersecting and of size strictly smaller than $|\mathcal{A}|$, contradicting the minimality of $\mathcal{A}$. If $|\mathcal{A}| > \binom{2n-1}{n}$, then in fact $|S_1| \geq 2n$ by the same argument.

Assume $m_1 = 1$. Fix $x \in S_1$. There are at least $2n-2$ other points in S_1, so at least $(2n-2)d_1$ sets in $\mathcal{A}$ not containing x. Fix $A \in \mathcal{A}$ with $x \in A$. If $n > 1$, these $(2n-2)d_1$ sets all meet $A \setminus \{x\}$, so one of the points of $A \setminus \{x\}$ has degree at least $(2n-2)d_1/(n-1) > d_1$, contradiction. □

Fact 4.11. *Let $\mathcal{A}$ be n-minimal, and let $d_1 = d_{1,1} > \cdots > d_{1,k}$ be the degrees of elements of X. Set $S_{1,i} := \{x : \deg_{\mathcal{A}}(x) = d_{1,i}\}$.*

1. *Every $A \in \mathcal{A}$ has elements of each degree (so $k \leq n$).*
2. *$m_i := |A \cap S_{1,i}|$ is independent of $A \in \mathcal{A}$.*
3. *$\dfrac{d_{1,i}|S_{1,i}|}{m_i} = |\mathcal{A}|$.*

Proof. The first two assertions are clear from the minimality of $\mathcal{A}$. The last one follows from a double counting argument:

$$\begin{aligned} d_{1,i}|S_{1,i}| &= \sum_{x \in S_{1,i}} d_{1,i} = \sum_{x \in X} \sum_{A \in \mathcal{A}} \chi_{S_{1,i}}(x)\chi_A(x) \\ &= \sum_{A} \sum_{x} \chi_{S_{1,i}}(x)\chi_A(x) = \sum_{A} |S_{1,i} \cap A| \\ &= m_i|\mathcal{A}|, \end{aligned}$$

and we are done. □

Fact 4.12. *Assume that $\psi(n-1) < \psi(n)$. Let $\mathcal{A} \subseteq [X]^{\leq n}$ be a minimal intersecting family with $|\mathcal{A}| > \psi(n-1)$. Let*

$$S_{n-1} := \{D \in [X]^{n-1} : \deg_{\mathcal{A}}(D) = d_{n-1}\}.$$

Then $d_{n-1} \leq n$ and if $n > 2$ and $d_{n-1} = n$, then S_{n-1} is 3-cc.

Proof. That $d_{n-1} \leq n$ follows from Lemma 3.4 and the fact that $|\mathcal{A}| > \psi(n-1)$.

Assume now that $d_{n-1} = n > 2$ and that $D_1, D_2, D_3 \in S_{n-1}$ are pairwise disjoint. We can then find x_1, x_3 such that $D_i \cup \{x_i\} \in \mathcal{A}$ and $x_i \notin D_2$, $i = 1, 3$. Since $d_{n-1} = n = |D_3 \cup \{x_3\}|$ and any extension of D_2 meets $D_3 \cup \{x_3\}$, it follows that, for any $a \in D_3$, $D_2 \cup \{a\} \in \mathcal{A}$ and therefore it meets $D_1 \cup \{x_1\}$, so $x_1 = a$. But this contradicts the fact that $|D_3| = n-1 > 1$. □

At this point, it is worth mentioning a reformulation of Lemma 3.4 for finite intersecting families $\mathcal{A}$. We first generalize a notion introduced earlier.

Definition 4.13. Let $\mathcal{A} \subseteq [X]^{\leq n}$. For $m \leq n$ let

$$S_m = \{D \in [X]^m : \deg_{\mathcal{A}}(D) = d_m\}.$$

Since for all $m < n$, $d_m \leq |\mathcal{A}|$, every S_m is definable from $\mathcal{A}$. With the nuisance of definability out of the way, Lemma 3.4 can be stated much more simply:

Lemma 4.14. *Let $\mathcal{A} \subseteq [X]^n$ be intersecting and finite. For all $m < n$ at least one of the following holds:*

1. *S_m is intersecting.*
2. *$d_m \leq nd_{m+1}$.* □

The above suggests that the larger $\mathcal{A}$ is, the smaller X seems required to be, if we want $\mathcal{A}$ to be minimal. This intuition is not completely accurate, as the remainder of this subsection shows.

Question 4.15. *Is a large $\mathcal{A} \subseteq [X]^{\leq n}$ necessarily low?*

Definition 4.16. A family $\mathcal{A} \subseteq [X]^n$ is *strongly minimal* or *sminimal* iff $\mathrm{Aut}(\mathcal{A})$ acts transitively on $\mathcal{A}$ and on $[X]^m$ for all $m < n$.

Question 4.17. *If $\mathcal{A}$ is large, is it sminimal?*

Theorem 4.18. *Suppose $\mathcal{A} \subseteq [X]^n$ is intersecting and $|\mathcal{A}| > \psi(n-1)$. Then the following are equivalent:*

1. *$\mathcal{A}$ is minimal.*
2. *For all $A \in [X]^{\leq n}$ either $A \notin \mathcal{A}$ and there is $\sigma \in \mathrm{Aut}(\mathcal{A})$ such that $A \cap \sigma \cdot A = \emptyset$, or else $A \in [X]^n$ and $|\mathrm{Aut}(\mathcal{A}) \cdot A| \geq |\mathcal{A}|$.*

Proof. (1⇒ 2) Suppose $\mathcal{A}$ is minimal so, in particular, it is finite. Let $A \in [X]^{\leq n}$. Let $\mathcal{B} = \mathcal{B}_A := \mathrm{Aut}(\mathcal{A}) \cdot A$ be the closure of A under $\mathrm{Aut}(\mathcal{A})$. By Lemma 2.12, $\mathcal{B}$ is definable from $\mathcal{A}$ and is contained in every subset of $\mathcal{P}(X) \cup X$ definable from $\mathcal{A}$ that contains A as an element. By minimality of $\mathcal{A}$, either $\mathcal{B}$ is not intersecting, which means that for some $\sigma \in \mathrm{Aut}(\mathcal{A})$, $A \cap \sigma \cdot A = \emptyset$, or else $|\mathcal{B}| = |\mathrm{Aut}(\mathcal{A}) \cdot A| \geq |\mathcal{A}|$. In the first case, $A \notin \mathcal{A}$. Otherwise, $\mathcal{B} \subseteq \mathcal{A}$ (since $\mathcal{A}$ is clearly definable from $\mathcal{A}$ and contains A as an element), but $\mathcal{B}$ is not intersecting, a contradiction. In the second case, it follows that $|A| = n$. Otherwise, there is an intersecting family $\mathcal{C}$ of $\leq (n-1)$-element sets with $|\mathcal{C}| \leq \psi(n-1) < |\mathcal{A}|$ definable from $\mathcal{B}$ (and therefore from $\mathcal{A}$), contradicting the minimality of $\mathcal{A}$.

(2⇒1) Let $\mathcal{A}$ satisfy the second condition, and suppose $\mathcal{A}$ is not minimal. Let $\mathcal{B}$ be an intersecting family of $\leq n$-element sets definable from $\mathcal{A}$ with $|\mathcal{B}| < |\mathcal{A}|$. For any $A \in \mathcal{B}$, $\mathcal{B}_A \subseteq \mathcal{B}$, so $\mathcal{B}_A$ is intersecting. It follows that $|A| = n$ and $|\mathcal{B}_A| \geq |\mathcal{A}|$, contradicting that $|\mathcal{B}| < |\mathcal{A}|$. □

A similar argument gives the following sufficient condition for minimality:

Lemma 4.19. *Let $\mathcal{A} \subseteq [X]^n$ be intersecting. Suppose that* $\mathrm{Aut}(\mathcal{A})$ *is transitive on $\mathcal{A}$ and for all $B \in [X]^{\leq n} \setminus \mathcal{A}$ there is a $\sigma \in \mathrm{Aut}(\mathcal{A})$ such that $B \cap \sigma \cdot B = \emptyset$. Then $\mathcal{A}$ is minimal.* □

Theorem 4.20. *For all even $n > 6$, $\psi(n) > (n/2)^{n/2}$ and there is a minimal intersecting $\mathcal{A} \subseteq [X]^n$ with $|X| \geq n^2/4$. In fact, for all rational $p \in (0,1)$, for all but finitely many n such that pn is an integer, $\psi(n) > (pn)^{(1-p)n+1}$.*

Proof. Assume $n = 2k$ is even. Let $X = \bigsqcup_{i=0}^{k} X_i$ be the disjoint union of sets X_i with $|X_i| = k$. For $I \subseteq \{0, \ldots, k\}$, a *transversal* of $\bigcup_{i\in I} X_i$ is a set B such that $|B| = |I|$ and $|B \cap X_i| = 1$ for each $i \in I$. Let $\mathcal{A}$ consist of all sets A of the form $X_i \cup B$ where $0 \leq i \leq k$ and B is a transversal of $\bigcup_{j\neq i} X_j$. Thus, $|\mathcal{A}| = (k+1)k^k > (n/2)^{n/2}$.

We only need the direction $\geq$ of the following lemma and leave its easy proof to the reader.

Lemma 4.21. $\mathrm{Aut}(\mathcal{A}) \cong S_{k+1} \times (S_k)^{k+1}$ *where S_i is the symmetric group on i letters.* □

Claim 4.22. *Suppose $k \geq 4$, $0 \leq a_0 \leq a_1 \leq \cdots \leq a_k \leq k$ are integers, and $\sum_{i=0}^{k} a_i \leq 2k$. Then exactly one of the following holds:*

1. $\forall i\, (a_i + a_{k-i} \leq k)$.
2. $a_0 = a_1 = \cdots = a_{k-1} = 1$ *and* $a_k = k$.

Proof. If the first condition fails, for some $i \leq k/2$ we must have $a_i + a_{k-i} > k$. First we show that $i = 0$: If $i = 1$, then $a_1 + a_{k-1} > k$ and also $a_2 + a_k > k$, since the a_j are increasing. Since $2 < k - 1$, this is a contradiction. Similarly, if $i \geq 2$, then $k - 1 > k - i$ and $a_k + a_{k-1} > k$, again a contradiction.

