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CLASSIFYING BOREL AUTOMORPHISMS

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§1. Introduction. This paper considers several complexity questions regarding *Borel automorphisms* of a Polish space. Recall that a Borel automorphism is a bijection of the space with itself whose graph is a Borel set (equivalently, the inverse image of any Borel set is Borel). Since the inverse of a Borel automorphism is another Borel automorphism, as is the composition of two Borel automorphisms, the set of Borel automorphisms of a given Polish space forms a group under the operation of composition. We can also consider the class of automorphisms of all Polish spaces. We will be primarily concerned here with the following notion of equivalence:

DEFINITION 1. *Two Borel automorphisms f and g of the Polish spaces X and Y are said to be Borel isomorphic, $f \cong g$, if they are conjugate, i.e. there is a Borel bijection $\varphi : X \to Y$ such that $\varphi \circ f = g \circ \varphi$.*

We restrict ourselves to automorphisms of uncountable Polish spaces, as the Borel automorphisms of a countable space are simply the permutations of the space. Since any two uncountable Polish spaces are Borel isomorphic, any Borel automorphism is Borel isomorphic to some automorphism of a fixed space. Hence, up to Borel isomorphism we can fix a Polish space and represent any Borel automorphism as an automorphism of this space. We will use the Cantor space 2^ω (with the product topology) as our representative space.

We may then represent a Borel automorphism by its graph, which is a subset of $(2^\omega)^2$. This graph is a Borel set, and may thus be coded as a real using a coding of Borel sets. The set of Borel automorphisms can then be viewed as a set of reals, and the relation of Borel isomorphism as an equivalence relation on this set. This allows us to analyze the complexity of this relation using descriptive set-theoretic techniques. Two natural questions arise:

1. How complicated is this equivalence relation descriptively; i.e., where does it fall in the Wadge hierarchy?
2. How complicated is this relation in the hierarchy of equivalence relations under Borel reducibility?

We will be able to completely answer the first question by showing that the isomorphism relation is $\boldsymbol{\Sigma}^1_2$-complete. We will be able to give a partial answer to the second question by showing that the relation is quite complicated: The equivalence relation of equality of Borel sets is Borel reducible to the isomorphism

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relation. In particular, this shows that any Borel equivalence relation is reducible to the isomorphism relation.

In [5] Hjorth considers analogous questions for the group of measure-preserving transformations of a measure space. There is a notable distinction between the two situations, since the group of measure-preserving transformations can be made into a Polish group and hence has a $\mathbf{\Sigma}^1_1$ conjugacy relation. This relation turns out to be $\mathbf{\Sigma}^1_1$-complete and not classifiable by countable structures, so that it is also quite complicated; however, it is strictly simpler than the conjugacy relation of Borel automorphisms. Here we see an example of the differences between measure-theoretic and descriptive contexts.

§2. Setup. Although we will be representing all Borel automorphisms by automorphisms of the Cantor space, it will be more convenient to define them on other spaces. Thus we will use (f, X) to indicate that f is an automorphism of the Polish space X. The Polish spaces we use will be sufficiently similar to each other that we will have a uniform way of producing isomorphic automorphisms of the Cantor space. We will discuss this below.

We will want a more general way of comparing automorphisms than isomorphism. We introduce the following partial order on the class of Borel automorphisms:

Definition 2. *We set* $(f, X) \preceq (g, Y)$ *if there is a Borel injection* $\varphi : X \hookrightarrow Y$ *such that* $\varphi \circ f = g \circ \varphi$.

Note that in the above definition, $\varphi[X]$ will be a g-invariant Borel set (since the injective image of a Borel set is Borel), so that we have the following equivalent form of the definition: $(f, X) \preceq (g, Y)$ if and only if there is a g-invariant Borel set $B \subseteq Y$ such that $(f, X) \cong (g \restriction B, B)$. Also notice that, by a standard Shroeder-Bernstein argument, we have:

$$(f, X) \cong (g, Y) \iff (f, X) \preceq (g, Y) \text{ and } (g, Y) \preceq (f, X).$$

We now explain our coding of Borel automorphisms. We first fix a good parameterization of the Borel subsets of $(2^\omega)^2$ (see [6]). This consists of sets $D \subseteq 2^\omega$ and $P, S \subseteq (2^\omega)^3$ where D is $\mathbf{\Pi}^1_1$, P is $\mathbf{\Pi}^1_1$, and S is $\mathbf{\Sigma}^1_1$ such that:

1. $d \in D \Longrightarrow P_d = S_d$ (so that these sections are Borel subsets of $(2^\omega)^2$)
2. $\{P_d : d \in D\}$ contains all Borel subsets of $(2^\omega)^2$
3. For any Polish space X and Borel set $A \subseteq X \times (2^\omega)^2$, there is a Borel function $p : X \to 2^\omega$ such that for all $x \in X$ we have $p(x) \in D$ and $A_x = P_{p(x)}$

The properties of this parameterization will be necessary for the definability considerations later.

Definition 3. *Let* $\mathcal{BA}$*, the set of codes for Borel automorphisms, be:*

$$\mathcal{BA} = \{d \in D : P_d \textit{ is the graph of a Borel automorphism of } 2^\omega\}.$$

This is a $\mathbf{\Pi}^1_1$ set. Up to isomorphism, every Borel automorphism has a code in $\mathcal{BA}$. We can then define the isomorphism relation on $\mathcal{BA}$ by letting two elements be equivalent if they code isomorphic Borel automorphisms.

