Equivalence relations which reduce all Borel equivalence relations

John D. Clemens

June 25, 2007

Abstract

We study equivalence relations E such that every Borel equivalence relation is Borel reducible to E, and consider the possibility of there being minimal such relations. We first study the relation of equality of Borel sets, and show that this is not a minimal such relation among co-analytic equivalence relations, nor is it universal for co-analytic relations. We then show that there is no minimum analytic equivalence relation to which all Borel equivalence relations are reducible by producing a "minimal pair" of analytic equivalence relations with the property that another equivalence relation is reducible to both of these precisely when it is Borel.

The theory of definable equivalence relations studies the complexity of equivalence relations defined on Polish spaces under the relation of Borel reducibility. This reducibility notion formalizes the idea of having a definable injection between the corresponding quotient spaces. By a definable equivalence relation we mean one which has a suitable definition in the descriptive set-theoretic sense, e.g., it is Borel, analytic, co-analytic, etc.

There are two significant aspects to the study of the Borel reducibility relation. First, one can consider the reducibility hierarchy it generates as a quasi-order and study the structure of this ordering. Second, one can consider natural equivalence relations arising in various branches of mathematics and determine where they fall in this hierarchy in terms of other canonical equivalence relations. Several equivalence have been shown to be very complicated in the sense that every Borel equivalence relation is reducible to them; two examples of this are the relation of Borel bi-reducibility among countable Borel equivalence relations considered by Adams and Kechris in [1], and the relation of conjugacy among Borel automorphisms considered by this author in [2] (Gao has shown in [4] that these two equivalence relations also reduce any analytic equivalence relation).

Let us introduce the key terminology.

Definition 1. Let E and F be equivalence relations on the Polish spaces X and Y. We say that E is *Borel reducible* to F, $E \leq_B F$, if there is a Borel-measurable function $f : X \to Y$ such that for all $x_1, x_2 \in X$ we have $x_1 \, E \, x_2$ if and only if $f(x_1) \, F \, f(x_2)$. That is, f induces a definable injection from the quotient space X/E into Y/F. We write $E <_B F$ when $E \leq_B F$ but not $F \leq_B E$, and we write $E \sim_B F$ when $E \leq_B F$ and $F \leq_B E$.

Definition 2. We say that an equivalence relation E is *Borel-universal* if for any Borel equivalence relation F we have $F \leq_B E$.

The purpose of this article is to study some examples of Borel-universal equivalence relations, particularly those which are as simple as possible descriptively (e.g., analytic or co-analytic such relations). The technique for showing that the two equivalence relations mentioned above are Borel-complete involved reducing to them the equivalence relation of equality of Borel sets. This leads us to wonder whether this is necessary, i.e., whether any Borel-universal equivalence relation must also reduce the relation of equality of Borel sets. More generally, we can ask whether there are any minimal or minimum Borel-universal equivalence relations within certain classes of equivalence relations.

In the next section we consider the relation of equality of Borel sets, and show that it is not a universal co-analytic equivalence relation. In Section 2 we show that equality of Borel sets is not a minimal Borel-universal equivalence relation among the co-analytic equivalence relations, and discuss several other examples of co-analytic Borel-universal relations. In the final section we consider analytic Borel-universal equivalence relations. We show that there is no minimum Borel-universal relation for the the class of analytic equivalence relations by producing a pair of incomparable analytic Borel-universal relations such that any equivalence relation which is Borel reducible to both of them must be Borel.

1 Equality of Borel sets

We begin by defining the equivalence relation of equality of Borel sets. There are several ways of formalizing this relation besides the one we use; however, these alternate presentations are equivalent to the one we give here. We begin by defining codes for Borel subsets of the Baire space ω^ω (where $\omega = \mathbb{N}$ is the set of natural numbers). We recall some definitions about trees.

Definition 3. A *tree* on a set A is a subset of $A^{<\omega}$ (the set of finite sequences from A) which is closed under taking initial segments. A *branch* through a tree T is a sequence $x \in A^\omega$ such that $x \restriction n \in T$ for all $n \in \omega$. We denote the set of branches through T by $[T]$. A tree T is *well-founded* if $[T] = \emptyset$. A node $t \in T$ is a *terminal node* if t has no extensions in T.

Definition 4. Given a well-founded tree T, we define the *rank function of* T, $\rho_T : T \to \mathrm{Ord}$, by recursion:

$$\rho(t) = \begin{cases} 0 & \text{if } t \text{ is a terminal node of } T \\ \sup\{\rho(s) + 1 : t \sqsubseteq s\} & \text{if } t \text{ is not a terminal node.} \end{cases}$$

The *rank* of a well-founded tree T is the ordinal $\rho_T(\langle\rangle)$, where $\langle\rangle$ is the empty sequence, i.e., the root of T. When A is a countable set $\mathrm{rank}(T)$ is a countable ordinal.

We are now ready to define codes for Borel sets.

Definition 5. A *Borel code* on ω^ω is a pair (T, f) where T is a well-founded tree on ω and f is a map from the terminal nodes of T to ω. We let $\mathcal{BC}$ denote the set of Borel codes on ω^ω.

