Isometry of Polish metric spaces with a fixed set of distances

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Abstract

We show that the distance set of a Polish metric space is far from being a complete invariant for isometry. Using the theory of Borel reducibility of equivalence relations, we determine the complexity of isometry of the class of Polish metric spaces with a fixed difference set, which depends on the distance set in question. In particular, we show that for certain distance sets this relation is as complicated as the isometry of all Polish metric spaces, and we determine for which sets this relation is classifiable by countable structures.

The aim of this paper is to study the extent to which the distance set of a Polish metric space (a separable, complete metric space) fails to be a complete invariant for isometry. The set of distances in a Polish metric space is an invariant for isometry, i.e., isometric spaces have the same distance set; however, the converse generally fails. This study is motivated by two distinct results. The first is the characterization of which sets of reals can be the distance set of a Polish metric spaces. In the article [1], we determined which sets of real numbers could be the set of distances of some Polish metric space (as well as characterizing which could be the distance sets of particular types of spaces), obtaining the following characterization:

Theorem. *Given a set of non-negative reals, $A \subseteq [0, \infty)$, A is the set of distances for some Polish metric space if and only if A is an analytic set containing 0 and either A is countable or 0 is a limit point of A.*

In the proof of this result, there is a notable lack of uniformity in the construction of a Polish metric space having a given set A as its distance set. For each such analytic set A we needed to pick a tree representation of A and a countable set of distinguished distances; different choices of trees and distances can produce different (non-isometric) metric spaces with A as their distance set. We do not then have a well-defined map from analytic sets to Polish metric spaces with the given distance sets. Another way to say this is that we did not produce a definable reduction of the equality relation on analytic sets to the isometry relation on Polish metric spaces. In fact, as we will discuss below, there can be no such map. Hence, we can ask how many non-isometric Polish metric spaces there can be having a given distance set.

A second motivation for this study is the problem of classifying Polish metric spaces up to isometry. We showed in [2] that any equivalence relation induced by a Borel action of a Polish group on a Polish space is Borel reducible to the equivalence relation of isometry of Polish metric spaces, a result obtained independently by Gao and Kechris in [5]. The techniques for obtaining these reductions typically involve coding information into the distances in the spaces constructed, producing spaces with different distance sets to witness non-isometry, so we can ask if this is necessary. Gao and Kechris also show the converse of the previous result, producing:

Theorem (Gao-Kechris, [5]). *The equivalence relation of isometry of Polish metric spaces is Borel bi-reducible with the universal orbit equivalence relation of a Borel action of a Polish group on a Polish space.*

The relation of equality of analytic sets is not Borel reducible to the orbit equivalence relation of a Polish group action, so in particular this tells us that we could not have produced a suitably definable map from analytic sets to Polish metric spaces in the result of [1].

We are thus led to consider the following questions. A set of distances is said to be *metrically rigid* for a class of metric spaces if it is the distance set for a unique space in the class (up to isometry). The authors of [9] consider this notion for certain classes of metric spaces. We can here ask whether there is an analytic set A which is metrically rigid for the class of Polish metric spaces. It is easy to see, though, that this is never the case when A is non-trivial; for instance, taking a disjoint union of two copies of one of the spaces constructed in [1] (with the metric appropriately defined) will generally produce a space which is not isometric to the original. We could instead ask how many non-isometric spaces there are with a given distance set, but again (except for trivial cases) we will always have a continuum of non-isometric spaces. This leads us to consider the definable cardinality of the set of spaces with a fixed distance set, that is, to consider how far the distance set is from being a complete invariant for isometry:

Question. *For a given analytic set A, how complicated is it to classify up to isometry those Polish metric spaces having A as their set of distances?*

In the next section we formalize the coding of Polish metric spaces and the isometry relation for spaces with a fixed distance set. We then review some results about the Borel reducibility relation which will be used for gauging the complexity of these problems in Section 2. In Section 3 we present the main results for classification of the complexity of the isometry relations, giving upper and lower bounds depending on the topological and algebraic properties of A. We show that for many such sets the resulting isometry problem is as complicated as the isometry problem for all Polish metric spaces. We also present a dichotomy which characterizes when the isometry problem for spaces with a given distance set is classifiable by countable structures. Finally, in Section 4 we consider countable spaces with a given (necessarily countable) distance set.

1 Coding of Polish metric spaces and isometry

We begin by formalizing the collection of Polish metric spaces and the equivalence relation of isometry; a longer exposition and alternate coding methods may be found in [2], [3], or [5].

Definition 1. Let the space $\mathcal{M}$ of *codes for Polish metric spaces* be:

$$\begin{aligned}\mathcal{M} = \{\langle d_{i,j}\rangle_{i,j\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}} \text{ such that:} \\ &\forall i\forall j\ (d_{i,j} \geq 0) \wedge \\ &\forall i\ (d_{i,i} = 0) \wedge \\ &\forall i\forall j\ (d_{i,j} = d_{j,i}) \wedge \\ &\forall i\forall j\forall k\ (d_{i,k} \leq d_{i,j} + d_{j,k})\ \}.\end{aligned}$$

This is a closed subspace of $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ and is hence a Polish space in the relative topology. An array $\langle d_{i,j}\rangle_{i,j\in\mathbb{N}} \in \mathcal{M}$ represents a pseudo-metric d on the countable space $\{x_i : i \in \mathbb{N}\}$ where $d(x_i, x_j) = d_{i,j}$; we view this as coding the Polish metric space (X, d) obtained as the completion of the resulting countable metric space.

We now define the isometry relation, $\cong_i$, on this space.

Definition 2. For $\langle d_{i,j}\rangle$ and $\langle \widetilde{d_{i,j}}\rangle$ in $\mathcal{M}$, we set $\langle d_{i,j}\rangle \cong_i \langle \widetilde{d_{i,j}}\rangle$ if and only if the metric spaces coded by the two arrays are isometric.

Given a metric space (X, d), we define its set of distances, $\mathrm{Dist}(X, d)$:

Definition 3. The *distance set* of a Polish metric space (X, d) is

$$\mathrm{Dist}(X, d) = \{d(x, y) : x, y \in X\}.$$

We can now define the equivalence relations we are interested in.