§3. Reducing equality of Borel sets. In their paper [3], Eigen, Hajian and Weiss show how to construct a continuum of non-isomorphic Borel automorphisms. By extending their technique we will show the following:

Theorem 4. *There is a Borel map from the Borel subsets of the Baire space* ω^ω *to Borel automorphisms of the Cantor space, sending* A *to* f_A*, such that:*

$$A \subseteq B \Longleftrightarrow f_A \preceq f_B.$$

That is, the partial order of inclusion among the Borel sets embeds into the partial order of $\preceq$ *among Borel automorphisms.*

We will explain the definablility of the map below. From the theorem we derive the following corollary:

Corollary 5. *The equivalence relation of equality of Borel subsets of* ω^ω *is Borel reducible to the equivalence relation of isomorphism of Borel automorphisms of* 2^ω*.*

We will discuss some other consequences of these results in the next section.

Let us first set out some notation. The transformation φ_0 will refer to the *odometer map* on 2^ω, given by adding 1 with carry (so that if $x = 1^k \frown 0 \frown \alpha$, then $\varphi_0(x) = 0^k \frown 1 \frown \alpha$, and $\varphi_0(1^\infty) = 0^\infty$). Except on the eventually constant sequences, φ_0 induces the equivalence relation E_0 (two sequences are E_0 equivalent if they differ on only finitely many coordinates). A set $B \subseteq 2^\omega$ will be called *smooth* or φ_0-smooth if there is a Borel transversal for the φ_0-saturation of B, i.e., a Borel set S such that for each orbit of φ_0 which meets B, S meets the orbit in exactly one point. This is equivalent to saying that $E_0 \restriction B$ is a smooth equivalence relation, i.e. reducible to the identity relation. Note that B is φ_0-smooth if and only if $B \in \mathcal{W}(\varphi_0)$, the σ-ideal generated by the wandering sets for φ_0 (a set B is *wandering* for a transformation T if $B \cap T^n[B] = \emptyset$ for all $n \in \omega$, $n \neq 0$). This is true since in the realm of countable Borel equivalence relations, being smooth is equivalent to having a Borel transversal, i.e. a set meeting each equivalence class in precisely one point. A transversal is a wandering set, and one easily constructs a transversal from a wandering set.

Finally, for $n \in \mathbb{N}$, we let $\mathrm{ord}_2(n)$ be the largest integer k such that 2^k divides n (so that $\mathrm{ord}_2(n)$ is the first non-zero coordinate in the binary expansion of n). Similarly, we can define $\mathrm{ord}_2(\alpha)$ for $\alpha \in 2^\omega$ to be the first non-zero coordinate of α. We let $\mathrm{ord}_2(0) = \infty$.

Proof of Theorem 4. Let $\{s_n : n \in \omega\}$ be a recursive enumeration of $\omega^{<\omega}$, the finite sequences of natural numbers, such that if $s_n \sqsubseteq s_m$ then $n \leq m$. For $\alpha \in \omega^\omega$, let:

$$\widetilde{\alpha} = \{n : s_n \sqsubset \alpha\}.$$

Thus, we can think of the nodes in the tree $\omega^{<\omega}$ as being labeled by natural numbers, and $\widetilde{\alpha}$ lists those nodes which the branch α passes through. Note that if $\alpha \neq \beta$, then $\widetilde{\alpha} \cap \widetilde{\beta}$ is finite.

Let $A \subseteq \omega^\omega$ be a given non-empty Borel set. We will define the Borel automorphism f_A to be the restriction of the odometer map φ_0 to some φ_0-invariant Borel set X_A (in the case of $A = \emptyset$ we will simply take the automorphism to be smooth and aperiodic, so it will embed in all the others produced here). Then $f_A = (\varphi_0, X_A)$ will be a Borel automorphism of the standard Borel space X_A. We will explain at the end how to uniformly represent this as a Borel automorphism of the Cantor space. We define X_A as follows:

$$X_A = \{x : x \text{ is infinite and } \exists \alpha \in A \text{ such that } x \setminus \widetilde{\alpha} \text{ is finite}\}.$$

We are here identifying subsets of ω with their characteristic functions in 2^ω. We check that X_A is a φ_0-invariant Borel set. If we let

$$W_A = \{x : x \text{ is infinite and } x \subseteq \widetilde{\alpha} \text{ for some } \alpha \in A\},$$

then we have:

$$X_A = \bigcup_{i \in \omega} \varphi_0^i[W_A].$$

So X_A is invariant, and it will suffice to check that W_A is Borel. Note that

$$x \in W_A \iff \exists \alpha (\alpha \in A \text{ and } x \subseteq \widetilde{\alpha} \text{ and } x \text{ is infinite}),$$

so that W_A is the projection onto the first coordinate of the set

$$U = \{(x, \alpha) \in 2^\omega \times \omega^\omega : x \text{ is infinite and } \alpha \in A \text{ and } x \subseteq \widetilde{\alpha}\}.$$

The set U is Borel since A is Borel, and for each x there is at most one α with $(x, \alpha) \in U$, since $\widetilde{\alpha} \cap \widetilde{\beta}$ is finite for $\alpha \neq \beta$. Thus W_A is the continuous injective image of a Borel set, and hence Borel.