A Borel code indicates how a Borel subset of ω^ω is built from basic open sets by taking countable intersections and negations (we could also have allowed countable unions, but this will not have a significant effect). The Borel set coded by (T, f) is defined recursively using the rank function ρ_T. Fix an enumeration $\langle C_i \rangle_{i\in\omega}$ of the basic clopen neighborhoods in ω^ω, i.e., sets of the form $N_s = \{x \in \omega^\omega : s \sqsubseteq x\}$ with $s \in \omega^{<\omega}$. Then we define, $B_{(T,f)}(t)$, the Borel set coded by a node t in T, as follows:

1. If t is a terminal node of T, then $B_{(T,f)}(t) = C_{f(t)}$.
2. If t is not a terminal node, then

$$B_{(T,f)}(t) = \bigcap\{(\omega^\omega \setminus B_{(T,f)}(t \frown i)) : i \in \omega \wedge t \frown i \in T\},$$

   where $t \frown i$ is the concatenation of the sequence t with i. Thus, $B_{(T,f)}(t)$ is the intersection of the complements of the Borel sets coded by the nodes immediately below t in T.

Definition 6. The Borel set coded by (T, f) is $B_{(T,f)} = B_{(T,f)}(\langle\rangle)$, i.e., the set coded by the root of T.

Since T is well-founded, this produces a well-defined Borel set coded by (T, f). Note that our coding produces only certain representations of sets (G_δ but not F_σ representations, for instance); we could introduce a more complicated coding to remedy this, but since we are working in a completely regular space this is not a problem. It simply affects slightly the ranks assigned to some Borel sets. Every Borel set is represented by a Borel code, since the Borel subsets of a zero-dimensional space can be generated from the clopen sets by taking complements and countable intersections. We will later use x to denote a Borel code, in which case we use B_x to denote the set coded by x.

Note that $\mathcal{BC}$ (appropriately coded) is a $\mathbf{\Pi}^1_1$ (co-analytic) set, since it is a $\mathbf{\Pi}^1_1$ property of a tree to be well-founded. For a Borel code (T, f), the relation "$x \in B_{(T,f)}$" is a Borel relation on ω^ω. We will later also use the fact that the map $(T, f) \mapsto \text{rank}(T)$ is a $\mathbf{\Pi}^1_1$-rank on $\mathcal{BC}$; see [8] or [10] for the definition of a $\mathbf{\Pi}^1_1$-rank.

Definition 7. For a given countable ordinal α we will let $\mathcal{BC}_\alpha$ denote the set of *Borel codes of rank less than* α, i.e., codes (T, f) where T is a well-founded tree of rank less than α.

The sets $\mathcal{BC}_\alpha$ correspond to the Borel sets of Borel rank (roughly) less than α. Each set $\mathcal{BC}_\alpha$ is a Borel set, since for a fixed α the property of being a well-founded tree of rank less than α is a Borel property. Note that a given Borel set will have codes of arbitrarily high rank, so one should remember that $\mathcal{BC}_\alpha$ refers to the set of codes of rank less than α and not to the set of all codes for Borel sets of rank less than α. We say a set is *true* $\mathbf{\Pi}^0_\alpha$ if it $\mathbf{\Pi}^0\alpha$ but not $\mathbf{\Sigma}^0_\alpha$.

We are now ready to define the relevant equivalence relations.

Definition 8. Let E_B be the equivalence relation of *equality of Borel sets*, defined on the Borel set

$$\{(T, f) : T \text{ is a tree and f is a function from the terminal nodes of } T \text{ to } \omega\}$$

by letting

$$x \, E_B \, y \iff x = y \ \vee \ (x, y \in \mathcal{BC} \ \wedge \ B_x = B_y).$$

For each countable ordinal α we also define the relation of *equality of Borel sets of rank less than* α, given by

$$x \, E_\alpha \, y \iff x = y \ \vee \ (x, y \in \mathcal{BC}_\alpha \ \wedge \ B_x = B_y).$$

We check that these are also all $\mathbf{\Pi}^1_1$ equivalence relations. Since it is Borel to ask whether a given element $z \in \omega^\omega$ is an element of the Borel set coded by a given parameter x, we have

$$x \mathrel{E_B} y \iff \forall z \in \omega^\omega \ [z \in B_x \iff z \in B_y].$$

As noted earlier, any Borel equivalence relation E on a Polish space X is Borel reducible to E_B since we can map a point $x \in X$ to a Borel code for its equivalence class $[x]_E$; we then have $x_1 \mathrel{E} x_2$ if and only if $[x_1]_E = [x_2]_E$. Let us record several other properties of the above relations:

- For $\alpha < \omega_1$, any $\mathbf{\Pi}^0_\alpha$ equivalence relation E satisfies $E \leq_B E_\alpha$.
- For $\alpha < \beta < \omega_1$, we have $E_\alpha <_B E_\beta$. This is proved by Hjorth in [6] using Stern absoluteness.
- For $\alpha < \omega_1$, $E_\alpha <_B E_B$. This follows from the previous result, since each relation E_β is reducible to E_B.

Let us also note that the Borel coding we use does not have a substantial effect; any standard system of Borel codes should produce an equality relation which is bi-reducible with the one given here.

Our first question is whether E_B is as complicated as possible, namely, whether every $\mathbf{\Pi}^1_1$ equivalence relation is Borel reducible to it. This is ostensibly possible, since Hjorth has shown in [7] that there is a universal $\mathbf{\Pi}^1_1$ equivalence relation (in fact, Hjorth's relation has an even stronger universality property than what we have described here). This turns out not to be the case, though.

Theorem 9. *The relation E_B is not a universal $\mathbf{\Pi}^1_1$ equivalence relation.*

We will show this by exhibiting a $\mathbf{\Pi}^1_1$ equivalence relation which is not reducible to E_B. We begin with a technical lemma.

