The set of distances in a Polish metric space

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Abstract

We show that a set of non-negative reals is the distance set of a separable complete metric space if and only if it is an analytic set which is either countable or has 0 as a limit point.

In this article we consider the possible sets of distances in Polish metric spaces. A *Polish metric space* is a pair (X, d), where X is a Polish space (a separable, completely-metrizable space) and d is a complete, compatible metric for X. Given a Polish metric space (X, d), we can consider its set of distances. $\mathrm{Dist}(X, d)$:

Definition 1. Let(X, d) be a Polish metric space. The *distance set* of (X, d), $\mathrm{Dist}(X, d)$, is:

$$\mathrm{Dist}(X, d) = \{d(x, y) : x, y \in X\}.$$

Our main result will characterize which sets of reals can be the distance set of some Polish metric space. We will also characterize the distance sets of specific classes of metric spaces, e.g. compact, locally compact, or zero-dimensional spaces. In the last section we will briefly consider the question of which sets of triangles (and larger configurations of points) can occur in a Polish metric space. Our interest in distance sets is partially motivated by the question of classifying metric spaces up to isometry, since distance sets form an isometry invariant, although generally not a complete invariant. An overview of the isometry problem may be found in [3].

1 The set of distances

Distance sets have been studied in several contexts. Much of the work has been on the distance sets of subsets of the spaces $\mathbb{Z}$, $\mathbb{R}$, or $\mathbb{R}^n$, with the usual metrics. One of the earliest results is Steinhaus's theorem (in [11]) that the distance set of a subset of $\mathbb{R}$ of positive measure contains a (right-) neighborhood of 0. Sierpiński also showed in [10] that distance sets could be more complicated than the set itself by producing a G_δ subset of $\mathbb{R}$ whose distance set is $\mathbf{\Sigma}^1_1$-complete.

In the case that the metric space in question is a subset of $\mathbb{Z}$ (with the usual metric), the distance set is generally known as a *difference set*. Schmerl in [9] has characterized how complicated the set of difference sets is, showing that in a descriptive sense there can be no better characterization of when a subset of $\mathbb{Z}$ is a difference set than the basic definition:

Theorem (Schmerl)**.** *The set*

$$\{A \subseteq \mathbb{Z} : A \textit{ is the difference set of some } B \subseteq \mathbb{Z}\}$$

is $\mathbf{\Sigma}^1_1$*-complete.*

There are also several results characterizing which sets can be the set of distances in some type of metric space. In the most general context, we have the following result (see [8]):

Theorem (Kelly and Nordhaus)**.** *Any set of non-negative reals containing 0 is the distance set of some metric space.*

Several results characterize which sets can be distance sets for subsets of $\mathbb{R}^n$ (see for instance [7]). More generally, in [8] the authors characterize which sets can be the set of distances of some separable metric space:

Theorem (Kelly and Nordhaus)**.** *A non-negative set of reals (containing 0) is the set of distances of some separable metric space if and only if it is either countable or has 0 as a limit point.*

Here we will be concerned with with the case of *Polish* metric spaces so we will need to produce complete metrics. Two properties of the distance set of a Polish metric space are clear. First, since d is a continuous map from the Polish space X^2 to $\mathbb{R}$, $\mathrm{Dist}(X, d)$ is an analytic set of non-negative reals containing 0. Second, if $\mathrm{Dist}(X, d)$ is uncountable, then X must also be uncountable, so in order for it to be separable $\mathrm{Dist}(X, d)$ must contain distances arbitrarily close to 0.

It turns out that these two conditions are sufficient for a non-negative set of reals to be the set of distances for some Polish metric space (X, d). In the next section we will consider some special classes of metric spaces which have more restrictive conditions on their sets of distances. Here we prove the main theorem:

Theorem 2. *A set $A \subseteq [0, \infty)$ is the set of distances for some Polish metric space if and only if A is an analytic set containing 0 and either A is countable or 0 is a limit point of A.*

Proof: First we consider the case where A is countable. If $A = \{0\}$ then we can take X to be the one-point space; otherwise, let $A = \{a_i : i \in \omega\}$ (where $\omega = \mathbb{N}$ is the set of natural numbers) with $a_0 = 0$ (if A is finite we allow the non-zero a_i's to repeat). We will construct (X, d) so that $\operatorname{Dist}(X, d) = A$. Our space X will have the underlying set $\{x_i : i \in \omega\}$. We let $d(x_i, x_j) = \max(a_i, a_j)$ for $i \neq j$. We see that d is a metric (in fact an ultrametric), since the two largest sides in any triangle will have equal length. We can see that d is complete by noting that if $\langle x_{i_n} : n \in \omega\rangle$ is a non-constant Cauchy sequence then $\lim a_{i_n} = 0$, so that the sequence converges to x_0. Since X is countable it is separable, and so (X, d) as constructed is a Polish metric space with $\operatorname{Dist}(X, d) = \{a_i : i \in \omega\} = A$.