It is clear that if $A \subseteq B$ then $X_A \subseteq X_B$, so taking φ to be the identity map on X_A shows that $(\varphi_0, X_A) \preceq (\varphi_0, X_B)$ in this case. Suppose on the other hand that $A \not\subseteq B$. We will show that $(\varphi_0, X_A) \not\preceq (\varphi_0, X_B)$.

If $A \not\subseteq B$, then there is some $\alpha \in A \setminus B$. Fix such an α, and suppose there were a Borel injection $\varphi : X_A \hookrightarrow X_B$ such that $\varphi \circ \varphi_0 = \varphi_0 \circ \varphi$. Let $Z = \varphi[X_A] \subseteq X_B$. Then Z is a φ_0-invariant Borel set. Let:

$$\begin{aligned} W_\alpha &= \{x : x \subseteq \widetilde{\alpha} \text{ and } x \text{ is infinite}\} \\ X_\alpha &= \bigcup_{i \in \omega} \varphi_0^i[W_\alpha]. \end{aligned}$$

Then X_α is a φ_0-invariant subset of X_A. Let

$$W_Z = \{x \in Z : x \cap \widetilde{\alpha} = \emptyset\}.$$

Then we have that $Z = \bigcup_i \varphi_0^i[W_Z]$ since any $x \in Z$ has finite intersection with $\widetilde{\alpha}$. Now, for $i, j \in \omega$, set:

$$V_{i,j} = \varphi^{-1} \circ \varphi_0^i[W_Z] \cap \varphi_0^j[W_\alpha].$$

Then each $V_{i,j}$ is a Borel set, and we have

$$\begin{aligned} X_\alpha &= X_\alpha \cap \varphi^{-1}[Z] \\ &= \left(\bigcup_j \varphi_0^j [W_\alpha] \right) \cap \varphi^{-1} \left[\bigcup_i \varphi_0^i [W_Z] \right] \\ &= \left(\bigcup_j \varphi_0^j [W_\alpha] \right) \cap \left(\bigcup_i \varphi^{-1} \circ \varphi_0^i [W_Z] \right) \\ &= \bigcup_{i,j} \left(\varphi^{-1} \circ \varphi_0^i [W_Z] \cap \varphi_0^j [W_\alpha] \right). \end{aligned}$$

Thus, $X_\alpha = \bigcup_{i,j} V_{i,j}$. We claim that each $V_{i,j}$ is a wandering set for φ_0. To see this, note that

$$\varphi_0^n[V_{i,j}] = \varphi_0^n[\varphi^{-1} \circ \varphi_0^i[W_Z] \cap \varphi_0^j[W_\alpha]] = \varphi^{-1} \circ \varphi_0^i[\varphi_0^n[W_Z]] \cap \varphi_0^j[\varphi_0^n[W_\alpha]],$$

so that

$$\begin{aligned} V_{i,j} \cap \varphi_0^n[V_{i,j}] &= \varphi^{-1} \circ \varphi_0^i[W_Z] \cap \varphi^{-1} \circ \varphi_0^i[\varphi_0^n[W_Z]] \\ &\quad \cap \varphi_0^j[W_\alpha] \cap \varphi_0^j[\varphi_0^n[W_\alpha]] \\ &= \varphi^{-1} \circ \varphi_0^i[\varphi_0^n[W_Z] \cap W_Z] \cap \varphi_0^j[\varphi_0^n[W_\alpha] \cap W_\alpha]. \end{aligned}$$

But now let $n \neq 0$ and set $k = \mathrm{ord}_2(n)$. Note that if $k \in \widetilde{\alpha}$ then every element of $\varphi_0^n[W_Z]$ has a 1 in the k-th coordinate. This is because applying φ_0^n can be thought of as adding the binary representation of n, which has its first k coordinates equal to 0 and the k-th coordinate equal to 1, and each element of W_Z has the k coordinate equal to 0 since $k \in \widetilde{\alpha}$ implies $k \notin x$ for $x \in W_Z$. Then, since every element of W_Z has its k coordinate equal to 0, we have that if $k \in \widetilde{\alpha}$, then $\varphi_0^n[W_Z] \cap W_Z = \emptyset$. Similarly, if $k \notin \widetilde{\alpha}$, we have that $\varphi_0^n[W_\alpha] \cap W_\alpha = \emptyset$. In any event, then, we have $V_{i,j} \cap \varphi_0^n[V_{i,j}] = \emptyset$, so $V_{i,j}$ is wandering for φ_0.

Thus, since X_α is a countable union of wandering sets, we have that X_α is φ_0-smooth, which is equivalent to saying $E_0 \upharpoonright X_\alpha$ is a smooth equivalence relation. This, however, is false, which we will see by showing that $E_0 \sqsubseteq_c E_0 \upharpoonright X_\alpha$. Since E_0 is non-smooth, this gives a contradiction.

We define the embedding as follows. Let the elements of $\widetilde{\alpha}$ be enumerated in increasing order as $\{a_n : n \in \omega\}$. For $x \in 2^\omega$ we define $f(x) \in 2^\omega$ by:

$$f(x)(k) = \begin{cases} x(n) & \text{if } k = a_{(2n)} \\ 1 & \text{if } k = a_{(2n+1)} \\ 0 & \text{if } k \notin \widetilde{\alpha} \end{cases}.$$

Then $f(x)$ is infinite and $f(x) \subseteq \widetilde{\alpha}$, so that $f(x) \in X_\alpha$. The map is clearly continuous and injective. Finally, it is clear that $x E_0 y \iff f(x) E_0 f(y)$, so that this is the desired embedding.