Definition 4. Let A be an analytic set containing 0 and having 0 as a limit point when A is uncountable. The space of *Polish metric spaces with distance set* A, $\mathcal{M}_A$, is:

$$\begin{aligned}\mathcal{M}_A &= \{(X,d) : (X,d) \text{ is a Polish metric space with } \mathrm{Dist}(X,d) = A\} \\ &= \{\langle d_{i,j}\rangle_{i,j\in\mathbb{N}} \in \mathcal{M} : \mathrm{Dist}(\overline{\{x_i : i\in\mathbb{N}\}}, \overline{\langle d_{i,j}\rangle_{i,j\in\mathbb{N}}}) = A\},\end{aligned}$$

where $(\overline{\{x_i : i\in\mathbb{N}\}}, \overline{\langle d_{i,j}\rangle_{i,j\in\mathbb{N}}})$ is the Polish metric space coded by $\langle d_{i,j}\rangle_{i,j\in\mathbb{N}}$. The *isometry relation for spaces with distance set* A, E_A, is

$$E_A = \cong_i \restriction \mathcal{M}_A.$$

When A is an uncountable set, any space having distance set A must be uncountable. On the other hand, when A is a countable set bounded away from 0 then any Polish metric space with distance set A must be countable. When A is a countable set with 0 as a limit point, though, we can have both countable and uncountable spaces with distance set A. We thus can also consider the restriction to countable spaces:

Definition 5. For a countable set A, let

$$\begin{aligned}\mathcal{M}_A^{\mathrm{ctble}} &= \{(X,d) : (X,d) \text{ is a countable Polish metric space} \\ &\qquad \text{with } \mathrm{Dist}(X,d) = A\} \\ E_A^{\mathrm{ctble}} &= \cong_i \restriction \mathcal{M}_A^{\mathrm{ctble}}.\end{aligned}$$

Note that $\mathcal{M}_A$ will in general be a $\mathbf{\Pi}^1_2$ subset of $\mathcal{M}$, since we have:

$$\begin{aligned}\langle d_{i,j}\rangle \in \mathcal{M}_A &\iff \forall z\in\mathbb{R}\ [z\in A \Leftrightarrow z \in \mathrm{Dist}(\overline{\{x_i : i\in\mathbb{N}\}}, \overline{\langle d_{i,j}\rangle})] \\ &\iff \forall z\in\mathbb{R}\ [z\in A \Leftrightarrow \exists f,g\in\mathbb{N}^{\mathbb{N}}\ (\langle x_{f(i)}\rangle_i \text{ and } \langle x_{g(i)}\rangle_i \text{ are} \\ &\qquad\qquad \text{Cauchy sequences with } d(\lim x_{f(i)}, \lim x_{g(i)}) = z)] \\ &\iff \forall z\in\mathbb{R}\ \Bigg[z\in A \Leftrightarrow (\exists f,g\in\mathbb{N}^{\mathbb{N}})(\forall k\geq 1)(\exists N)(\forall n,m\geq N) \\ &\qquad\qquad \left(d_{f(n),f(m)} < \frac{1}{k} \wedge d_{g(n),g(m)} < \frac{1}{k} \wedge d_{f(n),g(n)} < \frac{1}{k}\right)\Bigg].\end{aligned}$$

We will then be a bit imprecise when we speak of Borel reducibility below; when appropriate we will assume that we have Borel-measurable functions defined on all of $\mathcal{M}$ which behave as required when restricted to $\mathcal{M}_A$.

We wish to determine the complexity of the relation E_A for various sets A. The complexity will depend on the choice of the set A. To make this complexity precise, we next explain the notion of Borel reducibility between equivalence relations.

2 Borel reducibility of equivalence relations

The key comparison of equivalence relations is the following:

Definition 6. Let E and F be equivalence relations on the Polish spaces X and Y. We say that E is *Borel reducible* to F, $E \leq_B F$, if there is a Borel-measurable function $f : X \to Y$ such that for all $x_1, x_2 \in X$ we have $x_1 \, E \, x_2$ if and only if $f(x_1) \, F \, f(x_2)$. We say E is *bi-reducible* with F, $E \sim_B F$, when $E \leq_B F$ and $F \leq_B E$.

Thus, when E is Borel reducible to F we have a definable injection from the quotient space X/E into Y/F. We can thus think of this as a comparison of the definable cardinalities of the quotient spaces.

We now introduce some standard equivalence relations which we will use to gauge the complexity of E_A for various sets A.

Definition 7. An equivalence relation E on X is *smooth* or *concretely classifiable* if E is Borel reducible to the identity relation on some Polish space Y, i.e., there is a Borel function $f : X \to Y$ such that $x_1 \, E \, x_2$ iff $f(x_1) = f(x_2)$.

The smooth equivalence relations are the simplest equivalence relations. For a Polish space X we use $\Delta(X)$ to denote the smooth equivalence relation of equality on X. The second main class of equivalence relations which will appear below are those which admit isomorphism types of countable structures as complete invariants.

Definition 8. An equivalence relation E on X is *classifiable by countable structures* if there is a Borel map f assigning a (code for a) countable structure in some first-order language to each point of X so that $x_1 \, E \, x_2$ iff $f(x_1)$ and $f(x_2)$ code isomorphic structures.

We refer the reader to [4] and [6] for the theory of classification by countable structures. In particular we will use the relations of isomorphism of countable trees and isomorphism of countable graphs; these are bi-reducible with one another and are of maximum complexity among equivalence relations which are classifiable by countable structures. We refer to the maximum such relation as *graph isomorphism*, $\cong_{\text{graphs}}$. We shall also use the the relations $\mathbb{F}_n$ for $n \in \mathbb{N}$, the isomorphism of *trees of height* $n + 1$ (so $\mathbb{F}_0 \sim_B \Delta(\mathbb{N})$, $\mathbb{F}_1 \sim_B \Delta(\mathbb{R})$, and $\mathbb{F}_2$ is equivalent to equality of countable sets of reals). These are strictly increasing in terms of Borel reducibility and are all strictly simpler than graph isomorphism.

There are two more types of countable structures which we will use. The first is the collection of *reverse trees*. A reverse tree has a single bottom level of nodes, each node has a unique immediate successor, and any two nodes have a common successor; thus, it is like a well-founded tree with a single bottom level of terminal nodes without a root. The second type of structure is the collection of *pruned trees with countably many branches*, where a tree is pruned if it has no terminal nodes. The isomorphism relation on reverse trees is reducible to isomorphism of pruned trees with countably many branches, but the exact complexity of these two relations is unclear. Both are reducible to graph isomorphism, but it is not known if the are bi-reducible. Some discussion of the latter relation may be found in [5] following Theorem 8.10.

There are equivalence relations which are not classifiable by countable structures. The one which we will use here is the equivalence relation E_d induced by the action of the *density ideal* I_d on $\mathcal{P}(\mathbb{N}) = 2^{\mathbb{N}}$ by symmetric difference, where

$$I_d = \left\{ X \subseteq \mathbb{N} : \lim_{n\to\infty} \frac{|X \cap \{0, \ldots, n-1\}|}{n} = 0 \right\}.$$

This a turbulent action (see [6]), and hence is not classifiable by countable structures.