We next consider the case that A is analytic with 0 a limit point of A. We will first assume that $A \subseteq [0, 1)$, and handle the general case at the end. We may identify sequences from $\{0, 1\}$ with reals in $[0, 1]$ via the map

$$\alpha \mapsto \sum_{i \in \omega} \frac{\alpha(i)}{2^{i+1}}.$$

This is a continuous map, so the inverse image of an analytic set of reals is an analytic subset of the Cantor space $\mathcal{C} = 2^\omega$. This is a one-to-one map except for those points with eventually constant binary representations. We may thus represent A as an analytic subset of the Cantor space; we will identify this set with A, and we will consider elements of A both as real numbers and as binary sequences. We can then express A as the projection of a closed subset of $\mathcal{C} \times \mathcal{N}$ (where $\mathcal{N}$ is the Baire space ω^ω) and thus as the projection of a pruned tree on $2 \times \omega$. So let T be a pruned tree on $2 \times \omega$ such that

$$\alpha \in A \iff (\exists \beta)(\forall n)[(\alpha \restriction n, \beta \restriction n) \in T].$$

For technical reasons, we choose T so that there is a unique branch projecting to 0 (i.e. the infinite sequence 0^∞). We can do this since $A \setminus \{0\}$ is still an analytic set, so we can take a tree projecting to $A \setminus \{0\}$ and add back a unique branch projecting to 0. We now let

$$T^* = \{s \in 2^{<\omega} : (\exists a \in \omega^{<\omega})[(s, a) \in T]\}.$$

Then T^* is a pruned tree on 2 with $[T^*] = \bar{A}$, the closure of A in 2^ω (where for a tree T, $[T]$ is the set of infinite branches through T). For each $s \in T^*$, we can thus pick $d_s \in A$ with $s \sqsubset d_s$. Choose finally a decreasing sequence $\langle \epsilon_i \rangle_{i \in \omega}$ from among the d_s's with $\epsilon_i \in A$ and $\epsilon_i < \frac{1}{2^i}$, which is possible since 0 is a limit point of A.

We set $X = [T]$, and define d as follows. For $x_1 = (\alpha_1, \beta_1)$ and $x_2 = (\alpha_2, \beta_2)$ with $x_1 \neq x_2$ we let (t, a) be the maximal mutual predecessor of x_1 and x_2 in T. We then let:

$$d(x_1, x_2) = \begin{cases} \max(\alpha_1, \alpha_2) & \text{if } t = 0^k \\ \text{least } \epsilon_i \text{ s.t. } 2^{-|t|} \leq \epsilon_i \leq d_t & \text{if } t \neq 0^k \text{ and such an } \epsilon_i \text{ exists} \\ d_t & \text{otherwise} \end{cases}$$

We check that this is a metric, which amounts to verifying the triangle inequality. Fix (α_i, β_i) distinct for $i = 1, 2, 3$.

1. If x_1, x_2 and x_3 have a mutual maximal predecessor (t, a), then:
    (a) If $t = 0^k$ then each distance $d(x_i, x_j) = \max(\alpha_i, \alpha_j)$, so this is an ultrametric triangle (the longest two distances are equal).
    (b) If $t \neq 0^k$ then all the distances are the same, since they depend only on t.
2. Otherwise, two of the branches agree longer than they do with the third. We may assume that we have $(t, a) \sqsubset (s, b)$ with (s, b) the maximal mutual predecessor of x_1 and x_2, and (t, a) the maximal predecessor of x_3 and (s, b). We have three sub-cases:
    (a) If $s = 0^k$ then again this is an ultrametric triangle.
    (b) If $s \neq 0^k$ and $t = 0^j$ then

$$\begin{aligned} d(x_1, x_3) &= \max(\alpha_1, \alpha_3) \\ d(x_2, x_3) &= \max(\alpha_2, \alpha_3) \\ 2^{-|s|} \leq d(x_1, x_2) &\leq s + 2^{-|s|} \end{aligned}$$

where by s we mean here the real coded by the sequence s followed by all 0's. Note that we have

$$|\alpha_1 - \alpha_2| \leq s \leq \delta_{1,2} \leq s + 2^{-|s|} \leq \alpha_1 + \alpha_2.$$