Lemma 10. *There is a $\mathbf{\Pi}^1_1$ equivalence relation E such that for each x, its equivalence class $[x]_E$ is a $\mathbf{\Pi}^1_1$-complete set.*

Proof: Let $G \subseteq (\omega^\omega)^2$ be a universal $\mathbf{\Pi}^0_3$ set. Specifically, let

$$G(x, z) \iff \forall n \exists j \forall k \ [z \notin N_{s_{x(\langle n,j,k\rangle)}}],$$

where $\langle s_i \rangle_{i\in\omega}$ is an enumeration of $\omega^{<\omega}$, N_s is the basic open interval determined by s, and $(n, j, k) \mapsto \langle n, j, k\rangle$ is a recursive bijection between ω^3 and

ω. By universal, we mean that for every $\mathbf{\Pi}^0_3$ set $A \subseteq \omega^\omega$ there is an x such that $A = G_x = \{z : (x, z) \in G\}$. We now set $x\,E\,y$ if and only if $G_x = G_y$, i.e.,

$$x\,E\,y \iff \forall z\;[G(x, z) \iff G(y, z)].$$

This is clearly $\mathbf{\Pi}^1_1$. Now let $x_0 \in \omega^\omega$ and a $\mathbf{\Pi}^1_1$ set $A \subseteq \omega^\omega$ be given. We will show that $A \leq_W [x_0]_E$, i.e., A is the continuous preimage of $[x_0]_E$. Fix a tree T on $\omega \times \omega$ to represent A, i.e.,

$$y \in A \iff \neg\exists z\;[(z, y) \in [T]].$$

It now suffices to find a continuous function f such that for each y,

$$\exists z\;[(z, y) \in [T]] \iff G_{f(y)} \neq G_{x_0}.$$

Since G_{x_0} is a Borel set, it either contains or is disjoint from a perfect set, and hence there is a continuous embedding of ω^ω into either this set or its complement. We consider the two cases.

If G_{x_0} is disjoint from a perfect set, let $\varphi : s \mapsto t_s$ be a mapping of finite sequences inducing a continuous embedding of ω^ω into $\omega^\omega \setminus G_{x_0}$, in the sense that x is sent to $\varphi(x) = \bigcup_n t_{x\restriction n}$. We can choose this map so that if s_1 and s_2 are incomparable sequences then so are t_{s_1} and t_{s_2}. Let $\langle G_n \rangle_{n\in\omega}$ be the sequence of F_σ sets coded by x_0 according to our universal set G, where each $G_n = \bigcup_k G_{n,k}$ with $G_{n,k}$ closed. Thus, $G_{x_0} = \bigcap_n G_n$. We wish to express G_{x_0} as the intersection of a descending sequence of sets, so we let $\widetilde{G}_n = \bigcap_{k\leq n} G_k$. The sets $\widetilde{G}_n$ are still F_σ, and we can effectively write them in the coding scheme for our universal set (since there are recursive maps which, given indices for two F_σ sets, produce indices for their intersection and union). We still have $G_{x_0} = \bigcap_n \widetilde{G}_n$.

Now for $n \in \omega$ let

$$H_n = \widetilde{G}_n \cup \bigcup \{N_{t_s} : |s| = n \;\wedge\; (s, y \restriction n) \in T\},$$

and let $\langle H_{n,k} \rangle_{k\in\omega}$ enumerate the closed sets necessary to express this as an F_σ set according to the coding in G (we can inter-weave the closed sets making up G_n with codes for the N_{t_s}'s in increasing order of their indexing). We let $f(y)(\langle n, k\rangle) = H_{n,k}$ (i.e., the index for $H_{n,k}$), so that $G_{f(y)} = \bigcap_n H_n$. Note that each $H_n \supseteq \widetilde{G}_n$ so that $G_{f(y)} \supseteq G_{x_0}$. Thus, we need only check that for each y we have

$$\exists z\;[(z, y) \in [T]] \iff G_{f(y)} \setminus G_{x_0} \neq \emptyset.$$

If there is a z with $(z, y) \in [T]$, then each H_n contains the set $N_{t_{z \upharpoonright n}}$, so that $\varphi(z) \in G_{f(y)}$. We know $\varphi(z) \notin G_{x_0}$, so this suffices.

Conversely, suppose there is a $w \in G_{f(y)} \setminus G_{x_0}$ There must then be some n_0 such that for all $n \geq n_0$ we have $w \notin \widetilde{G}_n$. Thus, for all such n there must be some s_n with $(s_n, y \upharpoonright n) \in T$ and $w \in N_{t_{s_n}}$ (so $w \sqsupseteq t_{s_n}$). Thus all of the t_{s_n}'s must be comparable in T, so we must also have that the s_n's are all comparable in T. But then we can take $z = \bigcup_n s_n$ and we have that $(z, y) \in [T]$ as desired.

The case where G_{x_0} contains a perfect set is similar, except now we take elements away from G_{x_0} rather than adding them. Let φ be chosen in the same manner as before, but this time embedding ω^ω in G_{x_0}, and again let $G_{x_0} = \bigcap_n \widetilde{G}_n$ be given as the intersection of a descending sequence of F_σ sets. Let $H_{0,k}$ be the open set

$$\bigcup \{N_{t_s} : |s| = k \ \wedge \ (s, y \upharpoonright k) \in T\},$$

and let H_0 be the F_σ set

$$H_0 = \omega^\omega \setminus \bigcap_{k \in \omega} H_{0,k}.$$

We now let $H_{n+1} = \widetilde{G}_n$ for $n \in \omega$ and let $f(y)$ code the intersection of these sets, $\bigcap_{n \in \omega} H_n$. Clearly $G_{f(y)} \subseteq G_{x_0}$, so it suffices to check that for each y we have

$$\exists z \ [(z, y) \in [T]] \Longleftrightarrow G_{x_0} \setminus G_{f(y)} \neq \emptyset.$$

If there is a z with $(z, y) \in [T]$, then $\varphi(z)$ is in each of the $H_{0,k}$'s. Hence $\varphi(z)$ is not in H_0 and so $\varphi(z)$ is not in $G_{f(y)}$, whereas we know $\varphi(z) \in G_{x_0}$. Conversely, any element of G_{x_0} which is not in $G_{f(y)}$ must be in the complement of H_0 and hence in each of the $H_{0,k}$'s. As before, this gives us a z such that $(z, y) \in [T]$. □

We can now show that E_B is not a universal $\mathbf{\Pi}^1_1$ equivalence relation.