Now we show that $a_k = k$: Otherwise, say $a_k = k - j$. Then $a_0 \geq j + 1$. Then $\sum a_i \geq (j+1)k + (k-i) \geq 2k + (k-i) > 2k$, a contradiction.

Since $a_0 = 1 \leq a_i \leq a_k = k$ for all i and $\sum a_j \leq 2k$, it now follows that $a_1 = \cdots = a_{k-1} = 1$ as well, and we are done. □

To show that $\mathcal{A}$ is minimal, it suffices to verify that the condition of Lemma 4.19 holds. But this follows from Lemma 4.21 and Claim 4.22, letting (after a renumbering if necessary) $a_i = |A \cap X_i|$ for $A \in \mathcal{B}$ for $\mathcal{B}$ a putative intersecting family of $\leq n$-element sets definable from $\mathcal{A}$. The point is that if the a_i satisfy conclusion (1) of Claim 4.22, we can let σ be the following involution in S_X: σ exchanges setwise the blocks X_i and X_{k-i} for each i, and in such a way that $A \cap X_i$ is disjoint from $\sigma \cdot (A \cap X_{k-i})$, which is possible since $a_i + a_{k-i} \leq k = |X_i|$. Clearly, $\sigma \in \mathrm{Aut}(\mathcal{A})$, and $A \cap \sigma \cdot A = \emptyset$, so $\mathcal{B}$ was not intersecting after all (recall that $\mathcal{B} \supseteq \mathcal{B}_{\mathcal{A}}$, where $\mathcal{B}_{\mathcal{A}}$ is as in Theorem 4.18). Thus, the a_i satisfy conclusion (2) of Claim 4.22, so $A \in \mathcal{A}$ and since $\mathrm{Aut}(\mathcal{A})$ is transitive on $\mathcal{A}$, $\mathcal{B} \supseteq \mathcal{A}$.

To prove the general version, consider now the collection $\mathcal{A}$ of sets A as before, but with each block X_i of size pn and a total of $1 + (1-p)n$ many blocks. A modified version of the claim holds with essentially the same proof, and this gives the result as well. For the sake of exposition, we state a slightly weaker version of the modified statement: Consider blocks of size $n-k$ (for some fixed k) and a total of $k+1$ blocks. Now we require in the claim that $0 \le a_0 \le a_1 \le \cdots \le a_k \le n-k$ and $\sum_i a_i \le n$. The conclusion is that for almost all values of n, either $a_i + a_{k-i} \le n-k$ for all i, or else $a_0 = \cdots = a_{k-1} = 1$ and $a_k = n-k$. $\square$

Corollary 4.23. *For all $n > 3$, $\psi(2n) > \psi(n)$.* $\square$

Remark 4.24. Notice that the examples from Theorem 4.20 are low. Also, Corollary 4.23 holds for all n since $\psi(1) = 1$, $\psi(2) = 3$, $\psi(4) \ge \binom{7}{4} = 35$, and $\psi(3) = 10$ by Theorem 4.30 below.

Notice that if $\mathcal{A} \subseteq [X]^n$ is intersecting, then so is

$$\mathcal{A}_+ = \{B \in [X]^{n+1} : \exists A \in \mathcal{A}\,(A \subseteq B)\}.$$

If $\mathcal{A}$ is minimal and $\mathcal{B} \subseteq [X]^{\le n}$ is intersecting and definable from $\mathcal{A}_+$, then $|\mathcal{B}| \ge |\mathcal{A}|$, since $\mathcal{A}_+$ is itself definable from $\mathcal{A}$. Hence, whether $\mathcal{A}_+$ is $(n+1)$-minimal reduces to whether one can define from it a "small" intersecting subfamily of $[X]^{n+1}$.

Notice that $|\mathcal{A}_+| < |\mathcal{A}|$ is possible. For example, let us introduce the notation $[k]^n$ to denote the collection of n-element subsets of $\{0, 1, \dots, k-1\}$. If $\mathcal{A} = [3]^2$, then $\mathcal{A}_+ = [3]^3$, and if $\mathcal{A} = [5]^3$, then $\mathcal{A}_+ = [5]^4$.

Claim 4.25. *If n is sufficiently large ($n > 34$ suffices) and $\mathcal{A} \subseteq [X]^n$ is n-large, then $|\mathcal{A}_+| > |\mathcal{A}|$.*

Proof. If n is even, by Theorem 4.20, $|\mathcal{A}| \ge (n/2)^{n/2} > 4^n > \binom{2n+1}{n}$, so $|\bigcup A| > 2n+1$. If n is odd, a similar construction to that the one described in Theorem 4.20 gives the same result: Now consider $(n+1)/2$ blocks, each of size $(n-1)/2$, and consider the family $\mathcal{C}$ of sets that contain one of the blocks and meet the others in exactly one point. This family $\mathcal{C}$ is n-minimal as in Theorem 4.20 and

$$|\mathcal{C}| = \left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right)^{\frac{n-1}{2}}.$$

Consider now the set $C = \{(A, B) \in \mathcal{A} \times \mathcal{A}_+ : A \subseteq B\}$. The result follows from a double counting argument: since $|\bigcup A| > 2n+1$, each A in $\mathcal{A}$ can be extended to a B in $\mathcal{A}_+$ in more than $n+1$ ways, so $|C| > (n+1)|\mathcal{A}|$. On the other hand, each $B \in \mathcal{A}_+$ contains at most $n+1$ members of $\mathcal{A}$, so $|C| \le (n+1)|\mathcal{A}_+|$. $\square$

If one can show that n-largeness of $\mathcal{A}$ implies that $\mathcal{A}_+$ is $(n+1)$-minimal, then it follows that $\psi(n) < \psi(n+1)$. Some assumption on $\mathcal{A}$ is necessary, though. For example, if $\mathcal{A}$ is the Fano plane, then $\mathcal{A}_+$ has size 28 and its complement in $[7]^4$ is also intersecting and has size 7, so $\mathcal{A}_+$ is not 4-minimal. To establish minimality of $\mathcal{A}_+$, it seems that a better understanding of $\mathrm{Aut}(A)$ is required.

Remark 4.26. Notice that $\mathrm{Aut}(\mathcal{A}) \le \mathrm{Aut}(\mathcal{A}_+)$ since $\mathcal{A}_+$ is definable from $\mathcal{A}$.

Question 4.27. *If $\mathcal{A}$ is low, does it follow that $\mathcal{A}_+$ is also low?*

4.3 Minimal families of triples

We compile lists of n-minimal families for $n \leq 3$. Let $\mathcal{B}_n$ denote the collection of n-minimal intersecting families of sets, considered up to isomorphism. Then $\mathcal{B}_n$ forms a basis for intersecting families of $\leq n$-element sets in the sense that given any intersecting $\mathcal{A} \subseteq [X]^{\leq n}$ there exists an isomorphic copy of some $\mathcal{A}' \in \mathcal{B}_n$ which is definable from $\mathcal{A}$.

Recall that $[k]^n$ denotes the collection of n-element subsets of $\{0, 1, \ldots, k-1\}$. Given a set of integers $A \subseteq \{0, 1, \ldots, n-1\}$, let A_n denote the family of translations of A taken modulo n so, for example, $\{013\}_7$ is the Fano plane.

Theorem 4.28. *Let L be the family from Example 4.5 and*

$$\text{AOct} = \{012, 045, 135, 234\} \subseteq [6]^3.$$

Then:

1. $\mathcal{B}_1 = \{[1]^1\}$,
2. $\mathcal{B}_2 = \{[1]^1, [2]^2, [3]^2\}$,
3. $\mathcal{B}_3 = \{[1]^1, [2]^2, [3]^3, [4]^3, \text{AOct}, \{013\}_5, \{013\}_6, \{013\}_7, [5]^3, L\}$.

The family AOct can be viewed as the collection of red faces of an octahedron whose faces are colored red and blue in such a way that no two adjacent faces share the same color; we call it the *alternating octahedron*.

It is not hard to see that $\mathcal{B}_1$ and $\mathcal{B}_2$ are as listed. Furthermore, using the equivalent condition in Theorem 4.18, it is routine to verify that the families in the lists above are minimal. This subsection is devoted to the proof of Theorem 4.28. We proceed by stages: In Theorem 4.30 we show that $\psi(3) = 10$, in Proposition 4.40 we show that the only large members of $\mathcal{B}_3$ are $[5]^3$ and L, and in Proposition 4.43 we show that no member of $\mathcal{B}_3$ can have size 8 or 9. Along the way, we also show in Proposition 4.41 that $\xi(1, 3) = 7$; this is an immediate consequence of Theorem 4.28, but providing a direct argument at that point is shorter. We then analyze the remaining possible sizes of members of $\mathcal{B}_3$ to conclude the proof.

Remark 4.29. Notice that every minimal family in $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ is low. In fact, every minimal family we mention in this paper is also low, suggesting that this might be a general phenomenon.

Theorem 4.30. $\psi(3) = 10$.

Although our argument is specific to the case $n = 3$, we try to illustrate some of the complexities that are present in the analysis of a general large family $\mathcal{A}$ of n-element sets.

Proof. Assume $\mathcal{A} \subseteq [X]^{\leq 3}$ is minimal and $|\mathcal{A}| \geq 10$. Since $\psi(2) = 3$, it then follows that $\mathcal{A} \subseteq [X]^3$ and also that neither S_1 nor S_2 is intersecting. Thus, by Lemma 4.14 we know $d_2 \leq 3$ and $d_1 \leq 3d_2$.

Claim 4.31. $d_1 \geq 4$. *Hence,* $d_2 \geq 2$.