We also see that in all cases we have $|d(x_1, x_3) - d(x_2, x_3)| \leq |\alpha_1 - \alpha_2|$, so that we have

$$|d(x_1, x_3) - d(x_2, x_3)| \leq d(x_1, x_2) \leq d(x_1, x_3) + d(x_2, x_3),$$

which guarantees the triangle inequality.

(c) If $t \neq 0^j$ then $d(x_1, x_3) = d(x_2, x_3)$ since these distances depend only on t. If this distance is some ϵ_i then we have $d(x_1, x_2)$ less than or equal to this distance, since it will either be this ϵ_i or some smaller ϵ_j. If this distance is d_t, then we have $d(x_1, x_2) \leq d_s \leq d_t + 2^{-|t|} \leq 2 \cdot d_t$ since $t \neq 0^j$, so in all cases the triangle inequality holds here.

Thus d is a metric; we check that it is complete. Let $\langle x_i \rangle$ be a Cauchy sequence with $x_i = (\alpha_i, \beta_i) \in [T]$. If there is some subsequence of the α_i's approaching 0, then distances between the corresponding x_i's will be determined by case (1) of the metric, so that the sequence must in fact approach the branch with $\alpha = 0^\infty$. Otherwise, the α_i's are bounded away from 0, so that for the sequence to be Cauchy we must have some i_0 beyond which the distances are defined by case (2), so that in fact the sequence of (α_i, β_i)'s must converge to some (α, β) in the usual topology on $\mathcal{C} \times \mathcal{N}$. Since $[T]$ is closed, we have $(\alpha, \beta) \in [T]$ and we see that the Cauchy sequence approaches this point.

To check separability, we pick for each $(t, a) \in T$ some branch in $[T]$ extending this node. This yields a countable set, and for any other branch in $[T]$, if we pick one of these (t, a)'s which agrees up to a long enough initial segment we can come arbitrarily close.

Finally, it is clear that $\mathrm{Dist}(X, d) = A$, since all of the distances defined are in A, and distances from the branch with $\alpha = 0^\infty$ will include all elements of A. This completes the proof when $A \subseteq [0, 1)$.

For the general case of $A \subseteq [0, \infty)$, let $A_n = A \cap [0, n)$. Since the A_n's satisfy the hypotheses of the theorem, we can construct (X_n, d_n) as above (stretching the metric by n) such that $\mathrm{Dist}(X_n, d_n) = A_n$. Now choose a sequence $\langle \delta_n \rangle$ with $\delta_n \in A_n$, $\delta_n \geq \frac{1}{2} \sup A_n$, and $\delta_n \leq \delta_{n+1}$. We let X be the disjoint union $\bigsqcup_{n \geq 1} X_n$. We set $d = d_n$ on each X_n, and for $x \in X_n$, $y \in X_m$ with $n < m$ we let $d(x, y) = \delta_m$. The conditions on the δ_n's guarantee that this is a complete metric and adds no distances other than those in the A_n's, so that $\mathrm{Dist}(X, d) = \bigcup_{n \geq 1} A_n = A$. □

There is a lack of uniformity in the above construction. For a given set A, we needed to pick a tree representation of it in order to construct our space, and different trees may give rise to non-isometric spaces. In a subsequent article ([1]) we will consider how close the distance set is to being a complete invariant for isometry, and show that it is very far from being complete. In fact, for many analytic sets A the classification up to isometry of spaces with distance set A is as complicated as the classification of all Polish metric spaces.

2 Distance sets of special classes of spaces

We now consider various special classes of Polish metric spaces, and ask what the set of distances can look like. In general, the set of distances will retain some of the topological properties of the original space (in particular, any properties preserved under continuous images). The classes we will consider include compact, locally compact, σ-compact, ultrametric, discrete, perfect, zero-dimensional, connected and path-connected spaces. There is some correspondence between the topological complexity of distance sets and the complexity of the isometry problem for a given class of spaces; see [2], [3], or [4] for results on these classification problems. For brevity, we will always assume that 0 is contained in a putative set of distances.