Proposition 11. *Let E be the equivalence relation from Lemma 10. Then $E_B <_B E \times E_B$.*

Proof: Clearly $E_B \leq_B E \times E_B$. Suppose there were a Borel function f reducing $E \times E_B$ to E_B. Since for all x and y we have

$$[(x, y)]_{E \times E_B} = [x]_E \times [y]_{E_B},$$

we have that all $E \times E_B$ equivalence classes are $\mathbf{\Pi}^1_1$-complete since all E equivalence classes are. Thus, f must map each $E \times E_B$ class into a $\mathbf{\Pi}^1_1$-complete equivalence class of E_B, since each $E \times E_B$ class is the inverse images under f of an E_B class in the range of f. Thus, f can not map into any of the singleton classes of elements outside of $\mathcal{BC}$, and hence the range of f is a subset of $\mathcal{BC}$. Since $f[(\omega^\omega)^2]$ is now a $\mathbf{\Sigma}^1_1$ subset of $\mathcal{BC}$, we must have that $f[(\omega^\omega)^2] \subseteq \mathcal{BC}_\alpha$ for some $\alpha < \omega_1$ by the Boundedness Theorem (Theorem 35.23 of [8]), since the map sending a Borel code to its Borel rank is a $\mathbf{\Pi}^1_1$-rank on $\mathcal{BC}$. But then f would in fact witness that $E \times E_B \leq_B E_\alpha$ for this α, and hence we would have $E_B \leq_B E_\alpha$ which is a contradiction since $E_\alpha <_B E_B$. □

This then establishes Theorem 9. The above proof actually gives us a more general theorem, that there is a type of "jump" for certain $\mathbf{\Pi}^1_1$ equivalence relations.

Theorem 12. *Let E be a $\mathbf{\Pi}^1_1$ equivalence relation none of whose equivalence classes is Borel. Suppose that F is a $\mathbf{\Pi}^1_1$ equivalence relation such that F is not Borel reducible to $F \restriction A$ for any $\mathbf{\Sigma}^1_1$ set $A \subseteq \{x : [x]_F$ is not Borel$\}$. Then we have that $F <_B E \times F$.*

We have seen that E_B is not a universal $\mathbf{\Pi}^1_1$ equivalence relation. In the next section we will consider whether it can be minimal among those $\mathbf{\Pi}^1_1$ equivalence relations which reduce all Borel equivalence relations.

2 Minimal Borel-universal $\mathbf{\Pi}^1_1$ relations

Equality of Borel sets is a canonical example of an equivalence relation to which we can reduce all Borel equivalence relations. We wish to analyze what equivalence relations of this form can look like, and see if this property implies any sort of additional complexity. The type of additional complexity we are seeking is some single complicated equivalence relation E such that, for all equivalence relations F of some given type, if every Borel equivalence relation is reducible to F then E must also be reducible to F. An appropriate standard for complexity of E is that every Borel equivalence relation should be reducible to it. Note that by a result of Friedman and Stanley (see [3]) no Borel equivalence relation can be universal for all Borel equivalence relations, so we should look to more complicated equivalence relations such as analytic or co-analytic ones. There are $\mathbf{\Sigma}^1_1$ equivalence relations universal for all $\mathbf{\Sigma}^1_1$ equivalence relations, and $\mathbf{\Pi}^1_1$ equivalence relations universal for all $\mathbf{\Pi}^1_1$ equivalence relations (see [7]), so any of these relations is Borel-universal. We want to know if there are simpler Borel-universal relations in these classes, though. This suggests we investigate the following notions.

Definition 13. Given a collection $\mathcal{F}$ of equivalence relations, we say that an equivalence relation E is *minimum above Borel for* $\mathcal{F}$ if $E \in \mathcal{F}$, every Borel equivalence relation is reducible to E, and for every $F \in \mathcal{F}$ to which we can reduce all Borel equivalence relations we have $E \leq_B F$. We say that E is *minimal above Borel for* $\mathcal{F}$ if $E \in \mathcal{F}$, every Borel equivalence relation is reducible to E, and there is no equivalence relation $F \in \mathcal{F}$ with $F <_B E$ such that every Borel equivalence relation is reducible to F.

We wish to investigate the possibility of there being minimum or minimal equivalence relations above Borel for classes of definable equivalence relations. In this section we consider the class of co-analytic equivalence relations, and we consider the class of analytic relations in the next section. We will first show that E_B is not minimum above Borel for the class of $\mathbf{\Pi}^1_1$ equivalence relations. We will in fact present two Borel-universal $\mathbf{\Pi}^1_1$ equivalence relations to which we can not reduce E_B.