Thus, our map $A \mapsto (\varphi_0, X_A)$ has the desired property that $A \subseteq B$ if and only if $(\varphi_0, X_A) \preceq (\varphi_0, X_B)$. We must lastly check that we can produce codes for the automorphisms (φ_0, X_A) in a Borel way, that is, find representatives for them as automorphisms of the Cantor space.

Recall the good parameterization of Borel subsets of $(2^\omega)^2$ introduced in the last section. Fix now a similar parameterization of Borel subsets of the Baire space ω^ω. For each Borel set coded by a parameter d we have just produced an automorphism which is the odometer map on a certain subset of the Cantor space, which we will denote X_d. Uniformly in the parameter d we can produce a bijection from 2^ω onto X_d (note that this bijection in general will depend on the exact code d and not merely on the Borel set it codes). Let σ_d denote this map. If we then define:

$$\varphi_d = \sigma_d^{-1} \circ \varphi_0 \circ \sigma_d,$$

we will have that φ_d is a Borel automorphism of the Cantor space such that $(\varphi_0, X_d) \cong (\varphi_d, 2^\omega)$. Now we can produce a map p which takes a code d for a Borel subset of ω^ω and assigns a code $p(d)$ for a Borel automorphism of 2^ω, namely φ_d. Note that if d and e are distinct codes for the same Borel set, we do not necessarily have that φ_d and φ_e are the same automorphism, but we do have that they are isomorphic.

Since the automorphism coded by $p(d)$ is isomorphic to (φ_0, X_d), the map p is the desired Borel map claimed in the theorem. The proof of the corollary is immediate, as again p is a reduction of the equivalence relation of equality of Borel sets to the equivalence relation of isomorphism of Borel automorphisms. $\dashv$

Let us explain the underlying principle of the argument here. For a set $S \subseteq \mathbb{N}$, let $\mathrm{IP}\{S\}$ be the set consisting of all sums of finite subsets of S. Now let $\Omega_\alpha = \mathrm{IP}\{2^n : n \notin \widetilde{\alpha}\}$. These can be viewed as those integers whose binary representations (viewed as a finite subset of ω) are disjoint from $\widetilde{\alpha}$. Note that for $n_1 \neq n_2$ in Ω_α, we have $\mathrm{ord}_2(n_1 - n_2) \notin \widetilde{\alpha}$. From this we can see that $X_\alpha = \bigsqcup_{n \in \Omega_\alpha} \varphi_0^n[W_\alpha]$, where the union is disjoint. Thus, W_α is an exhaustive weakly wandering set for the transformation (φ_0, X_α) and the exhaustive weakly wandering sequence Ω_α (see [3] for these concepts). On the other hand, Ω_α can not be an exhaustive weakly wandering sequence for φ_0 restricted to any invariant subset of X_B, as would be necessary if (φ_0, X_α) were to embed in (φ_0, X_B), since exhaustive weakly wandering sequences are isomorphism invariants for Borel automorphisms.

Also note that the above argument will show that for two distinct Borel sets A and B, the automorphisms produced will not be conjugate via a universally measurable map, since E_0 has no universally measurable selector. Stronger results of this form also follow from stronger set-theoretic hypotheses.

§4. Consequences of the reduction.

Let us first note an important consequence of the above theorem.

Corollary 6. *Let E be a Borel equivalence relation. Then E is Borel reducible to isomorphism of Borel automorphisms.*

Proof. Given a Borel equivalence relation E, we have the map $x \mapsto [x]_E$ sending each element to its equivalence class, a Borel set. Given a Borel code for E, we define this map to uniformly produce Borel codes for the equivalence classes. Composing with the map p described above, we then have

$$x \mathrel{E} y \iff [x]_E = [y]_E \iff \varphi_{p([x])} \cong \varphi_{p([y])},$$

giving a reduction of E to the relation of isomorphism of Borel automorphisms. ⊣

Thus, the equivalence relation of isomorphism of Borel automorphisms, and the classification problem it represents, is quite complicated. For instance, one can not in any reasonably definable manner (universally measurable, e.g.) assign a real number as a complete invariant (the relation is not concretely classifiable). This answers a question from [2]. In fact, since the Borel equivalence relation E_1 is reducible to the isomorphism relation, it follows from results of [7] that this relation can not be reduced to the orbit equivalence relation of any Polish group action.

The exact complexity of the isomorphism relation is not known; however, Su Gao has improved the above result:

Theorem 7 (Gao [4]). *Let E be an analytic equivalence relation. Then E is Borel reducible to isomorphism of Borel automorphisms.*

This should be contrasted with the situation where we consider conjugacy in the group of Lipschitz automorphisms of the Cantor space. These are the isometries of the Cantor space under the standard metric

$$d(x,y) = \frac{1}{1+n(x,y)} \text{ where } n(x,y) \text{ is the least } n \text{ such that } x(n) \neq y(n).$$

Here, both the equivalence relation of conjugacy by Lipschitz automorphisms and the relation of conjugacy by arbitrary Borel automorphisms are concretely-classifiable (see [2]).