Theorem 3. *A set $A \subseteq [0, \infty)$ is the set of distances of some compact metric space if and only if either A is finite or A is compact and 0 is a limit point of A.*

Proof: Since X^2 is compact when X is, and $d : X^2 \to \mathbb{R}$ is continuous, we must have that $\mathrm{Dist}(X, d)$ is compact (the continuous image of a compact set is compact). Since the only discrete compact spaces are finite, we must have 0 as a limit point of $\mathrm{Dist}(X, d)$ if X is infinite, so the conditions are necessary.

Let A satisfy the conditions. The case where A is finite is handled in the same way that the countable case of Theorem 2 was, except that we only include finitely many points in the space.

For the uncountable case, we may assume $A \subseteq [0, 1]$ since A is bounded (and we may simply multiply the resulting metric by a constant). We can represent the set A as the branches through a tree on $\{0, 1\}$, i.e. a closed subset of the Cantor space, using as before the fact that the map $\alpha \mapsto \sum_{i \in \omega} \frac{\alpha(i)}{2^{i+1}}$ is a continuous surjection of $\mathcal{C}$ onto $[0, 1]$. The construction will resemble the original construction, using a tree on $\{0, 1\}$ rather than on $\{0, 1\} \times \omega$.

Let T be a pruned tree with $A = [T]$, and choose $\{d_s : s \in T\}$ and $\langle \epsilon_i \rangle$ as in Theorem 2, with T here taking the place of T^*. We set $X = [T]$ and define d exactly as before. Verifying that this is a separable complete metric space with $\mathrm{Dist}(X, d) = A$ is also the same, so it only remains to check compactness.

We will show that (X, d) is totally bounded. Fix $\epsilon > 0$, and let n_0 be such that there is some ϵ_i with $2^{-n_0} < \epsilon_i < \epsilon$. Set:

$$D_\epsilon = \{d_s : s \in T \text{ and } |s| \leq n_0\}$$

(where by d_s we mean here the branch coding the real). Then D_ϵ is a finite set, and we claim that every point in X is within distance ϵ of some point in D_ϵ. This is clear from our metric, since any point must agree with one of these branches through its first n_0 coordinates; if these are not all 0, we have an ϵ_i with $2^{-n_0} < \epsilon_i < \epsilon$, and if they are all 0 then the distance to $d_{0^{n_0}}$ is at most 2^{-n_0}. $\square$

Theorem 4. *The following are equivalent:*

1. $A \subseteq [0, \infty)$ *is the set of distances of some locally compact Polish metric space.*
2. A *is the set of distances of some* σ*-compact Polish metric space.*
3. A *is either countable or* A *is* K_σ *with 0 as a limit point.*

Proof: (1) $\Longrightarrow$ (2) follows from the fact that any locally compact Polish space is σ-compact. For (2) $\Longrightarrow$ (3), let $X = \bigcup_{i\in\omega} K_i$ where each K_i is compact and $K_i \subseteq K_{i+1}$. Then $(K_i, d \restriction K_i)$ is a compact metric space. Set $A_i = \mathrm{Dist}(K_i, d \restriction K_i)$, so that each A_i is compact. Notice that if $d_0 = d(x, y)$ for some $x, y \in X$, then there is some K_i such that $x, y \in K_i$, so that $d_0 \in A_i$. Thus, we get that $\mathrm{Dist}(X, d) = \bigcup_{i\in\omega} A_i$, so that $\mathrm{Dist}(X, d)$ is K_σ, and is either countable or contains 0 as a limit point since (X, d) is a Polish metric space.

For (3) $\Longrightarrow$ (1), let $A = \bigcup_{i\in\omega} A_i$ where each A_i is compact, $A_i \subseteq A_{i+1}$, and $0 \in A_0$. If 0 is a limit point of A, then we can choose these sets such that 0 is a limit point of A_0. Simply add 0 to A_0 along with a sequence approaching 0; this sequence, together with the point 0, is a compact set. We will also assume that if A contains more than one point, then A_0 contains a non-zero point. Then each A_i satisfies the hypotheses of Theorem 3, and so we can build compact metric spaces (X_i, d_i) with $\mathrm{Dist}(X_i, d_i) = A_i$. Let $M_i = \sup A_i < \infty$; this is an element of A_i by the compactness of A_i. Note that $M_i \leq M_{i+1}$, so for all i we have $M_i \geq M_0 > 0$. We now define our space to be the disjoint union of the X_i's, $X = \bigsqcup_{i\in\omega} X_i$. To define d, let

$$d(x, y) = \begin{cases} d_i(x, y) & \text{if } x, y \in X_i \text{ for some } i \\ M_j & \text{if } x \in X_i,\ y \in X_j \text{ for } i < j \end{cases}.$$