Proof. This follows from the pigeonhole principle and only requires that $|\mathcal{A}| \geq 8$: Given any $A \in \mathcal{A}$, at least one of its elements belongs to at least three other members of $\mathcal{A}$. □

Lemma 4.32. $\mathcal{A}$ *is regular.*

Proof. Recall that $m_1 = |A \cap S_1|$ for any $A \in \mathcal{A}$. By minimality, $\mathcal{A}$ is regular if and only if $m_1 = 3$. Towards a contradiction, assume $\mathcal{A}$ is not regular, so $m_1 = 2$ by Lemma 4.10. Define d'_2 as

$$d'_2 := \max\{\deg_{\mathcal{A}}(\{x, y\}) : \{x, y\} \in [S_1]^2\}.$$

By minimality of $\mathcal{A}$, for each $A \in \mathcal{A}$, $A \cap S_1$ is contained in d'_2 elements of $\mathcal{A}$. Define a graph on S_1 by: $x_1 G x_2$ iff $\deg_{\mathcal{A}}(\{x_1, x_2\}) = d'_2$. Clearly, $d'_2 \leq d_1$.

Claim 4.33. $d'_2 = 1$.

Proof. Assume otherwise. Suppose a, b, c, d are distinct points in S_1 and $ab, cd \in G$. Then there are $x, y \notin S_1$ such that $abx, aby \in \mathcal{A}$, since $d'_2 > 1$. If $\{c, d\} \subset A \in \mathcal{A}$, then A meets at most one of these two sets, contradicting that $\mathcal{A}$ is intersecting.

It follows that $G \subseteq [S_1]^2$ is intersecting. Since $\psi(2) = 3$, from G we can further define an intersecting family of size at most 3 of 2-element sets (or a singleton). Since this family is definable from $\mathcal{A}$ (because G is), this contradicts that $\mathcal{A}$ is minimal. □

Our next goal is to show that (S_1, G) is complete.

Claim 4.34. *In* (S_1, G)*, given any* 3 *points, there is at most one edge missing. In particular,* G *is connected of diameter at most* 2.

Proof. Let x, y, z be distinct points in S_1 and suppose x is G-connected to neither y nor z.

Case 1. Suppose first that yGz, i.e., there is $A \in \mathcal{A}$ with $\{y, z\} \subset A$, and let w be the third member of A. Let B be any of the d_1 sets in $\mathcal{A}$ with $x \in B$ and notice that $\{w\} = B \cap A$. It follows that $\deg_{\mathcal{A}}(w) \geq d_1 + 1$, a contradiction.

Case 2. Otherwise. Now let $A \in \mathcal{A}$ be any set containing y, so A contains neither x nor z and any set in $\mathcal{A}$ containing either of them meets A in a point other than y. By the pigeonhole principle at least one element of A must have degree at least $(2d_1 + 1)/2 > d_1$, again a contradiction. □

Claim 4.35. (S_1, G) *is complete.*

Proof. Suppose otherwise, and let yz be a missing edge. For any $x \in S_1 \setminus \{y, z\}$, both xy and xz are in G. Since $\deg_{\mathcal{A}}(y) = d_1$, there are precisely d_1 such possible vertices x and $|S_1| = d_1 + 2$. Let $A \in \mathcal{A}$ with $\{x, y\} \subset A$ and let a be the third element of A, so $a \notin S_1$. Then for any of the $d_1 - 1$ remaining vertices w of S_1, if $\{w, z\} \subset B \in \mathcal{A}$ then a is also the third element of B. If follows that $\deg_{\mathcal{A}}(a) \geq d_1$ so $a \in S_1$, a contradiction. □

We are almost done now: (S_1, G) is complete and $|S_1| = d_1+1$. Let $x, y \in S_1$ and let $a \notin S_1$ be such that $\{a, x, y\} \in \mathcal{A}$. Then a also belongs to any set in $\mathcal{A}$ containing 2 of the remaining $d_1 - 1$ elements of S_1. Hence,

$$\deg_{\mathcal{A}}(a) \geq \binom{d_1 - 1}{2} + 1 \geq d_1$$

since $d_1^2 - 5d_1 + 4 = (d_1 - 4)(d_1 - 1) \geq 0$ as $d_1 \geq 4$, a contradiction. □

Remark 4.36. For future reference, we explain how to weaken the assumption that $|\mathcal{A}| \geq 10$ in the proof of Lemma 4.32 to $|\mathcal{A}| \geq 8$. The only place where the assumption was used was to conclude that $m_1 \geq 2$ via Lemma 4.10. So, assume that $|\mathcal{A}| = 8$ or 9 and that $m_1 = 1$, thus any $A \in \mathcal{A}$ contains exactly one element of degree d_1. Clearly, $|S_1| > 1$, by minimality of $\mathcal{A}$. Arguing as in the proof of Lemma 4.10, if $|S_1| = k + 1$ and $x \in S_1$, then there are kd_1 sets in $\mathcal{A}$ not containing x. Let $A \in \mathcal{A}$ extend $\{x\}$. Those kd_1 sets meet $A \setminus \{x\}$ so one of its two points, say a, satisfies $\deg_{\mathcal{A}}(a) \geq kd_1/2$. Thus $kd_1/2 < d_1$ or $k < 2$, so $k = 1$ and $|S_1| = 2$ so, since $d_1|S_1| = |\mathcal{A}|$, we have that $|\mathcal{A}| = 8$ and $d_1 = 4$.

Let $S_1 = \{z, w\}$ and let $\{x, y, z\}$ extend $\{w\}$, so x or y meets at least two of the four sets extending $\{z\}$, and it follows that $d_{1,2} \geq 3$. Since $d_{1,2} < d_{1,1} = d_1$, then $d_{1,2} = 3$. Since $3|S_{1,2}|/m_2 = |\mathcal{A}| = 8$ by Fact 4.11, we must have $m_2 \geq 3$. But $m_2 \leq 2$, a contradiction.

Lemma 4.32 means that $X = S_1$, i.e., every point has degree d_1. By a standard double counting argument, we have $3|\mathcal{A}| = d_1|X|$. We use this fact in conjunction with a more refined version of an earlier counting argument to bound the size of X.

Claim 4.37. $|X| < 8$.

Proof. By the principle of inclusion/exclusion, we have

$$|\mathcal{A}| = \sum_{D \in [A]^1} \deg_{\mathcal{A}}(D) - \sum_{D \in [A]^2} \deg_{\mathcal{A}}(D) + \sum_{D \in [A]^3} \deg_{\mathcal{A}}(D)$$

for any $A \in \mathcal{A}$. Since $\mathcal{A}$ is a regular family of triples, we have

$$|\mathcal{A}| \leq 3d_1 - (d_2 + 2) + 1.$$

Now, since $3d_2 \geq d_1$ and $3|\mathcal{A}| = d_1|X|$, we conclude

$$(|X| - 8)d_1 \leq -3,$$

which implies that $|X| < 8$. □

Using a similar technique, we can show that $\mathcal{A}$ is not regular on pairs whenever $|X| > 6$. Define m_2' as

$$m_2' = |\{D \in [A]^2 : \deg_{\mathcal{A}}(D) = d_2\}|,$$

for any $A \in \mathcal{A}$ (minimality of $\mathcal{A}$ ensures that m_2 is independent of the choice of A).

Claim 4.38. *Suppose that $|X| > 6$. Then $m_2' < 3$.*

Proof. Suppose, towards a contradiction, that $m_2' = 3$. Again using

$$|\mathcal{A}| = \sum_{D \in [A]^1} \deg_{\mathcal{A}}(D) - \sum_{D \in [A]^2} \deg_{\mathcal{A}}(D) + \sum_{D \in [A]^3} \deg_{\mathcal{A}}(D),$$

we see

$$|\mathcal{A}| = 3d_1 - 3d_2 + 1.$$

As before, since $3d_2 \geq d_1$ and $3|\mathcal{A}| = d_1|X|$, we may manipulate this to yield

$$(|X| - 6)d_1 \leq 1,$$

which contradicts Claim 4.31 whenever $|X| > 6$. $\square$

We are now in a position to slightly improve our bound on the size of X.

Claim 4.39. $|X| \neq 7$.

Proof. Suppose towards a contradiction that $|X| = 7$. Since $3|\mathcal{A}| = d_1|X|$, we must have that $|\mathcal{A}|$ is a multiple of 7. By Proposition 3.5, we have $|\mathcal{A}| \leq 25$, so $|\mathcal{A}| \in \{14, 21\}$.

Suppose first that $|\mathcal{A}| = 21$. Since $\binom{7}{2} = 21$ and each element of $\mathcal{A}$ contains three pairs, the average degree of a pair is 3. Since $d_2 \leq 3$, this implies that every pair has degree 3, contradicting Claim 4.38.

We then have that $|\mathcal{A}| = 14$. Mimicking the counting done above, we see that the average degree of a pair is 2. To avoid reaching a contradiction with Claim 4.38, we must then have $d_2 = 3$. Let us now attempt to count the number of pairs of degree 3. Since each $A \in \mathcal{A}$ contains m_2' pairs of degree 3, $m_2'|\mathcal{A}|$ counts each such pair three times. Thus, there must be 28/3 pairs of degree 3, which is absurd. $\square$

Thus, we have $|X| \leq 6$. If $|X| \leq 5$, then certainly $|\mathcal{A}| \leq \binom{5}{3} = 10$, so we may assume $|X| = 6$. Then for any $A \in [X]^3$, at most one of $A, X \setminus A$ is in $\mathcal{A}$, so $|\mathcal{A}| \leq \binom{6}{3}/2 = 10$. $\square$

We organized the argument above in a way that allows us to characterize the large $\mathcal{A} \subseteq [X]^{\leq 3}$.

Proposition 4.40. *Let $\mathcal{A} \subseteq [X]^{\leq 3}$ be large. Then either $\mathcal{A}$ is the family of triples from a 5-element set, or $\mathcal{A}$ is isomorphic to the family L of Example 4.5.*

Proof. Using notation as above, we see from the argument of Theorem 4.30 that if $\mathcal{A}$ is a large family of 3-element subsets of X, then $|X| \leq 6$ and if $|X| < 6$, then $|X| = 5$ and $\mathcal{A} = [X]^3$.