Finally, we will use the orbit equivalence relations induced by Borel actions of Polish groups on Polish spaces. There is such an equivalence relation of maximum complexity; this is bi-reducible with the isometry relation on the collection of all Polish metric spaces.

3 Classifying the complexity of E_A

We now proceed with determining the complexity of E_A for various analytic sets A. The complexity of E_A will depend on the set A, and in fact will depend not only on the topological structure but also on the algebraic structure. We will tacitly assume that every set A below is an analytic set containing 0 which is either countable or has 0 as a limit point.

We begin by proving several upper and lower bounds which give a nearly complete classification of the complexity of E_A for various sets A. To do this we will show that various of the equivalence relations discussed above are Borel reducible to the various E_A's and vice versa. In showing these reductions, we will generally simply describe the space or structure being mapped to; the maps described can be formalized as Borel functions on the codes in a straightforward manner.

Note that when A is a countable set we can have countable metric spaces with distance set A, and thus can will consider both E_A and E_A^{ctble}. When A

is bounded away from 0, in fact, all spaces with distance set A are discrete (and hence countable), so $\mathcal{M}_A = \mathcal{M}_A^{\text{ctble}}$ and $E_A = E_A^{\text{ctble}}$.

We may consider a countable metric space as a countable structure for a language with binary relations R_q for $q \in \mathbb{Q}^+$, where the metric can be coded into such a structure by setting $x\ R_q\ y$ iff $d(x, y) < q$. Hence, we will have that E_A^{ctble} is Borel reducible to graph isomorphism for each countable set A. We thus have the following two conclusions:

Proposition 9. *If A is a countable set then E_A^{ctble} is Borel reducible to graph isomorphism.*

Corollary 10. *If A is bounded away from 0 then $\mathcal{M}_A = \mathcal{M}_A^{ctble}$ and $E_A = E_A^{ctble}$ is Borel reducible to graph isomorphism.*

We will say more about the relations E_A^{ctble} in the final section; here we focus primarily on E_A. We start by determining when graph isomorphism is reducible to E_A.

We begin with two trivial cases. First, if $A = \{0\}$ then the one point space is the unique space with distance set A (up to isometry). So we have:

Proposition 11. *If $A = \{0\}$ then $E_A \sim_B \Delta(1)$.*

Second, if A contains a single non-zero distance then a Polish metric space with distance set A is determined up to isometry by its cardinality, which must either be finite or countably infinite. Hence there are only countable many isometry classes, so we have:

Proposition 12. *If $A = \{0, d_0\}$ with $d_0 > 0$, then $E_A \sim_B \Delta(\mathbb{N})$.*

So we will now assume we have at least two non-zero distances in A. The next proposition gives a sufficient condition for reducing graph isomorphism:

Proposition 13. *If $A \supseteq \{d_0, d_1\}$ with $0 < d_0 < d_1 \le 2d_0$, then graph isomorphism is reducible to E_A.*

Proof: We will actually reduce isomorphism of countably infinite connected graphs (which is bi-reducible with graph isomorphism) to E_A. Fix $d_0, d_1 \in A$ with $0 < d_0 < d_1 \le 2d_0$. Let $A' = A \setminus \{d_0, d_1\}$, and let $A'_n = A' \cap [0, n \cdot d_1]$ for $n \ge 1$. From [1], we can then find metric spaces (X_n, d_n) with $\text{Dist}(X_n, d_n) = A'_n$ for $n \ge 1$ (where we can take $X_n = \emptyset$ when $A'_n = \{0\}$). Pick a sequence of elements $\delta_n \in A$ for $n \ge 1$ with $d_1 \le \delta_n \le \delta_{n+1}$ and $\delta_n \ge \frac{1}{2} \sup A'_n$ for each n. This all depends only on the set A.

To reduce graph isomorphism, let G be a countably infinite connected graph with vertices $\{v_i : i \in \mathbb{N}\}$ and edge relation E. We define the metric space (X_G, d_G) as follows. We first define the space (X_G^0, d_G^0) with

$$X_G^0 = \{x_i : i \in \mathbb{N}\}$$

and

$$d_G^0(x_i, x_j) = \begin{cases} d_0 & \text{if } E(v_i, v_j) \\ d_1 & \text{if } \neg E(v_i, v_j) \end{cases} \qquad \text{for } i \neq j.$$

This is a metric since $d_0 < d_1 \leq 2d_0$. We now let X_G be the disjoint union

$$X_G = X_G^0 \sqcup \bigsqcup_{n \geq 1} X_n.$$

We define d_G by letting $d_G = d_G^0$ on X_G^0 and $d_G = d_n$ on X_n for $n \geq 1$. For $x \in X_i$ and $y \in X_j$ with $i \neq j$ (where $X_G^0 = X_0$) we set $d_G(x, y) = \delta_{\max(i,j)}$. This is easily seen to be a metric, and we can produce a Borel function carrying a code for a graph G to a code for X_G. We have $\text{Dist}(X_G, d_G) = A$ since the distances d_0 and d_1 occur in X_G^0 and all the others occur in some X_n; these are the only distances which occur because the X_n's are bounded away from each other.

To check that this function is the desired reduction, note that we easily have $G_1 \cong G_2$ if and only if $(X_{G_1}^0, d_{G_1}^0) \cong_i (X_{G_2}^0, d_{G_2}^0)$, and this gives that $(X_{G_1}, d_{G_1}) \cong_i (X_{G_2}, d_{G_2})$ when $G_1 \cong G_2$. Conversely, any isometry of (X_{G_1}, d_{G_1}) with (X_{G_2}, d_{G_2}) must induce an isometry of $(X_{G_1}^0, d_{G_1}^0)$ with $(X_{G_2}^0, d_{G_2}^0)$ since the distance d_0 only occurs between pairs of points in X_G^0, and each point in X_G^0 is at distance d_0 from some other point since G is infinite and connected. Hence we have that $G_1 \cong G_2$ as needed. □

A second sufficient condition for reducing graph isomorphism to E_A is the following:

Proposition 14. *If 0 is a limit point of A then isomorphism of trees is reducible to E_A; hence, graph isomorphism is reducible to E_A.*

Proof: Choose a sequence $d_i \in A$ for $i \in \mathbb{N}$ with $d_{i+1} < d_i < 1$ for each i and $\lim_{i \to \infty} d_i = 0$ and such that the set

$$A' = A \setminus \{d_i : i \in \mathbb{N}\}$$

still has 0 as a limit point. For $n \geq 1$ let $A'_n = A \cap [0, n]$, and let (X_n, d_n) be a space with $\text{Dist}(X_n, d_n) = A'_n$. Also choose a sequence $\delta_i \in A$ for $i \geq 1$

such that $d_0 \leq \delta_i \leq \delta_{i+1}$ and $\delta_i \geq \frac{1}{2} \sup A'_n$. This all depends only on the set A.