This defines a Polish metric space, and it will be locally compact because for each $x \in X$ the open ball of radius M_0 centered at x is contained entirely within some X_i and hence has compact closure. $\square$

Recall that a metric d is said to be an *ultrametric* if for all x, y, z, we have

$$d(x, z) \leq \max(d(x, y), d(y, z)).$$

This is equivalent to saying that the longest two sides in any triangle have the same length.

Theorem 5. *A set $A \subseteq [0, \infty)$ is the set of distances of some ultrametric Polish metric space if and only if A is countable.*

Proof: For a countable set A, recall that the construction given in Theorem 2 for the case where A is countable in fact produces an ultrametric space. For the other direction, let (X, d) be a given ultrametric Polish metric space. Fix a countable dense set $\mathcal{D} \subseteq X$, $\mathcal{D} = \{x_i : i \in \omega\}$. We claim that

$$\mathrm{Dist}(X, d) = \{d(x_i, x_j) : i, j \in \omega\},$$

which is a countable set. To see this, let $x, y \in X$ with $x \neq y$, where $x = \lim x_{i_n}$ and $y = \lim x_{j_n}$ with $\langle x_{i_n} \rangle$ and $\langle x_{j_n} \rangle$ Cauchy sequences. Let $\delta = d(x, y) = \lim_{n \to \infty} d(x_{i_n}, x_{j_n})$. Now, since the two sequences are Cauchy and approach x and y respectively, there is an N such that

$$(\forall n \geq N) \left(d(x_{i_n}, x) < \frac{\delta}{2} \ \wedge \ d(x_{j_n}, y) < \frac{\delta}{2} \right)$$

and

$$(\forall n, m \geq N) \left(d(x_{i_n}, x_{i_m}) < \frac{\delta}{2} \ \wedge \ d(x_{j_n}, x_{j_m}) < \frac{\delta}{2} \right).$$

Now, for $n \geq N$, consider the arrangement of x, y, x_{i_n} and x_{j_n}. Because d is an ultrametric, we must have

$$d(x, x_{j_n}) = \max(d(x, y), d(y, x_{j_n})) = \delta,$$

since $d(y, x_{j_n}) < \delta$. Then, since $d(x, x_{i_n}) < \delta$, we have

$$d(x_{i_n}, x_{j_n}) = \max(d(x, x_{i_n}), d(x, x_{j_n})) = \delta.$$

From this we conclude that

$$d(x, y) = d(x_{i_N}, x_{j_N}) \in \{d(x_i, x_j) : i, j \in \omega\}$$

as desired. □

Theorem 6. *A set* $A \subseteq [0, \infty)$ *is the set of distances of some perfect, compact, ultrametric space if and only if A can be enumerated as a countable decreasing sequence* $\langle d_i : i \geq 0 \rangle$ *with* $\lim_{i \to \infty} d_i = 0$.

Proof: Suppose we have $A = \{d_i : i \geq 0\}$ with $d_i > d_{i+1}$ and $\lim_{i \to \infty} d_i = 0$. We take as our underlying set $X = 2^\omega$. For $\alpha, \beta \in 2^\omega$ with $\alpha \neq \beta$ we define

$$d(\alpha, \beta) = d_{n(\alpha,\beta)}$$

where

$$n(\alpha, \beta) = \text{the least } n \text{ such that } \alpha(n) \neq \beta(n).$$

This metric is equivalent to the usual metric on the Cantor space 2^ω given by $d(\alpha, \beta) = 2^{-n(\alpha,\beta)}$. Since the Cantor space is compact and perfect, so is this space. To see that the metric is an ultrametric, note that for α, β and γ we have $n(\alpha, \gamma) \geq \min(n(\alpha, \beta), n(\beta, \gamma))$ so $d(\alpha, \gamma) \leq \max(d(\alpha, \beta), d(\beta, \gamma))$.