Assume now that $|X| = 6$. Then $\mathcal{A}$ is low, by Lemma 4.7. Recall that an intersecting family $\mathcal{B} \subseteq \mathcal{P}(Y)$ is called *maximal* iff for any $\mathcal{C}$, if $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{P}(Y)$, either $\mathcal{C}$ is no longer intersecting, or else $\mathcal{C} = \mathcal{B}$. Following Meyerowitz [7], we say that an element B of $\mathcal{B}$ is *minimal* iff there is no $A \subset Y$ such that $A \subsetneq B$ and $A \in \mathcal{B}$. It is trivial that any intersecting family is contained in a maximal one and that, if Y is finite, any maximal family has size $2^{|Y|-1}$. In particular, the collection of supersets of sets in $\mathcal{A}$ is maximal and the members of $\mathcal{A}$ are the minimal elements of this maximal family.

In Meyerowitz [7, Proposition 3.1], a list of all 30 maximal intersecting families (up to isomorphism) of a set of size 6 is presented; the families are generated by stages starting from the family of supersets of a singleton $\{a\}$ by means of shifts, and are enumerated according to the number of shifts required to generate them. The families listed in Meyerowitz [7] as A, B, C, D, $E1$, $E2$, $F1$, $F2$, $G1$–$G3$, $H1$–$H4$, $I1$–$I6$, $J2$–$J4$ and $K1$ are not the families of supersets of the members of a large minimal $\mathcal{A}$ (in our sense) with $X = \bigcup \mathcal{A}$, since they all have at least one minimal element (in the sense of Meyerowitz [7]) of size other than 3. The families $J1$, $K2$ and $K3$ are not as required either, since from each of them an element of X is definable. The triple $\{a, b, c\}$ is definable in $K4$. The only remaining example is L, which means that $\mathcal{A}$ must be isomorphic to the family of Example 4.5, which is itself isomorphic to L (and provides a new explanation as to why all the families from Example 4.6 are isomorphic to the one from Example 4.5 as well). □

In §3 we stated that $\xi(1, 3) = 7$. Theorem 4.30 provides us with an easy way to show this.

Proposition 4.41. $\xi(1, 3) = 7$.

Proof. $\xi(1, 3) \geq 7$ by Theorem 3.10.2. Assume towards a contradiction that $\mathcal{A} \subseteq [X]^3$ is intersecting and such that any non-empty subset of X definable from $\mathcal{A}$ has size at least 8. By passing to a smaller definable intersecting family if necessary, we may assume that $\mathcal{A}$ is minimal, so $|\mathcal{A}| \leq 10$. We use notation as in Fact 4.11 and Theorem 4.30.

Claim 4.42. *$\mathcal{A}$ is regular.*

Proof. Assume otherwise.

Case 1. There are three distinct degrees of elements of X. Then

$$\begin{aligned} 3|\mathcal{A}| &= d_{1,1}|S_{1,1}| + d_{1,2}|S_{1,2}| + d_{1,3}|S_{1,3}| \geq 8(d_{1,1} + d_{1,2} + d_{1,3}) \\ &\geq 8(3 + 2 + 1) > 3 \times 10 \geq 3|\mathcal{A}|, \end{aligned}$$

a contradiction.

Case 2. There are two distinct degrees. Then

$$3|\mathcal{A}| = d_{1,1}|S_{1,1}| + d_{1,2}|S_{1,2}| \geq 8(2+1),$$

so $|\mathcal{A}| \geq 8$, so $d_{1,1} \geq 4$ by Claim 4.31, so

$$3|\mathcal{A}| \geq 8(4+1) > 3 \times 10,$$

again a contradiction. □

Then

$$8d_1 \leq |S_1|d_1 = 3|\mathcal{A}| \leq 30,$$

so $d_1 \leq 3$. On the other hand, by Proposition 3.6,

$$8 \leq |S_1| \leq 9 - \frac{6}{d_1},$$

or $6 \leq d_1$, a contradiction. □

Proposition 4.43. *There is no minimal $\mathcal{A} \subseteq [X]^3$ with $|\mathcal{A}| = 8$ or 9.*

Proof. Suppose towards a contradiction that we have a minimal $\mathcal{A} \subseteq [X]^3$ with $|\mathcal{A}| = 8$ or 9. Examining the proof of Theorem 4.30, we only used that $|\mathcal{A}| \geq 8$ to conclude that $d_1 \geq 4$, $\mathcal{A}$ is regular, and $|X| \leq 6$. Thus $|X| = 5$ or 6. A cursory examination of the equation

$$3|\mathcal{A}| = d_1|X|$$

reveals that the only solution fitting the above constraints is $|\mathcal{A}| = 8$, $|X| = 6$, and $d_1 = 4$. Using, once again, the inclusion/exclusion counting as above, we have for any $A \in \mathcal{A}$,

$$|\mathcal{A}| = 3 \times 4 - \sum_{D \in [A]^2} \deg_{\mathcal{A}}(D) + 1,$$

which implies that $\sum_{D \in [A]^2} \deg_{\mathcal{A}}(D) = 5$. In particular, every $A \in \mathcal{A}$ contains a pair D with $\deg_{\mathcal{A}}(D) = 1$; there must therefore be at least 8 such pairs. Thus, at most $\binom{6}{2} - 8 = 7$ pairs have degree greater than 1. Then

$$\begin{aligned} 24 &= 3|\mathcal{A}| \\ &= \sum_{D \in [X]^2} \deg_{\mathcal{A}}(D) \\ &\leq 8 + 7d_2, \end{aligned}$$

so $d_2 = 3$. Since $\sum_{D \in [A]^2} \deg_{\mathcal{A}}(D) = 5$, this means every $A \in \mathcal{A}$ contains one pair of degree 3 and two of degree 1. Thus, there must be a total of 16 pairs of degree 1, a contradiction. □

Remark 4.44. Let X be the set of vertices of an octahedron $\mathcal{H}$ and let $\mathcal{A} \subseteq [X]^3$ consist of the sets of vertices of the faces of $\mathcal{H}$. Then $|X| = 6$, $|\mathcal{A}| = 8$, $d_1 = 4$, and $\mathcal{A}$ is low. The family $\mathcal{A}$ is not intersecting, although it is 3-cc. This example may help explain why some care was required in the proof of Proposition 4.43.

We now conclude the proof of Theorem 4.28:

Proof. Suppose that $\mathcal{A} \subseteq [X]^{\leq 3}$ is minimal, and for convenience suppose that $X = \bigcup \mathcal{A}$. Obviously, if $|\mathcal{A}| = 1$ then $\mathcal{A} \in \{[1]^1, [2]^2, [3]^3\}$ and those families are certainly minimal. We thus assume that $|\mathcal{A}| > 1$. In this case $\mathcal{A} \subseteq [X]^3$: if $\mathcal{A}$ contains a set of size 2 or smaller, then we may define from $\mathcal{A}$ one of the elements of $\mathcal{B}_2$ above, and since $[3]^3$ is definable from $[3]^2$ we can in fact define from $\mathcal{A}$ a family of cardinality 1.

In turn, if the set $Y = \{x \in X : \deg_{\mathcal{A}}(x) = 1\}$ is non-empty, then the family $\mathcal{A}' = \{A \cap (X \backslash Y) : A \in \mathcal{A}\}$ is intersecting, definable from $\mathcal{A}$, and contains sets of cardinality less than 3. Thus, as above, a family of size 1 is then definable from $\mathcal{A}'$, and hence from $\mathcal{A}$. Consequently, we may assume that every point has degree at least 2. Then we have

$$3|\mathcal{A}| = \sum_{x \in X} \deg_{\mathcal{A}}(x) \geq 2|X|,$$

giving the crude bound $|X| \leq \frac{3}{2}|\mathcal{A}|$.

We now consider several cases based upon the value of $|\mathcal{A}|$, recalling that we have already handled $|\mathcal{A}| \geq 8$ and $|\mathcal{A}| = 1$.

$|\mathcal{A}| = 2$: By the above bound, we have $|X| \leq 3$, which prevents this case from being realized.

$|\mathcal{A}| = 3$: We have $|X| \leq 4$. If $|X| < 4$, or if a proper subset of X were definable from $\mathcal{A}$, then there would be $\mathcal{A}' \subseteq [X]^{\leq 3}$ definable from $\mathcal{A}$ with $|\mathcal{A}'| = 1$, contradicting minimality. Thus, $|X| = 4$ and $\mathcal{A}$ is regular, so we have $3|\mathcal{A}| = d_1|X|$, which has no solutions when $|\mathcal{A}| = 3$ and $|X| = 4$.

$|\mathcal{A}| = 4$: We have $|X| \leq 6$. If $|X| < 4$ or if a proper subset of $|X|$ were definable from $\mathcal{A}$, then there would be $\mathcal{A}' \subseteq [X]^{\leq 3}$ definable from $\mathcal{A}$ with $|\mathcal{A}'| = 1$, contradicting minimality. Thus $|X| \geq 4$ and $\mathcal{A}$ is regular, so we have $12 = d_1|X|$. This has two solutions: $|X| = 4, d_1 = 3$, which corresponds to $[4]^3$, and $|X| = 6, d_1 = 2$, which corresponds to AOct.

To see this, note first that if $|X| = 4$, then it is clear that $\mathcal{A}$ must be isomorphic to $[4]^3$, so we focus our attention on the case that $|X| = 6$. As in the proof of Theorem 4.30, we count $|\mathcal{A}|$ by inclusion/exclusion. That is, $|\mathcal{A}| = 4 = 3 \times 2 - \sum_{D \in [A]^2} \deg_{\mathcal{A}}(D) + 1$, so $\sum_{D \in [A]^2} \deg_{\mathcal{A}}(D) = 3$. Thus $d_2 = 1$, so any pair of elements of X is contained in at most one element of $\mathcal{A}$. Without loss of generality, suppose that $012 \in \mathcal{A}$. Some other element of $\mathcal{A}$ contains 0 but not 1 or 2, without loss let us assume it is 045. Similarly, an element of $\mathcal{A}$ contains 1 but not 0 or 2. It must also intersect 045, so, reversing the labels of 4 and 5 if necessary, it is 135. Since $d_1 = 3$, the last element of $\mathcal{A}$ must be 234, so $\mathcal{A} = \{012, 045, 135, 234\} = \text{AOct}$.