We begin by defining codes for Borel equivalence relations on ω^ω. Let $D \subseteq \omega^\omega$ and $P, S \subseteq (\omega^\omega)^3$ be a parameterization of the Borel subsets of $(\omega^\omega)^2$, i.e., D is $\mathbf{\Pi}^1_1$, P is $\mathbf{\Pi}^1_1$, and S is $\mathbf{\Sigma}^1_1$ such that:

1. If $z \in D$ then $P_z = S_z$ (and hence these are Borel subsets of $(\omega^\omega)^2$).
2. The set $\{P_z : z \in D\}$ contains all Borel subsets of $(2^\omega)^2$.

Such a parameterization exists; see for instance [8].

Definition 14. The set $B \subseteq D$ of *codes for Borel equivalence relations* is defined by

$$\begin{aligned} B(z) &\iff P_z = S_z \text{ is an equivalence relation} \\ &\iff D(z) \ \wedge\ \forall x \forall y \forall w\ [P_z(x,x)\ \wedge\ (S_z(x,y) \Longrightarrow P(y,x))\ \wedge \\ &\qquad\qquad ((S_z(x,y)\ \wedge\ S_z(y,w)) \Longrightarrow P_z(x,w))]. \end{aligned}$$

Thus, B (together with P and S) is a parameterization of all Borel equivalence relations on ω^ω. The set B is evidently $\mathbf{\Pi}^1_1$.

Lemma 15. *The set B is $\mathbf{\Pi}^1_1$-complete.*

Proof: Let A be a $\mathbf{\Pi}^1_1$ set, represented by

$$A(x) \iff \forall y \exists n \; [(x \restriction n, y \restriction n) \notin T],$$

where T is a tree on $\omega \times \omega$. For a given $x \in \omega^\omega$, we let $f(x)$ code the following binary relation R_x on ω^ω as a Borel subset of $(\omega^\omega)^2$. We can easily produce a parameter for this set from x, i.e there will be a Borel function f such that $f(x) \in D$ codes relation R_x. We define R_x by the following conditions:

1. For y_1 and y_2 with $y_1 \neq x$ and $y_2 \neq x$, let $y_1 \, R_x \, y_2$.
2. For all y, let $x \, R_x \, y$.
3. For all y, set $y \, R_x \, x \iff (x, y) \notin [T]$.
4. Nothing else is R_x-related.

If T_x has an infinite branch y (i.e., $x \notin A$), then R_x will not be symmetric since $x \, R_x \, y$ but not $y \, R_x \, x$, and hence R_x is not an equivalence relation. Otherwise all pairs of points are R_x-related, so R_x is trivially an equivalence relation. □

We now introduce our first relation.

Definition 16. Let E_P be the $\mathbf{\Pi}^1_1$ equivalence relation defined on $(\omega^\omega)^2$ by

$$(z_1, x_1) \, E_P \, (z_2, x_2) \iff z_1 = z_2 \;\wedge\; [x_1 = x_2 \;\vee\; (B(z_1) \;\wedge\; P(z_1, x_1, x_2))].$$

Thus, each z-slice of E_P either is the identity relation (if $z \notin B$), or it is the Borel equivalence relation parameterized by z when $z \in B$. We check that E_P is Borel-universal.

Lemma 17. *For any Borel equivalence relation E we have $E \leq_B E_P$.*

Proof: We can assume that E is a Borel equivalence relation on ω^ω since all uncountable Polish spaces are Borel isomorphic. Let $z \in B$ be a parameter for E, so $E = P_z$. We then define the map $f : \omega^\omega \to (\omega^\omega)^2$ by $f(x) = (z, x)$. It is then immediate that $x_1 \, E \, x_2$ iff $P(z, x_1, x_2)$ iff $(z, x_1) \, E_P \, (z, x_2)$, i.e., $f(x_1) \, E_P \, f(x_2)$. □

We also note that every equivalence class of E_P is Borel, so this is a relatively simple non-Borel $\mathbf{\Pi}^1_1$ equivalence relation in a descriptive sense. The following proposition is thus immediate:

Proposition 18. *If E is an equivalence relation such that $E \leq_B E_P$, then every equivalence class of E is Borel.*

This then gives:

Corollary 19. *The equivalence relation E_B is not Borel reducible to E_P.*

Proof: We can easily see that E_B contains equivalence classes which are not Borel. For instance, the equivalence class consisting of codes for the empty set is not Borel since it contains codes (T, f) with T of arbitrarily high rank, and no Borel set of Borel codes can have codes of arbitrarily high rank by boundedness. $\square$

Corollary 20. *The relation E_B is not minimum above Borel for the class of $\mathbf{\Pi}^1_1$ equivalence relations.*

Let us say a bit more about the relation E_P. Fix a $\mathbf{\Pi}^1_1$-rank ρ on B, and set

$$B_\alpha = \{x : x \in B \ \wedge \ \rho(x) < \alpha\}.$$

Then each B_α is Borel, $B_\alpha \subseteq B_\beta$ for $\alpha < \beta$, and $B = \bigcup_{\alpha<\omega_1} B_\alpha$. For $\alpha < \omega_1$, define the following relation on $(\omega^\omega)^2$:

$$\begin{aligned}(z_1, x_1)\, E^\alpha_P\, (z_2, x_2) &\iff z_1 = z_2 \ \wedge \ [x_1 = x_2 \ \vee \ (B_\alpha(z_1) \ \wedge \ P(z_1, x_1, x_2))] \\ &\iff z_1 = z_2 \ \wedge \ [x_1 = x_2 \ \vee \ (B_\alpha(z_1) \ \wedge \ S(z_1, x_1, x_2))].\end{aligned}$$

Thus each E^α_P is a Borel equivalence relation containing as slices all Borel equivalence relations with codes of rank less than α. We list several properties of the relations E^α_P:

1. For $\alpha < \beta$, we have $E^\alpha_P \leq_B E^\beta_P <_B E_P$.
2. For $\alpha < \beta$, we have $E^\alpha_P \subseteq E^\beta_P \subseteq E_P$, and $E_P = \bigcup_{\alpha<\omega_1} E^\alpha_P$.
3. Every Borel equivalence relation is reducible to some E^α_P, and so an equivalence relation F is Borel-universal if and only if every E^α_P is Borel reducible to F.