For the other direction, let (X, d) be a perfect, compact, ultrametric space. Since the space is perfect it must have distances arbitrarily close to 0. It will thus suffice to check that in a compact ultrametric space (X, d), for any $b > 0$ the set

$$\{d(x, y) : x, y \in X \text{ and } d(x, y) \geq b\}$$

is finite, since we can then take d_0 to be the largest distance, d_1 to be the next largest, and so forth.

To see this, suppose there were $b > 0$ such that the above set is infinite. We will find an infinite sequence of distinct points $\langle x_i : i \in \omega \rangle$ with $d(x_i, x_j) \geq b$ for $i \neq j$, contradicting total boundedness. Let $x_0 \in X$ be an arbitrary point, and suppose we have chosen $x_0, \ldots, x_k$. By our assumption we can choose points y and z with $d(y, z) \geq b$ and

$$d(y, z) \notin \{d(x_i, x_j) : 0 \leq i, j \leq k\}.$$

We claim that at least one of these two points will work for x_{k+1}. If not, there would be i and j with $d(x_i, y) < b$ and $d(x_j, z) < b$. Then considering the triangle x_i, y, z we would have $d(x_i, z) = \max(d, b) = d$. Hence $x_i \neq x_j$, but then the three distances in the triangle x_i, x_j, z must be distinct, contradicting that d is an ultrametric. Hence we can choose x_{k+1} to satisfy the requirements, and continue. □

Theorem 7. *A set* $A \subseteq [0, \infty)$ *is the set of distances of some discrete metric space if and only if A is countable.*

Proof: A discrete Polish space is of course countable and so has a countable set of distances. Conversely, let A be a countable set, and let $A \setminus \{0\} = \{a_i : i \in \omega\}$, where we allow repetitions in the case A is finite. Now let the space have underlying set

$$X = \{x_i : i \in \omega\} \cup \{y_i : i \in \omega\}$$

and define d by:

$$\begin{aligned} d(x_i, y_i) &= a_i \\ d(x_i, x_j) &= d(y_i, y_j) = d(x_i, y_j) = \max(a_0, a_i, a_j) \text{ for } i \neq j. \end{aligned}$$

It is straightforward to check that this defines a metric. It is complete since all distances are at least a_0, so there are no non-trivial Cauchy sequences, and it is discrete since for each x_i or y_i, no point has distance less than a_i from it. □

A *zero-dimensional* space is one in which there is a basis consisting of clopen sets. Although zero-dimensional spaces are a special class of Polish metric spaces, their distance sets can be as complicated as those of arbitrary Polish metric spaces:

Theorem 8. *A set $A \subseteq [0, \infty)$ is the set of distances of some zero-dimensional Polish metric space if and only if either A is countable or A is analytic with 0 as a limit point.*

Proof: In fact, the construction in Theorem 2 produces zero-dimensional spaces. For the countable case, this is immediate. For the uncountable case, every point other than the branch corresponding to 0 has a clopen basis, since the topology there is essentially the subspace topology of $\mathcal{C} \times \mathcal{N}$. Around the point 0, if we fix some sufficiently large n_0 and consider the set

$$\begin{aligned} G &= \{(\alpha, \beta) : \alpha < 2^{-n_0}\} \cup \{(\alpha, \beta) : \alpha \restriction n_0 = 0^{n_0 - 1} \frown 1\} \cup \\ &\quad\ \{(\alpha, \beta) : \alpha \restriction (n_0 + 1) = 0^{n_0} \frown 1\} \\ &= \{(\alpha, \beta) : \alpha \leq 2^{-n_0}\} \cup \{(\alpha, \beta) : \alpha \restriction n_0 = 0^{n_0 - 1} \frown 1\} \cup \\ &\quad\ \{(\alpha, \beta) : \alpha \restriction (n_0 + 1) = 0^{n_0} \frown 1\}, \end{aligned}$$

then G is clopen since the first expression is open and the second is closed (we need both of the last two sets in each expression to account for both the eventually 0 and eventually 1 representation of 2^{-n_0}) and $0 \in G \subseteq B_{2^{-(n_0-1)}}(0)$, so that there is a clopen basis at 0 as well. □

Theorem 9. *The following are equivalent:*

1. *A is the set of distances of a path-connected Polish metric space.*
2. *A is the set of distances of a connected Polish metric space.*
3. *A is an interval of the form* $[0, r)$*,* $[0, r]$*, or* $[0, \infty)$*.*