$|\mathcal{A}| = 5$: We have $|X| \leq 7$. If $|X| < 4$ or if a proper subset of $|X|$ were definable from $\mathcal{A}$, then there would be $\mathcal{A}' \subseteq [X]^{\leq 3}$ definable from $\mathcal{A}$ with $|\mathcal{A}'| = 1$, contradicting minimality. Thus $|X| \geq 4$ and $\mathcal{A}$ is regular, so we have $15 = d_1|X|$. This has a unique solution: $|X| = 5, d_1 = 3$.

We count $|\mathcal{A}|$ by inclusion/exclusion, so $5 = 3 \times 3 - \sum_{D \in [A]^2} \deg_{\mathcal{A}}(D) + 1$, or $\sum_{D \in [A]^2} \deg_{\mathcal{A}}(D) = 5$. It cannot be the case that each element of $\mathcal{A}$ contains two pairs of degree 1, since all 10 pairs in $[X]^2$ would have degree 1, contradicting the fact that $\mathcal{A}$ counts 15 pairs (including multiplicity). Thus, it must be the case that each element of $\mathcal{A}$ contains one pair of degree 1 and two pairs of degree 2. This means that five of the pairs in $[X]^2$ have degree 1 and five have degree 2. Suppose that 013 is in $\mathcal{A}$, with 01 having degree 1 and 13, 03 having degree 2. Some other element of $\mathcal{A}$ must contain 03; call it 023. Similarly, some other element contains 13, so it is either 134 or 123. If it were 123, then no triple can contain 4 and two of 0,1,2,3. Thus, $134 \in \mathcal{A}$. Then the only triples that can contain the pair 24 are 024 and 124; without loss of generality assume it is 024. After that, 1, 2, and 4 are left with degree less than 3, so the last element must contain all of them. Therefore, $\mathcal{A} = \{013, 124, 023, 134, 024\} = \{013\}_5$.

$|\mathcal{A}| = 6$: We have $|X| \leq 9$. If $|X| < 4$ or if a proper subset of $|X|$ were definable from $\mathcal{A}$, then there would be $\mathcal{A}' \subseteq [X]^{\leq 3}$ definable from $\mathcal{A}$ with $|\mathcal{A}'| = 1$, contradicting minimality. Thus $|X| \geq 4$ and $\mathcal{A}$ is regular, so we have $18 = d_1|X|$. This has two solutions: $|X| = 9, d_1 = 2$ and $|X| = 6, d_1 = 3$. If $|X| = 9$, then counting $|\mathcal{A}|$ by inclusion/exclusion yields $6 = 2 \times 3 - \sum_{D \in [A]^2} \deg_{\mathcal{A}}(D) + 1$, so $\sum_{D \in [A]^2} \deg_{\mathcal{A}}(D) = 1$, which is absurd.

So, $|X| = 6$. Again counting $|\mathcal{A}|$ we have $6 = 3 \times 3 - \sum_{D \in [A]^2} \deg_{\mathcal{A}}(D) + 1$, so we have $\sum_{D \in [A]^2} \deg_{\mathcal{A}}(D) = 4$. This means each $A \in \mathcal{A}$ contains two pairs of degree 1 and one pair of degree 2. Then a total of twelve pairs in $[X]^2$ have degree 1, so the other three must have degree 2. The three pairs of degree 2 must be disjoint, since their points of intersection would form a definable set of size at most 3. Without loss of generality, let us label the pairs of degree two 03, 14, 25, and assume that 013 is one of the elements of $\mathcal{A}$ containing 03. The pair 04 must be contained in some element of $\mathcal{A}$; since it must contain a pair of degree 2 it must be either 034 or 014. However, if it were 014, the pair 01 would have degree greater than 1, so we must have $034 \in \mathcal{A}$. No element of $\mathcal{A}$ containing 14 can contain 0 or 3 since we have already used the pairs 01, 04, 13, 34, so we must have 124 and $145 \in \mathcal{A}$. Continuing in this fashion gives $\mathcal{A} = \{013, 124, 235, 034, 145, 025\} = \{013\}_6$.

$|\mathcal{A}| = 7$: We have $|X| \leq 10$. If a set of size smaller than 5 were definable from $\mathcal{A}$, then an intersecting family of size at most 4 would be definable, contradicting minimality. Thus, if $\mathcal{A}$ is not regular it could have at most two distinct degrees, say $d_{1,1}$ and $d_{1,2}$, each witnessed by five points of X. But if this were the case, then we would have

$$21 = 3|\mathcal{A}| = \sum_{x \in X} \deg_{\mathcal{A}}(x) = 5d_{1,1} + 5d_{1,2},$$

which has no solutions. Thus, $\mathcal{A}$ is regular, so we have $21 = d_1|X|$, which implies that $|X| = 7$ and $d_1 = 3$.

Counting $|\mathcal{A}|$ by inclusion/exclusion, we see $7 = 3\times 3 - \sum_{D\in[A]^2} \deg_{\mathcal{A}}(D)+1$, so $\sum_{D\in[A]^2} \deg_{\mathcal{A}}(D) = 3$. This means each $A \in \mathcal{A}$ contains three pairs of degree one. Since a total of $3|\mathcal{A}| = 21$ pairs are contained in elements of $\mathcal{A}$, we conclude that every pair in $[X]^3$ is contained in exactly one element of $\mathcal{A}$. Without loss of generality, suppose that 013, 026, and 045 are the three elements of $\mathcal{A}$ containing the point zero. Some element of $\mathcal{A}$ contains the pair 12; ruling out the edges already used, the only possibilities are 124 and 125. Switching the labels of 4 and 5 if necessary, we may assume 124 is in $\mathcal{A}$. Now, some triple must contain 23, and the only possibility is 235. Proceeding in this fashion, the only triple which can contain 46 is 346, and the only triple which can contain 16 is 156. Thus, $\mathcal{A} = \{013, 124, 235, 346, 045, 156, 026\} = \{013\}_7$. □

5 Low intersecting families

In the previous section we started from an intersecting family $\mathcal{A} \subseteq [X]^{\leq n}$ and investigated bounds on the sizes of intersecting families $\mathcal{A}' \subseteq [X]^{\leq n}$ definable from $\mathcal{A}$. Here we restrict our attention to those $\mathcal{A}'$ that are themselves subsets of $\mathcal{A}$.

Definition 5.1. An intersecting family $\mathcal{A}$ is *quasi-low* (*qlow*) iff $\mathrm{Aut}(\mathcal{A})$ acts transitively on $\mathcal{A}$.

It would seem reasonable to study qlow families $\mathcal{A} \subseteq [X]^n$, since no proper subfamily of such an $\mathcal{A}$ is definable. However, if one is interested in bounding the size of $\mathcal{A}$, this is not the right notion to investigate, since no such bound exists:

Example 5.2. Given a set X, natural numbers $m < n$, and $B_0 \in [X]^m$, the family $\mathcal{A} = \{A \in [X]^n : B_0 \subseteq A\}$ is qlow.

Example 5.3. In Example 3.1, assume $|A| = N$. Then $\mathcal{A}$ is qlow of size $3N$.

Hence, we restrict our attention to low families.

Definition 5.4. Let $\rho(n) = \max\{|\mathcal{A}| : \mathcal{A} \subseteq [X]^n$ is low and intersecting$\}$.

From the previous section, $\binom{2n-1}{n} \leq \rho(n)$, so $\rho(3) \geq 10$. Similarly, $\rho(n) > (n/2)^{n/2}$ for all $n > 6$. Notice that ρ is well-defined:

Lemma 5.5. *Let $\mathcal{A} \subseteq [X]^n$ be low. If $n = 2$, then $|\mathcal{A}| \leq 3$. If $n \geq 3$, then $|\mathcal{A}| < \binom{n^2-2}{n-1}$.*

Proof. Let $\mathcal{A} \subseteq [X]^n$. If $|X| < 2n$, clearly $|\mathcal{A}| \leq \binom{|X|}{n} \leq \binom{2n-1}{n} \leq \rho(n)$. By Theorem 3.7, $|X| \leq n^2 - 1$. It follows that $\rho(2) = 3$.

If $|X| \geq 2n$ and $\mathcal{A}$ is intersecting, by the Erdős-Ko-Rado theorem we have that $|\mathcal{A}| \leq \binom{|X|-1}{n-1}$, so

$$\rho(n) \leq \binom{n^2-2}{n-1}.$$

In fact, if $n \geq 3$ the inequality is strict since $n^2 - 1 > 2n$ and equality in the Erdős-Ko-Rado theorem requires that the family $\mathcal{A}$ consists of all n-element subsets of a set X of size $n^2 - 1$ that contain a fixed element of X. This element is therefore definable from $\mathcal{A}$, so $\mathcal{A}$ is not low. □

Theorem 5.6. $\rho(3) = 10$.

Proof. Observe first that to adapt the argument of Theorem 4.30, it suffices to check that a low $\mathcal{A} \subseteq [X]^3$ with $|\mathcal{A}| \geq 10$ satisfies alternative (2) of Lemma 4.14 for $m \in \{1, 2\}$. Aside from these bounds on the degrees, the only use of minimality in the proof of Theorem 4.30 is to ensure that no proper subset of $\mathcal{A}$ is definable from $\mathcal{A}$; this holds whenever $\mathcal{A}$ is low.

As usual, we may assume $\mathcal{A}$ is finite. Suppose, towards a contradiction, that either S_1 or S_2 is intersecting. Then, by Theorem 3.7, we may define from $\mathcal{A}$ a subset $Y \subseteq X$ with $|Y| \leq 3$. Since $\mathcal{A}$ is low, we have $|X| \leq 3$, which contradicts $|\mathcal{A}| \geq 10$. □

We now introduce an operation $*$ that allows us to "lift" small families to larger ones.

