Research Themes on Traffic Flow on Networks

(*Alberto Bressan, July 2016*)

Models of traffic flow can be of two main types [1, 14, 17].

1 - Microscopic particle models, describing the position and velocity of each single car. If there are N cars on a road, one thus needs to write a system of N differential equations, one for each car. These ODEs specify how each driver adjusts his velocity depending on the distance and the velocity of the vehicle ahead.

2 - Macroscopic models, describing the evolution of the vehicle density (i.e. the number of cars per unit length of the road). Since the total number of cars is conserved, these models consist of one or more PDEs, usually in the form of a conservation law.

Particle models are easier to simulate numerically. On the other hand, macroscopic models are mathematically more interesting and yield a better qualitative understanding of traffic patterns.

Having written down a set of mathematical equations, various issues can be studied.

(i) The first and most fundamental concern is making sure that the model "well posed". This means that, given any initial configuration, the equations determine a unique solution for all future times.

(ii) Study the traffic behavior. How do traffic waves propagate along a highway? When can queues arise ? Can *shocks* be produced, i.e. can the traffic density change suddenly at specific locations? Needless to say, the model should yield realistic predictions, and solutions should never exhibit absurd features, such as vehicles moving backwards or car density with negative values. Next, one can study how this evolution changes depending on parameters (i.e. speed limits, drivers' behavior, cars' performance, etc...).

(iii) At a further stage, one can analyze an optimization problems, i.e. how to optimally adjust certain parameters in order to minimize a cost criterion. What if the scheduling of departures are scheduled by a central planner, or each driver chooses his own departure time and route to destination?

In connection with the macroscopic PDE model, these ideas will be expanded in the sections below.

1 Conservation law models on a network of roads

The simplest and most popular macroscopic model for traffic flow, due to Lighthill-Witham-Richards [20, 21], takes the form of a single conservation law

$$\partial_t \rho + \partial_x [\rho\, v(\rho)] \;=\; 0. \tag{1}$$

Here t is time, x is the space variable denoting points along a highway, while $\rho(t,x)$ is the **density** of cars (= number of cars per mile of highway) at time t at location x. Moreover, $v(\rho)$ describes the **velocity** of cars, which we assume is a decreasing function of the density ρ. Notice that the product $\rho\, v(\rho)$ is the **flux** of cars (=number of cars that cross a point x on the highway per unit time).

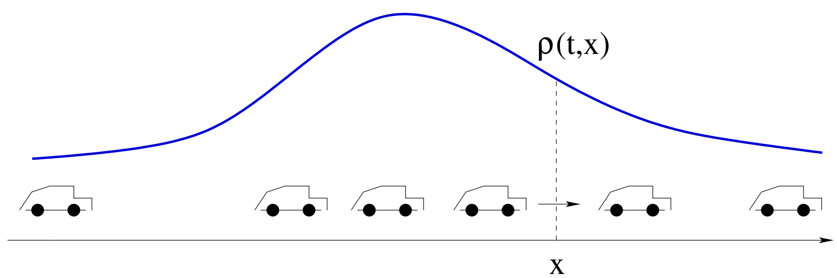

Figure 1: Modeling traffic flow by a PDE. Here ρ is the density of cars, whice the product $\rho \cdot v(\rho)$ gives the flux of cars.

To describe the evolution of traffic on a network of roads, one also needs to model behavior at intersections. In addition to a conservation law of the form (1) describing the traffic density on each road, one needs a set of equations relating the density on different roads, near the intersection. These equations need to take into account

- The conservation of the total number of cars (in all, there are as many cars arriving to the intersection as cars leaving the intersection).

- Drivers' preferences (i.e., assign the percentage of drivers arriving from a given road who will turn left or right at the intersection)

- Priority relations among incoming roads (assuming there is a crossroad light, among all roads leading to the intersection one needs to specify which one gets green light for a longer time).

An important feature of network models is the possibility of queues forming at an intersection and propagating backwards along one or more of the incoming roads.