Now, given a tree T on $\mathbb{N}$ we produce a Polish metric space (X_T, d_T) as follows. We can assume that T has at least one infinite branch (since we can uniformly add a single branch to a tree T without affecting the isomorphism relation). We start by defining a metric d^0_T on T (i.e., the set of nodes in the tree). For $s, t \in T$ with $s \neq t$, let u be the maximum mutual predecessor of s and t in T, and let $d^0_T(s,t) = d_{|u|}$. It is straightforward to check that this is a metric (in fact an ultrametric), since for three nodes in T the maximum mutual predecessors of the three pairs must either all be equal or the two shorter predecessors must be equal. We then let (X^0_T, d^0_T) be the completion of (T, d^0_T). Note that Cauchy sequences in T contain infinite branches of T, so that the points of T are isolated points and the limit points are branches through T.

We check that for trees T_1 and T_2 we have $T_2 \cong T_2$ if and only if $(X^0_{T_1}, d^0_{T_1}) \cong_i (X^0_{T_2}, d^0_{T_2})$. An isomorphism between T_1 and T_2 is immediately an isometry from $(T_1, d^0_{T_1})$ to $(T_2, d^0_{T_2})$ which extends to an isometry from $(X^0_{T_1}, d^0_{T_1})$ to $(X^0_{T_2}, d^0_{T_2})$. Conversely, since isolated points must map to isolated points under an isometry, an isometry of $(X^0_{T_1}, d^0_{T_1})$ with $(X^0_{T_2}, d^0_{T_2})$ induces an isometry of $(T_1, d^0_{T_1})$ with $(T_2, d^0_{T_2})$. This must be an isomorphism of T_1 and T_2 since for $s, t \in T$ we have $s \sqsubset t$ if and only if $d^0_T(s,t) = d_{|s|}$.

We now form the space (X_T, d_T) by letting

$$X_T = X^0_T \sqcup \bigsqcup_{n \geq 1} X_n$$

and defining d_T so that $d_T = d^0_T$ on X^0_T, $d_T = d_n$ on X_n, and for $x \in X_i$ and $y \in X_j$ with $i \neq j$ (where $X_0 = X^0_T$) we have $d_T(x,y) = \delta_{\max(i,j)}$. We have $\mathrm{Dist}(X_T, d_T) = A$ since all of the distances d_i occur in X^0_T (because we assumed T had an infinite branch, so it has nodes at all levels) and all the other distances in A occur in some X_n. For trees T_1 and T_2 we again immediately have that $(X_{T_1}, d_{T_!}) \cong_i (X_{T_2}, d_{T_2})$ when $T_1 \cong T_2$. Conversely, since the distances d_i occur only within X^0_T we must have that an isometry from $(X_{T_1}, d_{T_!})$ to (X_{T_2}, d_{T_2}) sends $X^0_{T_1}$ to $X^0_{T_2}$ and hence give an isomorphism of T_1 with T_2. $\square$

We then are left to consider sets A which are bounded away from 0 and such that for any two distinct $x, y \in A$ we have $2x < y$ or $2y < x$. Notice that any space with such a distance set must be discrete since the distances are bounded away from 0, and it must also be an ultrametric space since if we have a triangle with sides of length x, y, and z with $x \leq y \leq z$ we must have

$y = z$ (or otherwise $x + y < z$). Gao and Kechris show in Theorem 4.4 of [5] that isometry of ultrametric Polish metric spaces is bi-reducible with graph isomorphism; hence any such E_A is reducible to graph isomorphism. The exact complexity of isometry of discrete ultrametric spaces is not known.

The techniques of the next two propositions is adapted from the proof of Theorem 4.4 in [5]. We first consider finite sets A with the given condition.

Proposition 15. *If* $A = \{0\} \cup \{d_i : 0 \leq i < n\}$ *with* $2d_i < d_{i+1}$ *for each* $i < n - 1$*, then* E_A *is bi-reducible with isomorphism of trees of height* n*,* $\mathbb{F}_{n-1}$*.*

Proof: We first show that $E_A \leq_B \mathbb{F}_{n-1}$. Given a space $(X, d) \in \mathcal{M}_A$ we define the tree T_X as follows. The nodes of T_X are the closed balls of radius d_i for $0 \leq i < n$ or radius 0 (closed balls of radius 0 are the points of X). For each i there are only countably many such balls because d is an ultrametric and they partition the space. Note that two points have $d(x, y) \leq d_i$ iff they are in the same closed ball of radius d_i. For two such balls s and t we set $s \sqsubseteq t$ in T_X if the ball s contains the ball t. This forms a tree because a closed ball of radius d_{i-1} is entirely contained in a unique closed ball of radius d_i, and this tree has height n. The root of T_x is the unique closed ball of radius d_{n-1} (the whole space) and the terminal nodes are the points of X. An isometry of two spaces induces a permutation of the closed balls which preserves inclusion, and hence an isomorphism of the corresponding trees. Conversely, an isomorphism of T_{X_1} and T_{X_2} induces a bijection f of X_1 with X_2 with the property that $d_1(x, y) \leq d_i$ iff $d_2(f(x), f(y)) \leq d_i$ for each i, so that f is an isometry.

Second, we show that $\mathbb{F}_{n-1} \leq_B E_A$. We can assume that all of the trees are such that all of the terminal nodes are at depth n. Given a tree T of height n, we produce the space (X_T, d_T) by letting X_T be the terminal nodes of T. For $s, t \in T$ with $s \neq t$ we let $d_T(s, t) = d_{(n-1)-|u|}$, where u is the maximal mutual predecessor of s and t in T. This has distance set A, and it is straightforward to check that $T_1 \cong T_2$ if and only if $(X_{T_1}, d_{T_1}) \cong_i (X_{T_2}, d_{T_2})$. $\square$

Note that Proposition 12 is a special case of this when $n = 1$. Finally, if A is infinite with the given distance conditions we have:

Proposition 16. *If* $A = \{0\} \cup \{d_i : i \in \mathbb{N}\}$ *with* $2d_i < d_{i+1}$ *for each* i*, then* E_A *is bi-reducible with isomorphism of reverse trees.*

Proof: The technique is the same as for the previous proposition. To reduce E_A to isomorphism of reverse trees, let (X, d) be a space in $\mathcal{M}_A$. We form the reverse tree R_X where the nodes are the closed balls of radius d_i and points of X, with a node t extending a node s if the ball t is contained in a ball s. The bottom level nodes are again the points, and the tree is connected since any two points x and y are in the closed ball of radius $d(x, y)$. Checking that this is a reduction is the same as before.