Definition 5.7. Let $\mathcal{A} * \mathcal{B} = \{\bigcup_{x \in A}\{x\} \times B_x : B_x \in \mathcal{B}, A \in \mathcal{A}\}$.

Lemma 5.8. *Suppose $\mathcal{A} \subseteq [X]^m$ and $\mathcal{B} \subseteq [Y]^n$ are qlow, and let $\mathcal{C} = \mathcal{A} * \mathcal{B}$. Then $\mathcal{C} \subseteq [X \times Y]^{m \times n}$ is qlow and $|\mathcal{C}| = |\mathcal{A}||\mathcal{B}|^m$. If $\mathcal{A}$ and $\mathcal{B}$ are low, then so is $\mathcal{C}$.*

Proof. Notice that $\mathcal{C}$ is intersecting and has the claimed size. Let $C_1, C_2 \in \mathcal{C}$. For $C \in \mathcal{C}$ let $A_C := \mathrm{proj}_1(C)$ be its first coordinate projection, so $A_C = \{x : \exists y\, (x, y) \in C\}$. There is no loss of generality in assuming that for all $x \in A_{C_1}$, $B_{C_1,x} = B$ is a fixed element of $\mathcal{B}$, as some automorphism of $\mathcal{B}$ sends $B_{C_1,x}$ to B and this induces an automorphism of $\mathcal{C}$ that sends C_1 to $A_{C_1} \times B$.

Similarly, there is an automorphism of $\mathcal{C}$ sending C_2 to $A_{C_2} \times B$, and there is an automorphism of $\mathcal{A}$ that sends A_{C_1} to A_{C_2}. This automorphism lifts to an automorphism of $\mathcal{C}$, and appropriately composing these automorphisms we find one that sends C_1 to C_2, so $\mathcal{C}$ is qlow.

An easy extension of this argument shows that $\mathcal{C}$ is in fact low if $\mathcal{A}$ and $\mathcal{B}$ are low. □

Definition 5.9. The *powers* of $\mathcal{A}$ are given by $\mathcal{A}^{(1)} = \mathcal{A}$ and $\mathcal{A}^{(k+1)} = \mathcal{A}^{(k)} * \mathcal{A}$.

Fix n and a low $\mathcal{A} \subseteq [X]^n$ such that $|\mathcal{A}| > c^n$, say $|\mathcal{A}| \geq c_1^n$ where $c_1 > c$. Then

$$|\mathcal{A}^{(k)}| = |\mathcal{A}|^{\frac{n^k-1}{n-1}},$$

as a straightforward induction establishes, since $|A^{(k+1)}| = |\mathcal{A}^{(k)}||\mathcal{A}|^{n^k}$.

If $k > 1$, then $|\mathcal{A}^{(k)}| \geq (c_1^n)^{\frac{n^k-1}{n-1}} > c_1^{n^k}$ and $\mathcal{A}^{(k)} \subseteq [X]^{n^k}$. This gives us an infinite sequence of low families that "grow faster" than c^n, obtained by means of our lifting operation.

Remark 5.10. Notice that $*$ is not commutative, but it is associative (up to isomorphism), so $\mathcal{A}^{(k+1)} \cong \mathcal{A} * \mathcal{A}^{(k)}$.

Example 5.11. A "Sierpinski-like" sequence of low families can be obtained by starting with $n = n^1 = 2$ and $\mathcal{A}$ the collection of 2-element subsets of a set of size 3; our construction then produces for infinitely many values of n (namely, the powers of 2) a low $\mathcal{A}$ of n-element subsets, with $|\mathcal{A}| = 3^{n-1}$. In general, we can produce this way, for each k, low families $(\mathcal{A}_n : n \in \mathbb{N})$ with $|A_n| = \left(\sqrt[k-1]{\binom{2k-1}{k}} \right)^{k^n - 1}$ and $\mathcal{A}_n \subset [X]^{k^n}$.

Remark 5.12. In general, the operation $*$ does not preserve minimality. To see this, let $\mathcal{A}$ be minimal, let $|Y| = 3$ and consider $\mathcal{B} = \mathcal{A} * [Y]^2$ where each point of $X = \bigcup \mathcal{A}$ is replaced by a "block" of 3 points. By considering invariance under $\mathrm{Aut}(\mathcal{B})$, it follows that the family $\mathcal{C}$ consisting of those $A \in \mathcal{B}$ that contain one of these blocks of size 3 and exactly one point from each other block, is definable from $\mathcal{B}$, by Lemma 2.12, so $\mathcal{B}$ is not minimal. It is worth noting that an analysis of this example led to the lower bounds in Theorem 4.20.

Question 5.13. *Is* $\mathrm{Aut}(\mathcal{A} * \mathcal{B}) \cong \mathrm{Aut}(\mathcal{A}) * \mathrm{Aut}(\mathcal{B})^n$*, where n is the size of the sets in A?*

Recall that $\mathcal{A}$ is called k-wise t-intersecting iff the intersection of any k members of $\mathcal{A}$ has size at least t, so $\mathcal{A}$ is intersecting iff it is 2-wise 1-intersecting. It seems worthwhile to explore how the bounds obtained throughout this paper are affected if one now assumes that the family $\mathcal{A}$ is k-wise t-intersecting rather than $(k+1)$-cc or simply intersecting.

6 Ccc families of countable sets

Let $[X]^{\mathbb{N}}$ denote the family of countably infinite subsets of X. Let $\mathfrak{c} := |[\mathbb{N}]^{\mathbb{N}}|$, so $\mathfrak{c} = |\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$. In Example 2.9 we exhibited a low intersecting family $\mathcal{A} \subseteq [\aleph_1]^{\mathbb{N}}$ of cardinality $\mathfrak{c}$. We begin by extending this result:

Theorem 6.1. *Suppose that κ is an infinite cardinal. Then there is a low intersecting family $\mathcal{A} \subseteq [\kappa]^{\mathbb{N}}$ of cardinality $\kappa^{\aleph_0}$.*

Proof. Let $X = \mathbb{Z} \times \kappa$. As $|X| = \kappa$, it is sufficient to produce a low intersecting family $\mathcal{A} \subseteq [X]^{\mathbb{N}}$ of cardinality $\kappa^{\aleph_0}$. For each $f \in \kappa^{\mathbb{Z}}$, define $A_f \subseteq X$ by

$$A_f = \mathrm{graph}(f) = \{(n, f(n)) : n \in \mathbb{Z}\},$$

and let $\mathcal{A} = \{A_f : f \in \kappa^{\mathbb{Z}} \text{ and } \exists n \in \mathbb{Z} \forall m \leq n\ (f(m) = 0)\}$. It is clear that $\mathcal{A}$ is intersecting, $\mathcal{A} \subseteq [X]^{\mathbb{N}}$, and $|\mathcal{A}| = \kappa^{\aleph_0}$.

It remains to verify that $\mathcal{A}$ is low. Towards this end, suppose that $A_f, A_g \in \mathcal{A}$, and define $\pi : X \to X$ by

$$\pi(n, \alpha) = \begin{cases} (n, g(n)) & \text{if } \alpha = f(n), \\ (n, f(n)) & \text{if } \alpha = g(n), \\ (n, \alpha) & \text{otherwise.} \end{cases}$$

It is clear that $\pi \in \mathrm{Aut}(\mathcal{A})$ and π sends A_f to A_g.

Similarly, given $x = (n_x, \alpha_x)$ and $y = (n_y, \alpha_y)$, define $\pi : X \to X$ by

$$\pi(n, \alpha) = \begin{cases} (n + (n_y - n_x), \alpha_y) & \text{if } \alpha = \alpha_x \text{ and } n \geq n_x, \\ (n + (n_x - n_y), \alpha_x) & \text{if } \alpha = \alpha_y \text{ and } n \geq n_y, \\ (n, \alpha) & \text{otherwise.} \end{cases}$$

It is clear that $\pi \in \mathrm{Aut}(\mathcal{A})$ and π sends x to y. □

While Theorem 6.1 rules out the possibility of defining small sets from intersecting families of infinite sets in general, there is nevertheless a natural special case in which this is possible. First, we must return to families of finite sets:

Proposition 6.2. *Suppose that $\mathcal{A} \subseteq [X]^n$ is ccc and uncountable. Then there is a positive integer $m < n$ such that $\mathcal{A}^{(m,\aleph_0)}$ is non-empty and ccc.*

Proof. The argument to follow resembles closely the standard proof of the Δ-system (or sunflower) lemma of Erdős and Rado, and also the proof of Lemma 3.4. Our assumptions on $\mathcal{A}$ ensure that $n \geq 2$ and $\mathcal{A}^{(1,\aleph_0)}$ is non-empty. Fix $m < n$ largest such that $\mathcal{A}^{(m,\aleph_0)}$ is non-empty. (Note that $\mathcal{A}^{(m,\aleph_0)}$ coincides with the set of kernels of maximal size of uncountable Δ-systems of members of $\mathcal{A}$.) Suppose, towards a contradiction, that $\mathcal{A}^{(m,\aleph_0)}$ is not ccc, and fix an antichain $\langle B_\alpha \rangle_{\alpha < \omega_1}$ of elements of $\mathcal{A}^{(m,\aleph_0)}$. Given an ordinal $\alpha < \omega_1$ and a sequence $\langle A_\beta \rangle_{\beta < \alpha}$ of elements of $\mathcal{A}$, fix $\gamma < \omega_1$ such that B_γ is disjoint from the set $X_\alpha = \bigcup_{\beta < \alpha} A_\beta$. The maximality of m ensures that for each $x \in X_\alpha$, there are only countably many $A \in \mathcal{A}$ such that $\{x\} \cup B_\gamma \subseteq A$, thus there exists $A_\alpha \in \mathcal{A}$ such that $B_\gamma \subseteq A_\alpha$ and A_α is disjoint from X_α. Then $\langle A_\alpha \rangle_{\alpha < \omega_1}$ is an antichain contained in $\mathcal{A}$, the desired contradiction. □

Given a sequence $(m) = (m_0, \ldots, m_\ell)$ of positive integers, define

$$\mathcal{A}^{(m)} = ((\mathcal{A}^{(m_0,\aleph_0)})^{(m_1,\aleph_0)} \cdots)^{(m_\ell,\aleph_0)}.$$