Lastly, we should contrast the notion of Borel isomorphism of two Borel automorphisms with that of orbit equivalence. Given a Borel automorphism f, let E_f denote the associated orbit equivalence relation,

$$x \, E_f \, y \iff \exists n \in \mathbb{Z}\, [y = f^n(x)] \,.$$

Then we say that two automorphisms f and g are *orbit equivalent* if their orbit equivalence relations are isomorphic, i.e., there is a bijection between the underlying spaces such that two elements are in the same f-orbit if and only if their images are in the same g-orbit. For aperiodic, non-smooth automorphisms, there are only countably many isomorphism types, corresponding to the number of invariant probability measures that the automorphism admits (see [2]). All of the automorphisms produced in the above argument turn out to be orbit equivalent, since they admit no invariant probability measures. To see this, let a Borel set $A \subseteq \omega^\omega$ be given, and suppose μ were an invariant probability measure for (φ_0, X_A). For α in A, define

$$\begin{aligned} W_\alpha &= \{x : x \text{ is infinite and } x \subseteq \widetilde{\alpha}\ \} \\ X_\alpha &= \bigcup_i \varphi_0^i[W_\alpha], \end{aligned}$$

so that $W_A = \bigsqcup_{\alpha\in A} W_\alpha$ and $X_A = \bigsqcup_{\alpha \in A} X_\alpha$. Since $X_A = \bigcup_i \varphi_0^i[W_A]$ and μ is φ_0-invariant, we must have $\mu(W_A) > 0$. But this implies that the conditional probability measure of W_α on X_α must be positive for some α. This contradicts that (φ_0, X_α) has no invariant probability measure, as it admits a nontrivial weakly wandering sequence (see [3]). Thus, although all of the automorphisms produced are non-isomorphic, they are all orbit equivalent. It may be of interest to consider notions of equivalence intermediate between orbit equivalence and isomorphism (such as Kakutani equivalence) and determine the complexity of these equivalence relations. Some results of this type may be found in [8].

§5. Descriptive complexity. We now will calculate the descriptive complexity of the relations of isomorphism and embeddability of Borel automorphisms of the Cantor space. Recall the set $\mathcal{BA}$ of codes for Borel automorphisms. We now define:

$$\begin{aligned} \mathcal{E}_{\preceq} &= \{(d,e) \in D^2 : d, e \in \mathcal{BA} \ \& \ f_d \preceq f_e\} \\ \mathcal{E}_{\cong} &= \{(d,e) \in D^2 : d, e \in \mathcal{BA} \ \& \ f_d \cong f_e\} \end{aligned}$$

where we recall that f_d is the automorphism whose graph is coded by d. We start with a few straightforward calculations.

Proposition 8. *The set* $\mathcal{BA}$ *is* $\mathbf{\Pi}^1_1$.

Proof. Recall the good parametrization D, P, S of Borel subsets of $(2^\omega)^2$ introduced earlier. We then have:

$$\begin{aligned} \mathcal{BA}(d) \iff & D(d) \wedge \forall x \exists y P(d,x,y) \wedge \forall y \exists x P(d,x,y) \wedge \\ & \neg\exists x \exists y_1 \exists y_2 [y_1 \neq y_2 \wedge S(d,x,y_1) \wedge S(d,x,y_2)] \wedge \\ & \neg\exists x_1 \exists x_2 \exists y [x_1 \neq x_2 \wedge S(d,x_1,y) \wedge S(d,x_2,y)]. \end{aligned}$$

Notice that the existential witnesses in the first line must be unique, so that they can be calculated in a Δ^1_1 way. We then have:

$$\begin{aligned} \mathcal{BA}(d) \iff & D(d) \wedge \forall x \exists y \in \Delta^1_1(d,x) P(d,x,y) \wedge \\ & \forall y \exists x \in \Delta^1_1(d,y) P(d,x,y) \wedge \\ & \neg\exists x \exists y_1 \exists y_2 [y_1 \neq y_2 \wedge S(d,x,y_1) \wedge S(d,x,y_2)] \wedge \\ & \neg\exists x_1 \exists x_2 \exists y [x_1 \neq x_2 \wedge S(d,x_1,y) \wedge S(d,x_2,y)]. \end{aligned}$$

Since $\mathbf{\Pi}^1_1$ is closed under Δ^1_1-quantification, this gives the desired result. ⊣

Proposition 9. *The sets* $\mathcal{E}_{\preceq}$ *and* $\mathcal{E}_{\cong}$ *are both* $\mathbf{\Sigma}^1_2$.

Proof. A similar calculation to the previous one shows that the set $\mathcal{BI}$ of codes for Borel injections is also $\mathbf{\Pi}^1_1$. We then have

$$\begin{aligned} \mathcal{E}_{\preceq}(d,e) \iff & \mathcal{BA}(d) \wedge \mathcal{BA}(e) \wedge \exists c [\mathcal{BI}(c) \wedge f_c \circ f_d = f_e \circ f_c] \\ \iff & \mathcal{BA}(d) \wedge \mathcal{BA}(e) \wedge \exists c [\mathcal{BI}(c) \ \& \ \forall x \forall y \forall z \forall w \\ & [(S(d,x,y) \wedge S(c,x,w) \wedge S(c,y,z)) \implies P(e,w,z)]]. \end{aligned}$$

So we see that $\mathcal{E}_{\preceq}$ is $\mathbf{\Sigma}^1_2$. Then, since

$$\mathcal{E}_{\cong}(d,e) \iff \mathcal{E}_{\preceq}(d,e) \ \& \ \mathcal{E}_{\preceq}(e,d)$$

we also have that $\mathcal{E}_{\cong}$ is $\mathbf{\Sigma}^1_2$. ⊣

We now show that this calculation is optimal by showing that these sets are also $\mathbf{\Sigma}^1_2$-hard. The technique we use was introduced by Adams and Kechris in [1] to prove an analogous result about countable Borel equivalence relations. They first construct an equivalence relation for each element of ω^ω with certain properties; we will be able to use the automorphisms previously constructed in place of these.