The relation E_P seems to be a good candidate for a minimal above Borel $\mathbf{\Pi}^1_1$ equivalence relation, but we do not know whether this is true. As some evidence, we can see that $E_P \leq_B E_B$:

Proposition 21. *The relation E_P is Borel reducible to E_B.*

Proof: Since the set B of codes for Borel equivalence relations is $\mathbf{\Pi}^1_1$ and the set $\mathcal{BC}$ of Borel codes is easily $\mathbf{\Pi}^1_1$-complete, we may fix a continuous function g such that for all z we have $z \in B \iff g(z) \in \mathcal{BC}$. For $z \in B$, we have that the sets P_z and S_z are equal (hence $\mathbf{\Delta}^1_1$). We can thus apply the effective separation theorem (see [10]) to produce a Borel code for the set $P_z = S_z$, which is the equivalence relation E_z. Both the function producing this code and the function g can be taken to be total functions which produce Borel codes when necessary (and produce garbage otherwise). So we will treat the images under these functions as if they were Borel codes when appropriate.

Now, given a pair (z, x) we can produce the Borel code $g(z)$, a Borel code for the singleton $\{z\}$, and a Borel code for the set $[x]_{E_z}$ (which we get from the Borel code for E_z obtained above). From these, we can form the Borel code for the following set:

$$C_{z,x} = \{0 \frown z\} \cup \{1 \frown y : y \in B_{g(z)}\} \cup \{2 \frown y : y \in [x]_{E_z}\}.$$

When z codes a Borel equivalence relation this procedure produces a legitimate Borel code, and when z does not code a Borel equivalence relation it will produce something which is not a Borel code. Let f be the function producing this putative code for $C_{z,x}$. It is straightforward to check that for two of these codes $f(z_1, x_1)$ and $f(z_2, x_2)$ to be equivalent under E_B we must have had either that both z_1 and z_2 were not in B, or that $z_1 = z_2$ and x_1 is E_{z_1}-equivalent to x_2. Hence, f is a reduction from E_P to E_B. $\square$

Since we now have $E_P <_B E_B$, the following is immediate:

Corollary 22. *The relation E_B is not minimal above Borel for the class of $\mathbf{\Pi}^1_1$ equivalence relations.*

We will say a bit more about E_P at the end of the next section. We now give a second example of a Borel-universal $\mathbf{\Pi}^1_1$ equivalence relation to which we cannot reduce E_B. Recall that $\mathcal{WO}$ is the collection of codes for countable well-orders of ω.

Definition 23. Let E^*_B be the relation defined on $(\omega^\omega)^2$ by

$$(x, \alpha)\, E^*_B\, (y, \beta) \iff (x = y \ \wedge\ \alpha = \beta) \ \vee\ (\alpha, \beta \in \mathcal{WO} \ \wedge\ |\alpha| = |\beta| \ \wedge\ x, y \in \mathcal{BC}_{|\alpha|} \ \wedge\ B_x = B_y).$$

Thus, E^*_B is a "spreading out" of E_B, demanding not only that two codes produce the same Borel set but also that they be of the same given rank.

This is again a $\mathbf{\Pi}^1_1$ equivalence relation. Every Borel equivalence relation E is reducible to E^*_B because, having fixed a Borel code for E of rank α, we can produce codes for its equivalence classes which are all of rank α. We also have that each of the relations E_α is reducible to E^*_B, since we can map a code in $\mathcal{BC}_\alpha$ to the pair $(x, \widetilde{\alpha})$ where $\widetilde{\alpha} \in \mathcal{WO}$ is some fixed representation of α.

Proposition 24. *The relation E_B is not Borel reducible to E^*_B.*

Proof: Suppose $\varphi : x \mapsto (x^*, \alpha_x)$ were such a reduction. Then we have that if $x \in \mathcal{BC}$ then $x^* \in \mathcal{BC}_{|\alpha_x|}$, and if $B_x = B_y$ then $|\alpha_x| = |\alpha_y|$. For $\gamma < \omega_1$, we have that $\{\alpha_x : x \in \mathcal{BC}_\gamma\}$ is thus a $\mathbf{\Sigma}^1_1$ subset of $\mathcal{WO}$, so by boundedness there is some countable ordinal ξ_γ such that

$$x \in \mathcal{BC}_\gamma \Longrightarrow |\alpha_x| < \xi_\gamma.$$

We choose ξ_γ to be the least such ordinal with $\xi_\gamma > \gamma$. Now let

$$\begin{aligned} \gamma_0 &= 1 \\ \gamma_{n+1} &= \xi_{\gamma_n} \text{ for } n \in \omega \\ \gamma_\infty &= \sup_{n\in\omega} \gamma_n. \end{aligned}$$

Since the sequence of γ_n's is increasing, γ_∞ will be a limit ordinal and if $x \in \mathcal{BC}_\delta$ for some $\delta < \gamma_\infty$ then we will have $|\alpha_x| < \gamma_\infty$.