Corollary 6.3. *Suppose that $A \subseteq [X]^n$ is ccc and uncountable. Then there is a positive integer $\ell < n$ and positive integers $m_1 > \cdots > m_\ell$ such that, setting $(m) = (m_1, \ldots, m_\ell)$, the family $\mathcal{A}^{(m)}$ is non-empty and countable.*

Proof. This follows easily from Proposition 6.2. □

In particular, we obtain the following:

Proposition 6.4. *Suppose that $\mathcal{A} \subseteq [X]^{\mathbb{N}}$ is non-empty and there is a finite set $F \subseteq X$ such that $\forall A \in \mathcal{A}$ $(A \cap F \neq \emptyset)$. Then a non-empty countable subset of X is definable from $\mathcal{A}$.*

Proof. Fix $n \in \mathbb{N}$ such that the set $\mathcal{B} = \{B \in [X]^n : \forall A \in \mathcal{A}\ (A \cap B \neq \emptyset)\}$ is non-empty. As our hypotheses ensure that $\mathcal{A}$ is ccc, it follows that $\mathcal{B}$ is also ccc, and Corollary 6.3 implies that there is a finite sequence (s) such that the set $\mathcal{B}^{(s)}$ is non-empty and countable. □

Notice that no such result is possible if we weaken the assumption on F. For example, if $\mathcal{A}, X$ are as in Theorem 6.1, then $F = \{(n, 0) : n \in \mathbb{Z}\}$ is countable yet $\mathcal{A}$ is intersecting and low.

For any $n \in \mathbb{N}$, it is easy to provide examples where the situation of Proposition 6.4 holds, $\mathcal{A}$ is intersecting, any F meeting each $A \in \mathcal{A}$ has size larger than n, and no non-empty finite subset of X is definable from $\mathcal{A}$.

Next, we show that if $\mathcal{A} \subseteq [X]^{\mathbb{N}}$ is non-empty and ccc, and both X and $\mathcal{A}$ satisfy a natural definability condition, then a non-empty, countable subset of X is definable from appropriate witnesses to the definability of X and $\mathcal{A}$. For this, we require some basic notions from descriptive set theory. We refer the reader to either Kechris [6] or Moschovakis [9] for a detailed introduction to the subject, although we will now summarize briefly the descriptive set-theoretic facts that we will use.

Recall that a topological space X is *Polish* if it is separable and admits a complete metric. It is not difficult to see that if X is Polish, then $X^{\mathbb{N}}$ is Polish. Associated with each family $\mathcal{A} \subseteq [X]^{\mathbb{N}}$ is the set $\tilde{\mathcal{A}} = \{\mathbf{x} \in X^{\mathbb{N}} : \{\mathbf{x}(n)\}_{n \in \mathbb{N}} \in \mathcal{A}\}$.

For $s \in \mathbb{N}^{<\mathbb{N}}$ and $t \in \mathbb{N}^{\leq \mathbb{N}}$, we denote the *concatenation* of s and t by st, and the restriction of s to $\{i \in \mathbb{N} : i < n\}$ (where $n \leq |\operatorname{dom}(s)|$) by $s|n$. If $s \in \mathbb{N}^{<\mathbb{N}}$ and $i \in \mathbb{N}$, we use si to denote st where $\operatorname{dom}(t) = \{0\}$ and $t(0) = i$. We denote the lexicographic ordering on $\mathbb{N}^{<\mathbb{N}}$ by $<_{\text{lex}}$.

We identify $u \in 2^n$ with the function $\hat{u} \in \{0,1\}^n$ given by $\hat{u}(k) = 1$ iff the k^{th} position (from the right) in the base 2 representation of u is 1, and $\hat{u}(k) = 0$ otherwise.

For each sequence $\mathcal{X} = \langle X_s \rangle_{s \in \mathbb{N}^{<\mathbb{N}}}$ and $s \in \mathbb{N}^{<\mathbb{N}}$, let

$$\mathcal{X}_s = \bigcup_{t \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} X_{(st)|n}.$$

A *Souslin scheme* for a set $A \subseteq X$ is a sequence $\mathcal{X} = \langle X_s \rangle_{s \in \mathbb{N}^{<\mathbb{N}}}$ such that

$$A = \mathcal{X}_\emptyset \text{ and } \forall n \in \mathbb{N}\, \forall s \in \mathbb{N}^n\ (\operatorname{diam}(X_s) \leq 1/(n+1)),$$

where $\operatorname{diam}(X_s)$ denotes the *diameter* of X_s with respect to some fixed compatible, complete metric on X. We say that $\mathcal{X}$ is *regular* if in addition each X_s is closed in X. A subset A of X is said to be *analytic*, or $\boldsymbol{\Sigma}^1_1$, if it admits a regular Souslin scheme $\mathcal{X}$. This is equivalent to stating that A is the continuous image of a Polish space; the Souslin scheme provides us with an explicit *presentation* of the continuous map, see Kechris [6, §25]. The analytic sets form a natural family of concretely definable sets that includes all Borel subsets of X.

If A is analytic and non-empty, one can further require that each $X_s \neq \emptyset$, in which case there is a continuous function $f : \mathbb{N}^{\mathbb{N}} \to X$ such that

$$\{f(t)\} = \bigcap_n X_{t|n}$$

for all $t \in \mathbb{N}^{\mathbb{N}}$, so $f[\mathbb{N}^{\mathbb{N}}] = \mathcal{X}_{\emptyset} = A$; here, $\mathbb{N}^{\mathbb{N}}$ is given the product topology of countably many discrete copies of $\mathbb{N}$. We assume our Souslin schemes have these additional properties in what follows.

Theorem 6.5. *Suppose that X is a Polish space, $\mathcal{A} \subseteq [X]^{\mathbb{N}}$ is non-empty and $\mathfrak{c}$-cc, $\mathcal{X} = \langle X_s \rangle_{s \in \mathbb{N}^{<\mathbb{N}}}$ is a Souslin scheme for X, and $\mathcal{X}^{\mathcal{A}} = \langle X_s^{\mathcal{A}} \rangle_{s \in \mathbb{N}^{<\mathbb{N}}}$ is a Souslin scheme for $\tilde{\mathcal{A}}$. Then a non-empty, countable subset of X is definable from $\mathcal{X}$ and $\mathcal{X}^{\mathcal{A}}$.*

Proof. Given a sequence $\mathbf{x} \in X^{\mathbb{N}}$, we will also use $\mathbf{x}$ to denote the corresponding set $\{\mathbf{x}(n) : n \in \mathbb{N}\}$. We say that a pair $(s, t) \in \mathbb{N}^n \times (\mathbb{N}^k)^n$ is *stable* if

$$X_s^{\mathcal{A}} \cap \bigcap_{m<n} X_{t(m)} \neq \emptyset \text{ and } \exists F \subseteq X \text{ finite} \, \forall \mathbf{x} \in X_s^{\mathcal{A}} \cap \bigcap_{m<n} X_{t(m)} \; (\mathbf{x} \cap F \neq \emptyset).$$

Lemma 6.6. *The set $\bigcup_{k,n \in \mathbb{N}} \mathbb{N}^n \times (\mathbb{N}^k)^n$ contains a stable pair.*

Proof. Suppose, towards a contradiction, that there is no such pair. We will recursively find $\langle k_n \rangle \in \mathbb{N}^{\mathbb{N}}$, $\langle s_u \rangle \in (\mathbb{N}^n)^{2^n}$, and $\langle t_u \rangle \in ((\mathbb{N}^{k_n})^n)^{2^n}$ such that:

1. $\forall n \in \mathbb{N} \; (k_n < k_{n+1})$;
2. $\forall u \subseteq v \; (s_u \subseteq s_v \text{ and } t_u \subseteq t_v)$;
3. $\forall n \in \mathbb{N} \, \forall u \in 2^n \; (X_{s_u}^{\mathcal{A}} \cap \bigcap_{m<n} X_{t_u(m)} \neq \emptyset)$;
4. $\forall u, v \in 2^n \; (u \neq v \Rightarrow \forall i, j < n \; (t_u(i) \neq t_v(j)))$.

We begin by setting $k_0 = 0$ and $s_{\emptyset} = t_{\emptyset} = \emptyset$. Suppose now that we have found $k_n \in \mathbb{N}$, $\langle s_u \rangle \in (\mathbb{N}^n)^{2^n}$, and $\langle t_u \rangle \in ((\mathbb{N}^{k_n})^n)^{2^n}$. Condition (3) ensures that

$$\forall u \in 2^n \, \forall F \subseteq X \text{ finite} \, \exists \mathbf{x} \in X_{s_u}^{\mathcal{A}} \cap \textstyle\bigcap_{m<n} X_{t_u(m)} \; (\mathbf{x} \cap F = \emptyset),$$

thus we can recursively find a sequence $\langle \mathbf{x}_u \rangle \in X^{2^{n+1}}$ such that:

1. $\forall u \in 2^n \; (\mathbf{x}_{u0}, \mathbf{x}_{u1} \in X_{s_u}^{\mathcal{A}} \cap \bigcap_{m<n} X_{t_u(m)})$;
2. $\forall u, v \in 2^{n+1} \; (u <_{\text{lex}} v \Rightarrow \mathbf{x}_u(0), \ldots, \mathbf{x}_u(n) \notin \mathbf{x}_v)$.

Fix $k_{n+1} > k_n$ sufficiently large that

$$\forall u, v \in 2^{n+1} \, \forall i, j < n+1 \; (u \neq v \Rightarrow d(\mathbf{x}_u(i), \mathbf{x}_v(j)) > 1/k_{n+1}),$$

and for each $u \in 2^{n+1}$, fix $s_u \in \mathbb{N}^{n+1}$ and $t_u \in (\mathbb{N}^{k_{n+1}})^{n+1}$ such that $s_{u|n} \subseteq s_u$, $t_{u|n} \subseteq t_u$, and $\mathbf{x}_u \in X^{\mathcal{A}}_{s_u} \cap \bigcap_{m<n+1} X_{t_u(m)}$.