To reduce isomorphism of reverse trees, we send a reverse tree R to the space (X_R, d_R) where X_R consists of the points at the bottom level of R, and $d_R(x, , y) = d_{n(x,y)-1}$, where $n(x, y)$ is the distance up from the bottom level of the node where x and y connect in R. This is a reduction as before. □

This characterizes when graph isomorphism is a lower bound for the complexity of E_A, with the exception that we do not know whether isomorphism of reverse trees is bi-reducible with graph isomorphism. We now consider when it is an upper bound, i.e., when E_A is classifiable by countable structures. We will prove the following dichotomy:

Theorem 17. *E_A is classifiable by countable structures if and only if A is not dense in any half-neighborhood of 0.*

By a half-neighborhood of 0 we mean an interval of the form $(0, r)$ with $r > 0$. This dichotomy is a consequence of the next two propositions.

Proposition 18. *If A is not dense in any half-neighborhood of 0, then E_A is classifiable by countable structures.*

Proof: The proof is again based on the proof in [5] that isometry of ultrametric spaces is classifiable by countable structures. If A is not dense in a neighborhood of 0, there are mutually disjoint open intervals $(r_i - \delta_i, r_i + \delta_i)$ disjoint from A with $\delta_i > 0$ and $r_i \to 0$ (we can pick the r_i's and δ_i's to be rational if we wish). Let (X, d) be a space in $\mathcal{M}_A$. Consider the open ball of radius r_i around some point $x \in X$, $B_{r_i}(x)$. If y is another point in X with $d(x, y) < \delta_i$ and $z \in B_{r_i}(y)$ then $d(x, z) \leq d(x, y) + d(y, z) < \delta_i + r_i$. Since A is disjoint from $(r_i - \delta_i, r_i + \delta_i)$ this distance must in fact be less than r_i, so that $z \in B_{r_i}(x)$. By symmetry we have that $B_{r_i}(x) = B_{r_i}(y)$. Hence, if $\mathcal{D} \subseteq X$ is a countable dense set then

$$\{B_{r_i}(x) : x \in X\} = \{B_{r_i}(x) : x \in \mathcal{D}\},$$

so that there are only countably many such balls (which are in fact clopen). We now form the countable structure $\mathcal{A}_X$ whose underlying set is

$$\{B_{r_i}(x) : x \in \mathcal{D}, i \in \mathbb{N}\}$$

with the following relations:

$$\begin{aligned} S(B_1, B_2) &\Leftrightarrow B_1 \subseteq B_2 \\ R_q(B) &\Leftrightarrow \operatorname{diam}(B) < q, \text{ for } q \in \mathbb{Q}^+ \\ D_q(B_1, B_2) &\Leftrightarrow d_H(B_1, B_2) < q, \text{ for } q \in \mathbb{Q}^+. \end{aligned}$$

Here d_H is the Hausdorff distance, but any reasonable distance between the two balls will suffice.

We check that $(X, d_X) \cong_i (Y, d_Y)$ if and only if $\mathcal{A}_X \cong \mathcal{A}_Y$. An isometry of (X, d_X) and (Y, d_Y) induces a permutation of the closed balls preserving inclusion, diameter, and Hausdorff distance, and hence induces an isomorphism of $\mathcal{A}_X$ and $\mathcal{A}_Y$. For the converse, suppose $\varphi : \mathcal{A}_X \cong \mathcal{A}_Y$. We define the map $\tilde{\varphi} : X \to Y$ by letting

$$\tilde{\varphi}(x) = \bigcap \{\varphi(B) : B \in \mathcal{A}_X \wedge x \in B\}.$$

This intersection is a singleton since x is in the decreasing sequence of closed balls $B_{r_i}(x)$ of vanishing diameter, which map to a decreasing sequence of closed balls also of vanishing diameter. This map is a bijection since the inverse is given by

$$\tilde{\varphi}^{-1}(y) = \bigcap \{\varphi^{-1}(B) : B \in \mathcal{A}_Y \wedge y \in B\}.$$

It is an isometry since for $x_1, x_2 \in X$ we have

$$\begin{aligned} d_X(x_1, x_2) &= \lim_{i \to \infty} d_H(B_{r_i}(x_1), B_{r_i}(x_2)) = \lim_{i \to \infty} d_H(\varphi(B_{r_i}(x_1)), \varphi(B_{r_i}(x_2))) \\ &= \lim_{i \to \infty} d_H(B_{r_i}(\tilde{\varphi}(x_1)), B_{r_i}(\tilde{\varphi}(x_2))) = d_Y(\tilde{\varphi}(x_1), \tilde{\varphi}(x_2)). \end{aligned}$$

Thus $(X, d_X) \cong_i (YZ, d_Y)$. □

Note that this proof is effective, so that we can in fact get the following result:

Corollary 19. *Isomorphism of Polish metric spaces whose distance sets are not dense in some half-neighborhood of 0 is Borel reducible to graph isomorphism.*

Note that this is a Borel property of Polish metric spaces. We are now ready to prove the converse, that if A is dense in some half-neighborhood of 0 then E_A is not classifiable by countable structures. We will do this by reducing the equivalence relation E_d introduced in Section 2. We begin with a lemma showing that we can find an appropriate metric on I_d.

Lemma 20. *If A is dense in some neighborhood $(0,r)$ then there is a compatible invariant metric d_A on I_d such that $d_A(\alpha,\beta) \in A \cap [0,r)$ for all $\beta,\gamma \in I_d$.*

Proof: We will find a metric d_A of the form $d_A(\beta,\gamma) = \varphi_A(\beta\Delta\gamma)$ where φ_A is a finite lower semi-continuous submeasure on I_d with $I_d = \mathrm{Exh}(I_d)$, i.e., a function $\varphi : \mathcal{P}(\mathbb{N}) \to [0,\infty)$ such that:

0. $\varphi(\emptyset) = 0$.
1. $\varphi(\{n\}) > 0$ for each $n \in \mathbb{N}$.
2. $\varphi(X) \leq \varphi(Y)$ when $X \subseteq Y$.
3. $\varphi(X \cup Y) \leq \varphi(X) + \varphi(Y)$.
4. $\varphi(X) = \lim_{n\to\infty} \varphi(X \cap \{0,\ldots,n\})$.
5. $I_d = \{X \subseteq \mathbb{N} : \lim_{n\to\infty} \varphi(X \setminus \{0,\ldots n\}) = 0\}$.