Now consider $x \in \mathcal{BC}_{(\gamma_\infty+1)}$. If $B_x \in \mathbf{\Pi}^0_\delta$ for some $\delta < \gamma_\infty$ then B_x has a code $x' \in \mathcal{BC}_\delta$, so we have $|\alpha_{x'}| = |\alpha_x| < \gamma_\infty$. Conversely, if $|\alpha_x| < \gamma_\infty$ then $B_x \in \mathbf{\Pi}^0_\delta$ for some $\delta < \gamma_\infty$. Hence, for $x \in \mathcal{BC}_{(\gamma_\infty+1)}$ we have

$$B_x \text{ is true } \mathbf{\Pi}^0_{\gamma_\infty} \Longleftrightarrow |\alpha_x| \geq \gamma_\infty.$$

This would imply that the set

$$C = \{x \in \mathcal{BC}_{(\gamma_\infty+1)} : B_x \text{ is true } \mathbf{\Pi}^0_{\gamma_\infty}\}$$

is a Borel set. However, we claim that C is $\mathbf{\Sigma}^1_1$-hard (i.e., every $\mathbf{\Sigma}^1_1$ set is its continuous inverse image), so that it can not be Borel.

To see this, let A be a true $\mathbf{\Pi}^0_{\gamma_\infty}$ set and fix a Borel code in $\mathcal{BC}_{(\gamma_\infty+1)}$ for the set $A \times \omega^\omega$ (which is also true $\mathbf{\Pi}^0_{\gamma_\infty}$). Here we are identifying $(\omega^\omega)^2$ with ω^ω by some fixed homeomorphism, so that basic neighborhoods referred to below will really be neighborhoods in $(\omega^\omega)^2$. We will describe a continuous reduction of the $\mathbf{\Sigma}^1_1$-complete set of ill-founded trees to the set C.

Given a tree T, we will map T to a Borel code $x_T \in \mathcal{BC}_{(\gamma_\infty+1)}$ for the set $A\times[T]$. We already have a Borel code for the set $A\times\omega^\omega$, so we need only find a code for the intersection of this set with $\omega^\omega \times [T]$. We can easily produce a code of rank less than γ_∞ for the latter set in a continuous fashion; we now simply append a copy of this code to a new node added in the first level of the tree of the original Borel code for $A \times \omega^\omega$. This new code will then be a Borel code for the intersection of these two sets. Let x_T be this code. Then we will have that the set coded by x_T is empty if T is well-founded, and hence not true $\mathbf{\Pi}^0_{\gamma_\infty}$. On the other hand, if T is ill-founded then [T] will be a non-empty closed set, so clearly $A \leq_W A \times [T]$ and $A \times [T]$ is true $\mathbf{\Pi}^0_{\gamma_\infty}$. We thus have that T is ill-founded if and only if $x_T \in C$. □

So we know that E_B is not a minimum or even minimal above Borel for $\mathbf{\Pi}^1_1$ equivalence relations. We do not know if there can be any minimal such equivalence relations, though E_P or E_B^* seem reasonable candidates.

Question 25. *Is there a $\mathbf{\Pi}^1_1$ equivalence relation which is minimum or minimal above Borel for the set of $\mathbf{\Pi}^1_1$ equivalence relations? What if we allow more general reductions, such as $\sigma(\mathbf{\Sigma}^1_1)$-measurable ones?*

We say that $E \leq_{\sigma(\mathbf{\Sigma}^1_1)} F$ if there is a reducing function which is measurable with respect to the σ-algebra generated by the $\mathbf{\Sigma}^1_1$ sets. Some simple test questions for E_P and E_B^* are:

Question 26. *Is $E_B^* \leq_B E_B$ or $E_B^* \leq_{\sigma(\mathbf{\Sigma}^1_1)} E_B$? Is $E_P \leq_B E_B^*$? Is $E_B^* \leq_B E_P$?*

We also do not know if E_B can be minimal under more general reductions. Here good test questions are the following:

Question 27. *Is $E_B \leq_{\sigma(\mathbf{\Sigma}^1_1)} E_P$? Is $E_B \leq_{\sigma(\mathbf{\Sigma}^1_1)} E_B^*$?*

A question along similar lines to minimality above Borel is this:

Question 28. *Is there a minimum $\mathbf{\Pi}^1_1$ equivalence relation above all of the E_α's?*

3 A minimal pair of $\mathbf{\Sigma}^1_1$ equivalence relations

We now consider the analogous questions for $\mathbf{\Sigma}^1_1$ equivalence relations. Here we can show that there is no $\mathbf{\Sigma}^1_1$ relation which is a minimum above Borel for the class of $\mathbf{\Sigma}^1_1$ equivalence relations. Recall the set B introduced earlier parameterizing Borel equivalence relations on ω^ω.

Definition 29. Let E_S and E'_S be the equivalence relations defined on $(\omega^\omega)^2$ by

$$(z_1, x_1)\ E_S\ (z_2, x_2) \Longleftrightarrow z_1 = z_2 \ \wedge\ [\neg B(z_1)\ \vee\ S(z_1, x_1, x_2)]$$
$$(z_1, x_1)\ E'_S\ (z_2, x_2) \Longleftrightarrow [\neg B(z_1)\ \wedge\ \neg B(z_2)]\ \vee\ [z_1 = z_2\ \wedge\ S(z_1, x_1, x_2)].$$

These are both $\mathbf{\Sigma}^1_1$ equivalence relations. They are similar to E_P except that instead of making slices outside of B into the identity relation, in the case of E_S we make each slice into a single equivalence class and in the case of E'_S we lump all of these slices together into a single class. We thus have $E_p \subseteq E_S \subseteq E'_S$. Again, we see that every Borel equivalence relation is Borel reducible to both E_S and E'_S. Also note that we can not have any Borel equivalence relation E with $E_P \subseteq E \subseteq E'_S$, since it would also have to be universal for all Borel equivalence relations. A chief difference between E_S and E'_S is that every equivalence class of E_S is a Borel set whereas E'_S contains one $\mathbf{\Sigma}^1_1$-complete class.