Proof: (1) $\Longrightarrow$ (2) is immediate. For (2) $\Longrightarrow$ (3), we have that the continuous image of a connected set is connected, so $\mathrm{Dist}(X, d)$ must be a connected subset of $\mathbb{R}$ containing 0, and hence an interval. To show (3) $\Longrightarrow$ (1), for the first type of interval $[0, r)$ we can take the space to be $X = \mathbb{R}$ with the metric $d(x, y) = r\frac{|x-y|}{1+|x-y|}$. For an interval $[0, r]$, we can take $X = [0, r]$ with the standard metric, and for $[0, \infty)$ we can take $X = \mathbb{R}$ with the standard metric. $\square$

3 Larger point configurations

As shown in [1], the distance set of a Polish metric space is very far from being a complete invariant for isometry. We can try to improve on this by considering configurations of a larger number of points. Having characterized the possible sets of distances in a Polish metric space, one can ask what sets of triangles are possible, or in general what sets of n-point configurations are possible for a given n.

Definition 10. For (X, d) a Polish metric space and $n \geq 2$, let the *n-point spectrum* be

$$\begin{aligned} &\mathrm{Spec}_n(X, d) = \\ &\qquad \{\langle d_{i,j}\rangle_{i<j<n} : (\exists x_0, \ldots, x_{n-1} \in X)(\forall i < j < n)[d_{i,j} = d(x_i, x_j)]\}. \end{aligned}$$

Note that $\mathrm{Spec}_2(X, d) = \mathrm{Dist}(X, d)$. In general $\mathrm{Spec}_n(X, d)$ completely determines $\mathrm{Spec}_m(X, d)$ for $m < n$, and $\mathrm{Spec}_n(X, d)$ is an analytic subset of $\mathbb{R}^{\frac{n(n1)}{2}}$. There is not such a simple characterization of the possible n-point spectra as there was in the case $n = 2$. We note herre one additional necessary condition in the case $n = 3$, the set of triangles.

Suppose the space contains a triangle with one side of length d_0, and suppose $d_1 > d_0$ is any distance occurring in the space. Then, by considering points x_1 and x_2 with $d(x_1, x_2) = d_0$ and points y_1 and y_2 with $d(y_1, y_2) = d_1$ (where one of the y's may be the same as one of the x's), we see that

$$\begin{aligned} d(x_1, y_1) + d(x_1, y_2) &\geq d_1 \\ d(x_2, y_1) + d(x_2, y_2) &\geq d_1. \end{aligned}$$

Thus $d(x_1, y_1) + d(x_1, y_2) + d(x_2, y_1) + d(x_2, y_2) \geq 2d_1$, from which we see that at least one of these four distances must be at least as large as $\frac{d_1}{2}$. In other words, if $d_1 > d_0$ are any two distances in (X, d), then (X, d) contains a triangle with one side of length d_0 and another side of length at least $\frac{d_1}{2}$. So the behavior of the set of triangles is not "local"; adding one triangle will necessitate adding a number of others. So we can ask the following:

Question 1. *For $n \geq 3$, what sets can be $\mathit{Spec}_n(X, d)$ for some Polish metric space (X, d)?*

Although Polish metric spaces are not in general characterized up to isometry by their sets of distances, or even by the sequence $\langle \mathrm{Spec}_n(X, d) : n \in \omega \rangle$, there are two cases in which this is true. One example is the case of compact metric spaces: Two compact metric spaces are isometric if and only if they have the same n-point spectra for all $n \geq 2$. In this case, the spectra are fairly concrete: $\mathrm{Spec}_n(X, d)$ is a compact subset of $\mathbb{R}^{\frac{n(n-1)}{2}}$. Thus, we can take a sequence of compact sets as a complete invariant for isometry, which is essentially a real. This gives a proof of the result of Gromov (see [5]) that the isometry relation on compact metric spaces is concretely classifiable.

Another case where the spectra form complete invariants is that of *ultrahomogeneous* spaces, those in which any isometry between finite subsets of the space extends to an isometry of the whole space. Having equal spectra here is sufficient to use a back-and-forth argument to construct an isometry. Here the spectra are no longer compact, so we do not get a concrete classification. It would be interesting to know the possible spectra in this case, as an indication of the complexity of the isometry relation of ultrahomogeneous Polish metric spaces.

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