A discussion of such submeasures may be found in [10]. I_d admits such a submeasure φ_0 which takes values in the dyadic rationals in the unit interval on I_d, namely

$$\varphi_0(X) = \sup\left\{\frac{|X \cap \{0,\ldots,2^n-1\}|}{2^n} : n \in \mathbb{N}\right\}.$$

The supremum is always achieved when $X \in I_d$, and hence $\varphi_0(X) \in \mathbb{Q}_2^*$ for $X \in I_d$, where

$$\mathbb{Q}_2^* = \left\{\frac{k}{2^n} : n \in \mathbb{N}, 0 \leq k \leq 2^n\right\}$$

is the set of dyadic rationals in the unit interval. We will find φ_A of the form $\varphi_A(X) = \rho \circ \varphi_0(X)$ for an appropriate choice of $\rho : \mathbb{Q}_2^* \to A \cap [0,r)$. In order for this to be a lower-semicontinuous submeasure with $I_d = \mathrm{Exh}(\varphi_A)$, it will be sufficient to choose ρ so that ρ is continuous, strictly increasing, concave, and satisfies $\rho(0) = 0$. We can construct such a ρ by induction. We start by letting $\rho(0) = 0$ and $\rho(1) = r_1$ for some $r_1 > 0$ with $r_1 \in A \cap [0,r)$. At stage $n \geq 1$ we then define $\rho\left(\frac{k}{2^n}\right)$ for odd k with $0 < k < 2^n$. The point will be to ensure that there is always an open interval available from which to pick points for values for ρ; the density of A in $[0,r)$ then allows us to do so. We can start with $\rho\left(\frac{1}{2}\right) = r_{\frac{1}{2}}$ where $\frac{1}{2} \cdot r_1 < r_{\frac{1}{2}} < r_1$. Given our initial values, it will suffice to continue to ensure concavity. As we choose each successive value $\rho(q)$, we need to ensure two conditions on ρ:

1. If $q_1, q_2 < q$ are the two nearest values less than q for which we have already defined ρ, then the point $(q, \rho(q))$ lies below the line through $(q_1, \rho(q_1))$ and $(q_2, \rho(q_2))$, and similarly for the two nearest values greater than q.
2. If $q_1 < q < q_2$ are the two nearest values on either side for which we have defined ρ, then $(q, \rho(q))$ is above the line through $(q_1, \rho(q_1))$ and $(q_2, \rho(q_2))$.

This will always leave us an open interval from which to choose $\rho(q)$, so we can pick a value in A satisfying these conditions. $\square$

Proposition 21. *If A is dense in some half-neighborhood of 0, then $E_d \leq_B E_A$, where E_d is the orbit equivalence relation induced by the action of the density ideal on $2^{\mathbb{N}}$ by symmetric difference. In particular, E_A is not classifiable by countable structures and is strictly above graph isomorphism.*

Proof: When A is dense in a neighborhood of 0 then 0 is a limit point of A, so we know graph isomorphism is reducible to E_A by Proposition 14. We now wish to reduce E_d. Let A be dense in the neighborhood $(0, r)$. Choose a point $a_0 \in A$ with $0 < a_0 < \frac{1}{2}r$. By the previous lemma we can find a compatible metric φ_A on I_d taking values in the interval $[0, \frac{1}{4}a_0]$. Next choose a sequence of distinct points $d_i \in A$ for $i \geq 1$ with $a_0 < d_i < \frac{3}{2}a_0$ for all i. Also choose a sequence of distinct points $c_i \in A$ for $i \geq 0$ with $\frac{3}{4}a_0 < c_i < \frac{3}{4}a_0 + \varphi_A(\{i\}) < a_0$ for all i (so also $c_i < a_0$).

Now let $A' = A \setminus \{d_i : i \geq 1\}$ and let $A'_n = A' \cap [0, n \cdot r]$ for $n \geq 1$. Each A'_n is still an analytic set with 0 as a limit point, so by the results of [1] we can find Polish metric spaces (X_n, d_n) such that $\mathrm{Dist}(X_n, d_n) = A'_n$ for $n \geq 1$. We can also find a sequence of elements $\delta_n \in A'_n$ such that $\delta_n \geq \frac{1}{2}\sup A'_n$ and $\frac{r}{2} \leq \delta_n \leq \delta_{n+1}$ for $n \geq 1$. All of this depends only on the set A.

We now explain the reduction of E_d to E_A. Given $\alpha \in 2^{\mathbb{N}} = \mathcal{P}(\mathbb{N})$ we will produce a space (X_α, d_α) such that for all $\alpha_1, \alpha_2 \in 2^{\mathbb{N}}$ we have $\alpha_1 \Delta \alpha_2 \in I_d$ if and only if $(X_{\alpha_1}, d_{\alpha_1}) \cong_i (X_{\alpha_2}, d_{\alpha_2})$. We first define the space X^0_α by letting

$$X^0_\alpha = \{x_\beta : \beta \in I_d\} \cup \{x^*_i : i \in \mathbb{N}\}.$$

We define the metric d^0_α on X^0_α by setting:

$$\begin{aligned}
d^0_\alpha(x_\beta, x_\gamma) &= \varphi_A(\beta\Delta\gamma) && \text{for } \beta,\gamma \in I_d\\
d^0_\alpha(x^*_i, x^*_j) &= d_{\min(i,j)} && \text{for } i,j \in \mathbb{N} \text{ with } i \neq j\\
d^0_\alpha(x^*_i, x_\beta) &= \begin{cases} c_i & \text{if } \alpha\Delta\beta(i) = 1\\ \frac{3}{4}a_0 & \text{if } \alpha\Delta\beta(i) = 0 \end{cases} && \text{for } i \in \mathbb{N} \text{ and } \beta \in I_d.
\end{aligned}$$

Verifying that this is a metric is straightforward since the c_i's and d_i's are all in the interval $\left(\frac{3}{4}a_0, \frac{3}{2}a_0\right)$. The only case which needs some care is verifying the triangle inequality for three points x^*_i, x_β, and x_γ. Here the key observation is that $d^0_\alpha(x^*_i, x_\beta)$ and $d^0_\alpha(x^*_i, x_\gamma)$ will each be either $\frac{3}{4}a_0$ or c_i, and they will be equal if $\beta(i) = \gamma(i)$. Otherwise we have $\beta(i) \neq \gamma(i)$, so $\{i\} \subseteq \beta\Delta\gamma$ and hence $c_i - \frac{3}{4}a_0 \leq \varphi_A(\{i\}) \leq \varphi_A(\beta\Delta\gamma) = d^0_\alpha(x_\beta, x_\gamma)$ as needed. Note that the diameter of X^0_α is less than r.