Similarly to what we did with E_P, we can define E^α_S for $\alpha < \omega_1$ by letting

$$\begin{aligned}(z_1, x_1)\ E^\alpha_S\ (Z_2, x_2) &\Longleftrightarrow z_1 = z_2\ \wedge\ [\neg B_\alpha(z_1)\ \vee\ P(z_1, x_1, x_2)]\\ &\Longleftrightarrow z_1 = z_2\ \wedge\ [\neg B_\alpha(z_1)\ \vee\ S(z_1, x_1, x_2)],\end{aligned}$$

so that each E^α_S is a Borel equivalence relation, $E^\alpha_S \supseteq E^\beta_S$ for $\alpha < \beta < \omega_1$, and $E_S = \bigcap_{\alpha<\omega_1} E^\alpha_S$. Note that again every Borel equivalence relation is Borel reducible to some E^S_α. In fact, for each α we have that E^α_S is Borel bi-reducible with E^α_P. We can also express E'_S in a similar manner.

Our main result in this section is the following:

Proposition 30. *For any equivalence relation E on a Polish space, E is Borel if and only if $E \leq_B E_S$ and $E \leq_B E'_S$.*

Proof: One direction is immediate. For the other, let f be a Borel function reducing E to E'_S. Since E'_S has only one non-Borel equivalence class, at most one E class can be mapped to it via f. Let A be the inverse image of the complement of this class under f, so A is E-invariant. Since E is reducible to E_S, all of its classes must be Borel sets, and since A omits at most one equivalence class, A must be Borel. Therefore, $f[A]$ is the Borel image of a Borel set and hence $\mathbf{\Sigma}^1_1$. Letting π_0 be projection onto the first coordinate, we have that $\pi_0 \circ f[A]$ is also $\mathbf{\Sigma}^1_1$. Since this image is a subset of B, we must in fact have that $\pi_0 \circ f[A] \subseteq B_\alpha$ for some α by boundedness. Thus, we must have that $E \restriction A \leq_B E'_S \restriction (B_\alpha \times \omega^\omega)$ for such an α (using the given function f). Since $E'_S \restriction (B_\alpha \times \omega^\omega) = E^\alpha_S \restriction (B_\alpha \times \omega^\omega)$, we actually have that $E \upharpoonright A \leq_B E_S^\alpha$. This is a Borel equivalence relation, so $E \upharpoonright A$ must be also. Then E itself must be Borel since $E = E \upharpoonright A \cup (\omega^\omega \setminus A)^2$. $\square$

Since each of these two equivalence relations is reducible to itself and neither is Borel, we then have:

Corollary 31. *The relations E_S and E'_S are $\leq_B$-incomparable.*

We also immediately have the promised:

Theorem 32. *There is no $\mathbf{\Sigma}_1^1$ equivalence relation which is minimum above Borel for $\mathbf{\Sigma}_1^1$ equivalence relations.*

We do not know, though, if there can be minimal such relations.

Question 33. *Is there a $\mathbf{\Sigma}_1^1$ equivalence relation which is minimal above Borel for the class of $\mathbf{\Sigma}_1^1$ equivalence relations? What if we allow more general reductions?*

Although we have seen that E_S and E'_S are $\leq_B$-incomparable, they are in fact bi-reducible if we allow more general reductions. In fact:

Proposition 34. *We have $E_S \sim_{\sigma(\mathbf{\Sigma}_1^1)} E'_S \sim_{\sigma(\mathbf{\Sigma}_1^1)} E_P$.*

Proof: First, to see that $E'_S \leq_{\sigma(\mathbf{\Sigma}_1^1)} E_S$ let z_0 be a point not in B. Now define the reduction f_1 by

$$f_1(z,x) = \begin{cases} (z_0, x) & \text{if } z \notin B \\ (z, x) & \text{if } z \in B. \end{cases}$$

Next, to see that $E_S \leq_{\sigma(\mathbf{\Sigma}_1^1)} E_P$ define f_2 by

$$f_2(z,x) = \begin{cases} (z, z) & \text{if } z \notin B \\ (z, x) & \text{if } z \in B. \end{cases}$$

Finally, to see that $E_P \leq_{\sigma(\mathbf{\Sigma}_1^1)} E'_S$ fix a $z_1 \in B$ such that $P_{z_1} = S_{z_1} = \Delta(\omega^\omega)$, the identity relation. Now let f_3 be given by

$$f_3(z,x) = \begin{cases} (z_1, 0 \frown x) & \text{if } z \notin B \\ (z_1, 1 \frown x) & \text{if } z = z_1 \\ (z, x) & \text{if } z \in B \text{ and } z \neq z_1. \end{cases}$$

It is easy to verify that these functions are reductions and are $\sigma(\mathbf{\Sigma}_1^1)$-measurable. Compositions give the rest of the needed reductions (and we note that these compositions will be $\sigma(\mathbf{\Sigma}_1^1)$-measurable as well). Note that although the given functions are not injective, one can modify them to make them embeddings. $\square$

The case of E_S and E'_S in the above proposition suggests that we should consider more general reductions at this level than just Borel-measurable ones. These two relations are essentially the same, they are bi-reducible under $\sigma(\mathbf{\Sigma}^1_1)$-measurable reductions, and yet they are $\leq_B$-incomparable.

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Department of Mathematics
Penn State University
University Park, PA 16802, USA

email: clemens@math.psu.edu