This completes the recursive construction. For each $\alpha \in \mathbb{N}^{\mathbb{N}}$, define

$$\{\mathbf{x}_\alpha\} = \bigcap_{n\in\mathbb{N}} X^{\mathcal{A}}_{s_{\alpha|n}} = \bigcap_{n\in\mathbb{N}} \bigcap_{m<n} X_{t_{\alpha|n}(m)},$$

and note that $\langle \mathbf{x}_\alpha \rangle_{\alpha\in 2^{\mathbb{N}}}$ is an antichain contained in $\mathcal{A}$, a contradiction. □

Proposition 6.4 ensures that if (s,t) is stable, then a non-empty, countable subset of X is definable from $\mathcal{A}$ with parameters $\mathcal{X}$ and $\mathcal{X}^{\mathcal{A}}$ (since s and t are fixed, finite tuples of numbers), and the theorem follows. □

We close with a special case in which the Souslin schemes are no longer necessary. Recall that we call a set $\mathcal{A} \subseteq \mathcal{P}(X)$ a *weak antichain* iff

$$\forall A, B \in \mathcal{A}\ (A \neq B \Rightarrow A \nsubseteq B).$$

We say that a family $\mathcal{A} \subseteq \mathcal{P}(X)$ is *strongly κ-cc* if it does not contain a weak antichain of size κ.

Theorem 6.7. *Suppose that X is a Polish space, $\mathcal{A} \subseteq [X]^{\mathbb{N}}$ is strongly $\mathfrak{c}$-cc, and $\tilde{\mathcal{A}}$ is analytic. Then $\bigcup \mathcal{A}$ is countable.*

Proof. Since $\tilde{\mathcal{A}}$ is analytic, we can of course fix a Souslin scheme

$$\mathcal{X}^{\mathcal{A}} = \langle X^{\mathcal{A}}_s \rangle_{s\in\mathbb{N}^{<\mathbb{N}}}$$

for $\tilde{\mathcal{A}}$, even though such an $\mathcal{X}^{\mathcal{A}}$ may itself fail to be definable. Define $\pi : \mathbb{N}^{\mathbb{N}} \to X^{\mathbb{N}}$ by

$$\pi(\alpha) = \text{ the unique element of } \bigcap_{n\in\mathbb{N}} X^{\mathcal{A}}_{\alpha|n},$$

and let $\leq$ be the quasi-order on $\mathbb{N}^{\mathbb{N}}$ given by

$$\alpha \leq \beta \iff \pi(\alpha) \subseteq \pi(\beta).$$

Then $\leq$ is a strongly $\mathfrak{c}$-cc Borel quasi-order on $\mathbb{N}^{\mathbb{N}}$, and Harrington-Marker-Shelah [4, Theorem 5.1] ensures that there are Borel sets $B_n \subseteq \mathbb{N}^{\mathbb{N}}$ such that $\mathbb{N}^{\mathbb{N}} = \bigcup_{n\in\mathbb{N}} B_n$ and

$$\forall n \in \mathbb{N}\, \forall \alpha, \beta \in B_n\ (\alpha \leq \beta \text{ or } \beta \leq \alpha).$$

Set $\mathcal{A}_n = \pi[B_n]$. By Harrington-Marker-Shelah [4, Corollary 1.5], there are Borel sets $\mathcal{B}_n \supseteq \mathcal{A}_n$ such that $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ and

$$\forall n \in \mathbb{N}\, \forall \mathbf{x}, \mathbf{y} \in \mathcal{B}_n\ (\{\mathbf{x}(n)\}_{n\in\mathbb{N}} \subseteq \{\mathbf{y}(n)\}_{n\in\mathbb{N}} \text{ or } \{\mathbf{y}(n)\}_{n\in\mathbb{N}} \subseteq \{\mathbf{x}(n)\}_{n\in\mathbb{N}}).$$

Harrington-Marker-Shelah [4, Corollary 3.2] then implies that there are no well-founded, increasing sequences of sets in $\mathcal{A}_n$ of length ω_1, so the sets $\bigcup \mathcal{A}_n$ are countable, thus so too is $\bigcup \mathcal{A}$. □

6.1 An application

We close this section by mentioning an alternate proof of a result of Miller [8] utilizing the theory we have developed.

Fix a countable Borel equivalence relation E on a Polish space X. We say that a graph $\mathcal{G} \subseteq X \times X$ is a *graphing* of E iff the connected components of $\mathcal{G}$ are precisely the equivalence classes of E. We say that two rays (infinite paths) r and s through $\mathcal{G}$ are *end equivalent* if there is no finite subset $Y \subseteq X$ such that r and s are disconnected in $\mathcal{G} \setminus (Y \times Y)$. As a caveat to the reader, we remark that in some settings end equivalence is defined in terms of the induced subgraph associated with $X \setminus Y$, i.e., replacing $\mathcal{G} \setminus (Y \times Y)$ in the above definition with $\mathcal{G} \cap (X \setminus Y \times X \setminus Y)$. While the difference between the definitions is subtle, the notions are quite dissimilar when $\mathcal{G}$ is not locally finite.

Suppose that κ is a finite number or $\aleph_0$. We say that there is a Borel way of choosing κ many ends of $\mathcal{G}$ iff there is a Borel function $f : X \to [X^{\mathbb{N}}]^{\kappa}$ that assigns to each point $x \in X$ a set of κ many end inequivalent rays starting at x such that

$$\forall x \forall y E x \, \forall r \in f(x) \, \exists ! s \in f(y) \, (r \text{ and } s \text{ are end equivalent}),$$

where $\exists !$ stands for "there is a unique."

We then have the following:

Theorem 6.8. *If there is a Borel way of choosing two ends of $\mathcal{G}$, then E is hyperfinite.*

Theorem 6.9. *If there is a Borel way of choosing more than two but finitely many ends of $\mathcal{G}$, then E is smooth.*

The main idea behind the new proof of the latter theorem is that the collection of finite sets which witness end inequivalence forms an intersecting family on each E-class, which then allows us to isolate in a definable way finitely many points from each E-class, guaranteeing smoothness.

An advantage of this approach is that it generalizes somewhat to the uncountable context. We say that a graphing $\mathcal{G}$ of (a now possibly uncountable) E is a *quasitreeing* of E iff every $x \in X$ is contained it at most countably many cycles of $\mathcal{G}$ (here, cycles are taken to be sequences of distinct vertices in the natural way). In this situation one has an analogue of the above result, by first observing that on each E-class the collection of countable sets of vertices witnessing end inequivalence is $\mathfrak{c}$-cc. The results in the last section then ensure that we may define a countable, non-empty subset of each E-class, effectively reducing the problem to the countable case.

7 Possible extensions

It would be interesting to provide topological analogs of the results and questions studied here. Actually, the main issue is coming up with the right notion of definability in this new context. To illustrate the subtleties involved, we mention several examples of intersecting families of compact subsets of $\mathbb{R}^n$:

1. Take as $\mathcal{A}$ the family of all closed arcs of length π on the unit circle X. This is a low family of size $\mathfrak{c}$. But if one considers the arcs as subsets of $X = \mathbb{R}^2$, then it is reasonable to argue that the set $Z = \{(0,0)\}$ is "definable" if we allow our language to refer to something like Hausdorff distance between compact sets, see Kechris [6, §4.F] (Z is then the set of points minimizing the Hausdorff distance to all the sets in $\mathcal{A}$). In this example there is at least a definable compact subset of X.

2. Now take as $\mathcal{A}$ all line segments with endpoints $(0,q)$ and $(1/q,0)$ for q a positive rational and $X = \mathbb{R}^2$. For this, it is not clear that there is a compact definable subset of X (although there is a countable one, the set of intersection points). Again, if one allows Hausdorff distance (or diameter, or any reasonable manifestation of the metric structure of $\mathbb{R}^2$), then one can obtain a definable compact set (e.g., the segment with endpoints $(0,1)$ and $(1,0)$ has minimum diameter).

3. Helly's well known theorem (see, for example, Bollobás [1]) states that if $\{X_1,\ldots,X_n\}$ is a $(d+1)$-wise intersecting family of convex subsets of $\mathbb{R}^d$, then $\bigcap_i X_i \neq \emptyset$. If the X_i are in addition assumed compact, then the same holds even if we start with infinitely many X_i. Hence, we can define from any such $(d+1)$-wise intersecting family a non-empty (compact) convex subset of $\mathbb{R}^d$.

It would be interesting to formulate the appropriate logical setting in which these examples naturally reside. Here are two suggestions:

1. Rather than $\hat{\mathcal{A}}_X$, one could study structures of the form

$$(K(X) \cup X, \mathcal{A}, \in, d)$$

where X is a Polish space, $K(X)$ is the collection of compact subsets of X and d is a predicate coding Hausdorff distance, so $d(A,B,C,D)$ holds in this structure iff A,B,C,D are compact subsets of X and the Hausdorff distance between A and B is strictly smaller than the Hausdorff distance between C and D.

2. Similarly to the structures above, one could instead consider those of the form

$$(H(X) \cup X, \mathcal{A}, \in, d),$$

where $H(X)$ is the collection of convex (or convex and compact) subsets of X.

Part of what looking at these structures entails is to understand the definable subsets when $\mathcal{A}$ is empty, which in general depends on the underlying space X.

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Andrés Eduardo Caicedo
California Institute of Technology
Department of Mathematics — Mail code 253-37
Pasadena, CA 91125
Homepage: `http://www.its.caltech.edu/~caicedo/`
Email: `caicedo@caltech.edu`

John D. Clemens
Penn State University
Mathematics Department
University Park, PA 16802
Homepage: `http://www.math.psu.edu/clemens/`
Email: `clemens@math.psu.edu`

Clinton Conley
University of California, Los Angeles
Mathematics Department — Box 951555
Los Angeles, CA 90095-1555
Homepage: http://www.math.ucla.edu/∼clintonc/
Email: clintonc@math.ucla.edu

Benjamin Miller
(No address given)