We now define the space X_α as the disjoint union

$$X_\alpha = X^0_\alpha \sqcup \bigsqcup_{n\geq 1} X_n.$$

We define the metric d_α by letting $d_\alpha = d^0_\alpha$ on X^0_α and $d_\alpha = d_n$ on X_n for $n \geq 1$. If we have points $x \in X_i$ and $y \in X_j$ (where $X_0 = X^0_\alpha$) with $i \neq j$ we let $d_\alpha(x,y) = \delta_{\max(i,j)}$. Verifying that d_α is a metric is now simple, noting that $\delta_n \geq \frac{1}{2}\text{diameter}(X_n)$.

It is clear that $\text{Dist}(X_\alpha, d_\alpha) = A$, and we can can find a Borel function producing a code for this space from α. We need to check that this function is a reduction of E_d to E_A, i.e., for α_1 and α_2 in $2^{\mathbb{N}}$ we have $\alpha_1\Delta\alpha_2 \in I_d$ if and only if $(x_{\alpha_1}, d_{\alpha_1}) \cong_i (X_{\alpha_2}, d_{\alpha_2})$. Suppose first that $\alpha_1\Delta\alpha_2 \in I_d$. We define an isometry $\psi : X_{\alpha_1} \to X_{\alpha_2}$ by making ψ the identity everywhere except for on the copy of I_d in $X^0_{\alpha_1}$ and setting $\psi(x_\beta) = x_{\beta\Delta\alpha_1\Delta\alpha_2}$ on I_d. The only part of the metric which depended on α was the case of $d_\alpha(x^*_i, x_\beta)$, and there we have:

$$\begin{aligned}
d^0_{\alpha_1}(x^*_i, x_\beta) &= \begin{cases} c_i & \text{if } \alpha_1\Delta\beta(i) = 1\\ \frac{3}{4}a_0 & \text{if } \alpha_i\Delta\beta(i) = 0 \end{cases}\\
&= \begin{cases} c_i & \text{if } \alpha_2\Delta(\beta\Delta\alpha_1\Delta\alpha_2)(i) = 1\\ \frac{3}{4}a_0 & \text{if } \alpha_2\Delta(\beta\Delta\alpha_1\Delta\alpha_2)(i) = 0 \end{cases}\\
&= d^0_{\alpha_2}(x^*_i, x_{\beta\Delta\alpha_1\Delta\alpha_2})\\
&= d^0_{\alpha_2}(x^*_i, \psi(x_\beta)).
\end{aligned}$$

For the converse, suppose ψ is an isometry of X_{α_1} with X_{α_2}. Since the distances d_i occur only between pairs of x_i^*'s and x_i^* is the unique such point whose distances to the other x_j^*'s are equal to d_i for all but finitely many j, these points must be fixed by ψ. Since the copy of I_d in X_α^0 is closer to these points than to any of the X_n's we must have that the copy of I_d in X_{α_1} is carried to the copy of I_d in X_{α_2}, so that ψ induces an isometry of I_d. For each $i \in \mathbb{N}$ and $\beta \in I_d$ we then have:

$$d^0_{\alpha_1}(x_i^*, x_\beta) = \begin{cases} c_i & \text{if } \alpha_1 \Delta \beta(i) = 1 \\ \frac{3}{4}a_0 & \text{if } \alpha_i \Delta \beta(i) = 0 \end{cases} =$$

$$d^0_{\alpha_2}(x_i^*, \psi(x_\beta)) = \begin{cases} c_i & \text{if } \alpha_2 \Delta \psi(\beta)(i) = 1 \\ \frac{3}{4}a_0 & \text{if } \alpha_2 \Delta \psi(\beta)(i) = 0. \end{cases}$$

Hence, for each $i \in \mathbb{N}$ we have $\alpha_1 \Delta \beta(i) = \alpha_2 \Delta \psi(\beta)(i)$, so that $\alpha_1 \Delta \beta = \alpha_2 \Delta \psi(\beta)$. But then $\alpha_1 \Delta \alpha_2 = \beta \Delta \psi(\beta) \in I_d$, since β and $\psi(\beta)$ are both in I_d which is closed under symmetric difference. $\square$

Our final result in this section gives a sufficient condition for E_A to be of maximal possible complexity, that is, for E_A to be bi-reducible with the general isometry relation on Polish metric spaces.

Proposition 22. *If A contains a half-neighborhood of 0, then E_A is bi-reducible with the universal orbit equivalence relation of a Polish group on a Polish space.*

Proof: It will suffice to reduce the full isometry relation of Polish metric spaces, $\cong_i$, to E_A. Suppose A contains the interval $[0, r)$. Choose a sequence $\delta_n \in A$ for $n \geq 1$ such that $\frac{r}{2} \leq \delta_n \leq \delta_{n+1}$ and $\delta_n \geq \frac{1}{2}\sup(A \cap [0, n \cdot r])$. Let $A' = A \setminus \{\delta_n : n \geq 1\}$, and let $A'_n = A' \cap [0, n \cdot r]$. Choose spaces (X_n, d_n) for $n \geq 1$ with $\mathrm{Dist}(X_n, d_n) = A'_n$ This depends only on A.

Now, given a Polish metric space (X, d) we form the space $(X', d') \in \mathcal{M}_A$ as follows. Let $X'_0 = X$ and $d'_0 = \frac{r}{2}\frac{d}{1+d}$. This is a metric on X with $d'_0 < \frac{r}{2}$, and it is immediate that for spaces (X, d_X) and (Y, d_Y) we have $(X, d_X) \cong_i (Y, d_Y)$ if and only if $(X'_0, (d_X)'_0) \cong_i (Y'_0, (d_Y)'_0)$. We now form the space (X', d') by letting

$$X' = X'_0 \sqcup \bigsqcup_{n \geq 1} X_n$$

and setting $d' = d'_0$ on X'_0, $d' = d_n$ on X_n for $n \geq 1$, and for $x \in X_i$ and $y \in X_j$ with $i \neq j$ (where $X_0 = X'_0$) setting $d'(x, y) = \delta_{\max(i,j)}$. This is a

Polish metric space with distance set A. When $(X, d_X) \cong_i (Y, d_Y)$ we easily have $(X', d'_X) \cong_i (Y', d'_Y)$. Conversely, note that an isometry of (X', d'_X) and (Y', d'_Y) must send each X_i to some Y_j since the distances δ_n occur only between points in different X_i's. We must then have $X'_0 = X$ sent to $Y'_0 = Y$ since these are the only two components with diameter less than or equal to $\frac{r}{2}$. This gives an isometry of $(X'_0, (d_X)'_0)$ with $(Y'_0, (d_Y)'_0)$, and hence $(X, d_X) \cong_i (Y, d_Y)$. □

We summarize the above results in the following theorem.