Theorem 10. *The sets $\mathcal{E}_\preceq$ and $\mathcal{E}_\cong$ are both $\mathbf{\Sigma}^1_2$-complete.*

Proof. We will in fact show that there is a single automorphism (φ^*, X^*) such that the following two sets are $\mathbf{\Sigma}^1_2$-complete:

$$\begin{aligned}\mathcal{E}^*_\preceq &= \{d : d \in \mathcal{BA} \ \& \ (\varphi^*, X^*) \preceq f_d\}\\ \mathcal{E}^*_\cong &= \{d : d \in \mathcal{BA} \ \& \ (\varphi^*, X^*) \cong f_d\}.\end{aligned}$$

As these two sets are clearly Wadge-reducible to $\mathcal{E}_\preceq$ and $\mathcal{E}_\cong$, respectively, this will establish the theorem.

Our proof relies on a result of Steel about trees with full Borel uniformizations. Given a tree T on $\omega \times \omega$, let $[T]$ denote its set of branches, which is then a closed subset of $(\omega^\omega)^2$. We say that T has a full Borel Uniformization if the set $[T]$ can be uniformized by a total Borel function, i.e. there is a total Borel function f such that for all x in ω^ω we have $(x, f(x)) \in [T]$. We let $\mathcal{FBU}$ denote the set of trees with full Borel uniformizations. Then we have the following (for a proof, see [1]):

Theorem 11 (Steel). *The set $\mathcal{FBU}$ is $\mathbf{\Sigma}^1_2$-complete.*

We will first define (φ^*, X^*), and then define a Borel map $T \mapsto (\varphi_T, X_T)$ sending trees to automorphisms such that

$$T \in \mathcal{FBU} \iff (\varphi^*, X^*) \preceq (\varphi_T, X_T) \iff (\varphi^*, X^*) \cong (\varphi_T, X_T).$$

This map thus gives a Borel reduction of the set $\mathcal{FBU}$ to the sets $\mathcal{E}_\preceq$ and $\mathcal{E}_\cong$, establishing the theorem. For technical reasons we will work only with trees having at least one branch; this presents no difficulty since we can uniformly transform a given tree to one having at least one branch without affecting whether or not the tree has a full Borel uniformization. We also note that the set of trees on $\omega \times \omega$ is easily topologized so as to be a Polish space. For $\alpha \in \omega^\omega$ recall the map $\alpha \mapsto \widetilde{\alpha}$ and the sets

$$\begin{aligned}W_\alpha &= \{x : x \text{ is infinite and } x \subseteq \widetilde{\alpha}\ \}\\ X_\alpha &= \bigcup_{i \in \omega} \varphi_0^i[W_\alpha],\end{aligned}$$

where φ_0 again denotes the odometer map on 2^ω. Let also $X^*_\alpha = \omega^\omega \times X_\alpha$. We now define $X^* \subseteq \omega^\omega \times \omega^\omega \times 2^\omega$ by:

$$X^* = \bigsqcup_{\alpha \in \omega^\omega} X^*_\alpha,$$

i.e. $(\alpha, x, y) \in X^* \iff y \in X_\alpha$. We define φ^* by:

$$\varphi^*(\alpha, x, y) = (\alpha, x, \varphi_0(y)).$$

Next, for a tree T on $\omega \times \omega$ we define $X_T \subseteq (\omega^\omega)^2 \times \omega^\omega \times 2^\omega$ by:

$$X_T = \bigsqcup_{(\alpha,\beta)\in[T]} X^*_\alpha,$$

i.e. $(\alpha, \beta, x, y) \in X_T \iff (\alpha, \beta) \in [T]$ and $y \in X_\alpha$. This union is non-empty because we stipulated that $[T]$ be non-empty. We then set

$$\varphi_T(\alpha, \beta, x, y) = (\alpha, \beta, x, \varphi_0(y)).$$

This completes the definitions of the various automorphisms. It will suffice now to establish the following four lemmas.

Lemma 12. *For all trees T we have $(\varphi_T, X_T) \preceq (\varphi^*, X^*)$.*

Lemma 13. *For $T \in \mathcal{FBU}$ we have $(\varphi^*, X^*) \preceq (\varphi_T, X_T)$.*

Lemma 14. *If $(\varphi^*, X^*) \preceq (\varphi_T, X_T)$ then $T \in \mathcal{FBU}$.*

Lemma 15. *There is a Borel function p sending a tree T to a code $p(T) \in \mathcal{BA}$ such that $(\varphi_T, X_T) \cong (f_{p(T)}, 2^\omega)$.*

Proof of lemma 12. Let $(x, y) \mapsto \langle x, y\rangle$ be a Borel bijection of $(\omega^\omega)^2$ with ω^ω and set

$$\psi(\alpha, \beta, x, y) = (\alpha, \langle\beta, x\rangle, y).$$

Then ψ is a Borel injection from X_T into X^* and it is easy to check that $\psi\circ\varphi_T = \varphi^* \circ \psi$, witnessing the embedding. ⊣

Proof of lemma 13. Let f be a total Borel function uniformizing $[T]$, so that for all α we have $(\alpha, f(\alpha)) \in [T]$. Set

$$\psi(\alpha, x, y) = (\alpha, f(\alpha), x, y).$$

Then ψ is an injection from X^* into X_T with $\psi \circ \varphi^* = \varphi_T \circ \psi$. ⊣

Proof of lemma 14. Let ψ be a Borel injection from X^* to X_T such that $\psi \circ \varphi^* = \varphi_T \circ \psi$. We want to exhibit a Borel function f such that for all α we have $(\alpha, f(\alpha)) \in [T]$. Let α be given and let φ_α be the injection from X_α into X_T given by

$$\varphi_\alpha(y) = \psi(\alpha, 0^\infty, y)$$

(where 0^∞, the constant 0 sequence, is chosen arbitrarily). Note that

$$\begin{aligned}\varphi_\alpha(\varphi_0(y)) = \psi(\alpha, 0^\infty, \varphi_0(y)) = \psi(\varphi^*(\alpha, 0^\infty, y)) = \varphi_T(\psi(\alpha, 0^\infty, y))\\ = \varphi_T(\varphi_\alpha(y)),\end{aligned}$$

so $\varphi_\alpha \circ \varphi_0 = \varphi_T \circ \varphi_\alpha$, i.e. φ_α witnesses that $(\varphi_0, X_\alpha) \preceq (\varphi_T, X_T)$. Let a_α, b_α, x_α, and y_α be the (unique) functions such that

$$\varphi_\alpha(y) = (a_\alpha(y), b_\alpha(y), x_\alpha(y), y_\alpha(y)).$$

Then we have

$$\begin{aligned}\varphi_\alpha \circ \varphi_0(y) &= (a_\alpha(\varphi_0(y)), b_\alpha(\varphi_0(y)), x_\alpha(\varphi_0(y)), y_\alpha(\varphi_0(y)))\\ \varphi_T \circ \varphi_\alpha(y) &= (a_\alpha(y), b_\alpha(y), x_\alpha(y), \varphi_0(y_\alpha(y))),\end{aligned}$$

so that, comparing coordinates, we have

$$\begin{aligned}
a_\alpha(\varphi_0(y)) &= a_\alpha(y)\\
b_\alpha(\varphi_0(y)) &= b_\alpha(y)\\
x_\alpha(\varphi_0(y)) &= x_\alpha(y)\\
y_\alpha(\varphi_0(y)) &= \varphi_0(y_\alpha(y)).
\end{aligned}$$

Thus, the functions a_α, b_α, and x_α are all invariant under φ_0.

Recall from above that φ_0 on X_α induces the equivalence relation $E_0 \restriction X_\alpha$, and we have a uniform way of embedding E_0 into $E_0 \restriction X_\alpha$. We can then uniformly lift Lebesgue measure (which is E_0-invariant) from 2^ω to X_α and associate to each α a quasi-invariant, non-atomic, ergodic probability measure μ_α for φ_0 on X_α. The ergodicity of μ_α and the invariance of a_α, b_α, and x_α under φ_0 implies that these functions must be constant on a φ_0-invariant set $M_\alpha \subseteq X_\alpha$ of μ_α-measure 1. Call the (necessarily unique) constant values of these functions α_0, β_0, and x_0, respectively.

First, we claim that $\alpha_0 = \alpha$. Notice that on M_α we have

$$\varphi_\alpha(y) = (\alpha_0, \beta_0, x_0, y_\alpha(y)),$$

so that $y_\alpha(y) \in X_{\alpha_0}$. Since $y_\alpha \circ \varphi_0 = \varphi_0 \circ y_\alpha$, this shows that in fact $(\varphi_0, M_\alpha) \preceq (\varphi_0, X_{\alpha_0})$. The set X_α here is the same as was considered earlier, and the same argument shows that if $\alpha_0 \neq \alpha$ then M_α would be a wandering set (smooth set) for φ_0. But now we have that μ_α restricted to M_α is still a quasi-invariant non-atomic probability measure for φ_0, so M_α can not be φ_0-smooth. Thus $\alpha_0 = \alpha$.

Next, notice that we must have $(\alpha_0, \beta_0) \in [T]$, so that if we define our function f by $f(\alpha) = \beta_0$, we will have that $(\alpha, f(\alpha)) \in [T]$ for all α in ω^ω, so that f gives a full uniformization of $[T]$. We need only to check that f so defined is Borel-measurable. For this, we check that its graph is a Borel set. Let $G_f \subseteq (\omega^\omega)^2$ denote this graph.

For each α, let $\langle \widetilde{\alpha}_n \rangle_{n\in\omega}$ be the enumeration of $\widetilde{\alpha}$ in increasing order. We now define a function $e : \omega^\omega \times 2^\omega \to 2^\omega$ by setting:

$$e(\alpha, x)(k) = \begin{cases} x(n) & \text{if } k = \widetilde{\alpha}_{(2n)} \text{ for some } n \\ 1 & \text{if } k = \widetilde{\alpha}_{(2n+1)} \text{ for some } n \\ 0 & \text{if } k \notin \widetilde{\alpha} \end{cases}.$$