Theorem 23. *Let A be an analytic set which is either countable or has 0 as a limit point. Then:*

1. *If A contains a neighborhood of 0 then $E_A \sim_B \cong_i$.*
2. *If A is dense in a neighborhood of 0 but A does not contain a neighborhood of 0 then E_d and $\cong_{graphs}$ are reducible to E_A and E_A is reducible to isometry of 0-dimensional Polish metric spaces.*
3. *If A is not dense in a neighborhood of 0 and A either has 0 as a limit point or contains two distances d_0 and d_1 with $d_0 < d_1 \leq 2d_0$, then $E_A \sim_B \cong_{graphs}$.*
4. *If $A = \{0\} \cup \{d_i : i \in \mathbb{N}\}$ with $2d_i < d_{i+1}$ for each i, then E_A is bi-reducible with isomorphism of reverse trees.*
5. *If $A = \{0, d_0, \cdots, d_{n-1}\}$ with $2d_i < d_{i+1}$ for each i, then E_A is bi-reducible with isomorphism of trees of height n.*

This leaves one case where we do not know the exact complexity of E_A.

Question 1. *What is the exact complexity of E_A when A is dense in some neighborhood of 0 but does not contain a neighborhood of 0?*

In this case we know that E_A is strictly above graph isomorphism since both graph isomorphism and E_d are reducible to it. The spaces in $\mathcal{M}_A$ are all 0-dimensional, so we also have E_A reducible to isometry of 0-dimensional Polish metric spaces for all such A. We do not know know the exact complexity of isometry of 0-dimensional spaces, though.

Question 2. *What is the complexity of isometry of 0-dimensional Polish metric spaces?*

We note one additional gap in the above results. In all of the cases where we have determined the exact complexity of E_A, we know that when $A \subseteq A'$ then $E_A \leq_B E_{A'}$, but we do not actually have a direct proof of this. In order to add missing distances in the above proofs we were forced to use bounded metrics in parts of our reductions, and we see no way to simply add in additional distances. Although it seems almost certainly true that complexity increases with distance sets, we do not know how to prove this. More generally, if we let $E_{\subseteq A}$ denote isometry of spaces whose distance set is contained in A, we do not know the complexity of $E_{\subseteq A}$. We can thus ask:

Question 3. *If $A \subseteq A'$, is $E_A \leq_B E_{A'}$? Is $E_{\subseteq A} \sim_B E_A$?*

4 Countable metric spaces

We end by characterizing the complexity of E_A^{ctble} for countable sets A. As noted above, we have $E_A^{\text{ctble}} \leq_B \cong_{\text{graphs}}$ for all countable sets A. As also noted, when A is bounded away from 0 we have $E_A^{\text{ctble}} = E_A$ so that the classifications given in those cases for E_A remain the same. Next, observe that in the proof of Proposition 13 we can take the spaces X_n to be countable when A'_n is countable, so that we can produce a countable space X_G. Hence we .get:¡F13¿

Corollary 24. *If A is countable and $A \supseteq \{d_0, d_1\}$ with $0 < d_0 < d_1 \leq 2d_0$ then E_A^{ctble} is bi-reducible with graph isomorphism.*

The one difference is in the proof of Proposition 14. The spaces (X_T, d_T) produced in reducing isomorphism of trees to E_A will only be countable when T has at most countably many branches, so the proof only shows that isomorphism of trees with countably many branches is reducible to E_A^{ctble}. This turns out to be the precise bound when A does not contain two distances as above:

Proposition 25. *If $A = \{0\} \cup \{d_i : i \in \mathbb{N}\}$ or $A = \{0\} \cup \{d_i : i \in \mathbb{Z}\}$ with $d_i > 2d_{i+1}$ for each i, then E_A is bi-reducible with isomorphism of pruned trees with countably many branches.*

Proof: To reduce isomorphism of trees with countably many branches, we proceed as in the proof of Proposition 14. We can take the spaces X_n to be countable since each A'_n is countable, and since we start with a tree with countable many branches the space (T, d_T^0) will have countably many limit points; hence X_T will be a countable space.

To show that E_A is reducible to isomorphism of pruned trees with countably many branches, we first consider the case of $A = \{0\} \cup \{d_i : i \in \mathbb{N}\}$ with $d_i > 2d_{i+1}$. Given a countable space $(X, d) \in \mathcal{M}_A$ we can form the tree T_X as in the proof of Proposition 15; this will now be a tree of infinite height. Since each branch of T_X corresponds to a point of X, T_X will have only countably many branches, and we have $T_X \cong T_Y$ iff $(X, d_x) \cong_i (Y, d_Y)$.

For the case of $A = \{0\} \cup \{d_i : i \in \mathbb{Z}\}$, first form trees T_X^n for $n \geq 0$ as above but using only closed balls of radius d_i for $i \geq -n$. We then form the tree T_X by attaching a chain of length n followed by T_X^n to the root for each $n \geq 0$. Again we easily have $T_X \cong T_Y$ when $(X, d_X) \cong_i (Y, d_Y)$. For the reverse direction, an isomorphism of T_X with T_Y must induce an isomorphism from T_X^n to T_Y^n for each n. We can also find maps from T_X^n to T_X^{n+1} which are isomorphisms except that they do not include the closed balls of radius $d_{-(n+1)}$ in the range. Forming the limit of these isomorphisms allows us to construct an isomorphism from a bi-infinite tree for X (including balls of radius d_i for all $i \in \mathbb{Z}$) with the corresponding bi-infinite tree for Y, which induces an isomorphism from (X, d_X) to (Y, d_Y). $\square$

We summarize the classification of the complexity of E_A^{ctble}.

Theorem 26. *Let A be a countable set. Then:*

1. *If A contains two distances d_0 and d_1 with $d_0 < d_1 \leq 2d_0$, then $E_A^{ctble} \sim_B \cong_{graphs}$.*

2. *If $A = \{0\} \cup \{d_i : i \in \mathbb{N}\}$ or $A = \{0\} \cup \{d_i : i \in \mathbb{Z}\}$ with $2d_{i+1} < d_i$ for each i, then E_A^{ctble} is bi-reducible with isomorphism of pruned trees with countably many branches.*

3. *If $A = \{0\} \cup \{d_i : i \in \mathbb{N}\}$ with $2d_i < d_{i+1}$ for each i, then E_A^{ctble} is bi-reducible with isomorphism of reverse trees.*

4. *If $A = \{0, d_0, \cdots, d_{n-1}\}$ with $2d_i < d_{i+1}$ for each i, then E_A^{ctble} is bi-reducible with isomorphism of trees of height n.*

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