Thus, if we define functions e_α by $e_\alpha(x) = e(\alpha, x)$, we have that each e_α is an embedding of the equivalence relation E_0 into $E_0 \restriction X_\alpha$. The function e is Borel, in fact continuous. We can now uniformly define measures μ_α on X_α by setting

$$\mu_\alpha(A) = \mu_L(e_\alpha^{-1}[A]),$$

where μ_L is Lebesgue measure on 2^ω. Let $\pi_1 : X_T \to \omega^\omega$ be projection onto the second coordinate. We then compute:

$$\begin{aligned}(\alpha,\beta) \in G_f &\iff \mu_\alpha(\{y : b_\alpha(y) = \beta\}) = 1\\ &\iff \mu_L(\{x : b_\alpha(e_\alpha(x)) = \beta\}) = 1\\ &\iff \mu_L(\{x : \pi_1 \circ \varphi_\alpha(e_\alpha(x)) = \beta\}) = 1\\ &\iff \mu_L(\{x : \pi_1 \circ \psi(\alpha, 0^\infty, e(\alpha,x)) = \beta\}) = 1.\end{aligned}$$

Now, noting that the set

$$\{(x,\alpha,\beta) : \pi_1 \circ \psi(\alpha, 0^\infty, e(\alpha,x)) = \beta\}$$

is Borel, we can apply Theorem 17.25 of [6] to conclude that G_f is Borel. $\dashv$

PROOF OF LEMMA 15. First we define canonical Borel bijections σ_α between 2^ω and X_α. To do this, let

$$\begin{aligned}X_0 = \{y \in 2^\omega :\ & y \text{ has infinitely many 1's and}\\ & y \text{ has only finitely many non-zero odd coordinates}\}\end{aligned}$$

and fix a Borel bijection σ_0 between 2^ω and X_0. Let $\langle \widetilde{\alpha}_n \rangle$ enumerate $\widetilde{\alpha}$ as before, and let $\langle \widetilde{\gamma}_n \rangle$ enumerate $\omega \setminus \widetilde{\alpha}$. Define a bijection $\sigma_{0,\alpha}$ between X_0 and X_α by

$$\sigma_{0,\alpha}(x)(k) = \begin{cases} x(2n) & \text{if } k = \widetilde{\alpha}_n \\ x(2n+1) & \text{if } k = \widetilde{\gamma}_n \end{cases}.$$

We then let $\sigma_\alpha = \sigma_{0,\alpha} \circ \sigma_0$.

We also have that for a tree T with $[T] \neq \emptyset$ there is a canonical bijection between $[T] \times \omega^\omega$ and ω^ω. Combining all of these maps, we can produce a canonical bijection σ_T between 2^ω and X_T. Now let $\widetilde{\varphi}_T = \sigma_T^{-1} \circ \varphi_T \circ \sigma_T$, so that $\widetilde{\varphi}_T$ is an automorphism of 2^ω with $(\widetilde{\varphi}_T, 2^\omega) \cong (\varphi_T, X_T)$. We then have that the relation

$$G(T,x,y) \iff T \text{ is a tree and } y = \widetilde{\varphi}_T(x)$$

is Borel and G_T is the graph of $\widetilde{\varphi}_T$ on 2^ω. So, using property (3) of our good parameterization of Borel subsets of $(2^\omega)^2$, we have a Borel function p sending a tree to an element of 2^ω such that $p(T) \in D$ and $G_T = P_{p(T)}$. This implies that $p(T) \in \mathcal{BA}$, and that $f_{p(T)} = \widetilde{\varphi}_T$ as required. $\dashv$

This finishes the proof of the theorem. $\dashv$

Again following Adams and Kechris, we can draw a further result. We say that $(f,X) \preceq_{\sigma(\boldsymbol{\Sigma}^1_1)} (g,Y)$ if there is an embedding of f into g which is measurable with respect to the σ-algebra generated by the $\boldsymbol{\Sigma}^1_1$ sets, and similarly for $\cong_{\sigma(\boldsymbol{\Sigma}^1_1)}$. The Jankov-von Neumann uniformization theorem says that a closed plane set has a full projection if and only if it admits a full $\sigma(\boldsymbol{\Sigma}^1_1)$-measurable uniformization. Let $\mathcal{FP}$ be the set of trees T on $\omega \times \omega$ such that $[T]$ has full projection. Inspecting the above proof, we see that we have the following:

$$(\varphi^*, X^*) \preceq_{\sigma(\boldsymbol{\Sigma}^1_1)} (\varphi_T, X_T) \iff (\varphi^*, X^*) \cong_{\sigma(\boldsymbol{\Sigma}^1_1)} (\varphi_T, X_T) \iff T \in \mathcal{FP}.$$

Clearly $\mathcal{FBU} \subseteq \mathcal{FP}$; however, $\mathcal{FP}$ is easily seen to be $\boldsymbol{\Pi}^1_2$-complete (see [1]). From this we see that the relations $\preceq_{\sigma(\boldsymbol{\Sigma}^1_1)}$ and $\cong_{\sigma(\boldsymbol{\Sigma}^1_1)}$ are $\boldsymbol{\Pi}^1_2$-hard. We also know that the two sets $\mathcal{FP}$ and $\mathcal{FBU}$ are not equal, so there is a tree T which has full projection but does not have a full Borel uniformization. Letting $f = (\varphi_T, X_T)$ for such a tree T, and $g = (\varphi^*, X^*)$, we conclude the following:

Corollary 16. *There is a pair of Borel automorphisms f and g which are conjugate via a $\sigma(\mathbf{\Sigma}^1_1)$-measurable automorphism, but are not conjugate via a Borel automorphism.*

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