At present, the mathematical theory for the conservation equation (1) is well established, including the existence, uniqueness, and qualitative properties of its solutions [2, 15, 22]. However, various issues remain still open for conservation laws on networks. For a discussion of mathematical models of traffic flow near an intersection, see [3, 12, 17]. The counterexamples in [11] point uot several difficulties in providing a robust model of traffic flow at intersections. These difficulties have recently been resolved in [7] using an intersection model with buffers, where uniqueness and continuous dependence of solutions can be established for general bounded, measurable initial data.

Relations between models with and without buffers have been explored in [9]. Namely, as the size of the buffer approaches zero, the solution of the Riemann problem admits a well defined limit, described by a specific Riemann solver.

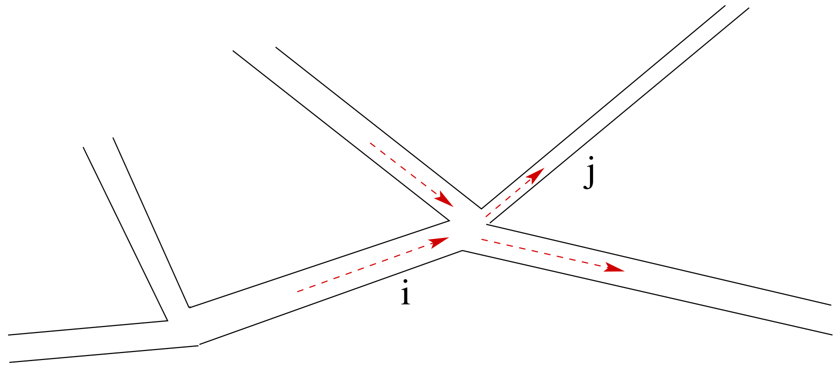

Figure 2: A road intersection.

2 Global optima and Nash equilibria

One can study vehicle flow also from a different point of view, regarding traffic patterns as the outcome of the decision problem. We assume that each individual driver has a cost $\varphi(\tau^d)$ for early departure and an additional cost $\psi(\tau^a)$ for late arrival. On a general network of roads, the arrival time τ^a is determined by (i) the departure time τ^d, (ii) the route taken to reach destination, and (iii) the overall traffic pattern, which of course depends globally on the decisions of all other drivers. The objective of minimizing the total cost $\varphi(\tau^d) + \varphi(\tau^a)$ leads to two distinct mathematical problems.

(P1) - Global Optimization Problem. Find departure times and routes to destinations in order to minimize the sum of all costs to all drivers.

(P2) - Nash Equilibrium Problem. Find departure times and routes to destinations in such a way that no driver can lower his own cost $\varphi(\tau^d) + \varphi(\tau^a)$ by changing his departure time or switching to a different route.

Note that (P1) is relevant in the case of a central planner who can decide the departure time and the route of every car. On the other hand, (P2) models the more realistic situation where each driver is free to choose his own departure time and route, in order to minimize his own personal cost. The existence, uniqueness, and characterization of solutions to the above problems has been recently studied in the case of a single road [4]. In the case of several groups of drivers on a network of roads, existence of a global optimum and a Nash equilibrium has been proved in [5], under very simplified assumptions on the dynamics at intersections. In connection with the intersection model with buffers introduced in [7], a general existence theorem appears in [8]. This is a more realistic model, which also accounts for the backward propagation of queues along roads leading to a crowded intersection.

3 Some open problems

Future directions include new ways for controlling and optimizing traffic flow, with the eventual goal of eliminating stop-and-go waves, and avoiding time lost in queues. For example, one could impose time-dependent toll fees to discourage driving at peak hours, regulate the access to highways, or automatically adjust the speed of cars, depending on local and global traffic conditions. Some more specific mathematical problems are discussed below.

1 - Necessary conditions for global optimality. In [4], necessary conditions were derived for a globally optimal solution, for a single group of drivers on a single road. Can one derive similar necessary conditions, characterizing the globally optimal solution constructed in [5] for several groups of drivers on a network of roads? Can one apply these conditions and describe the optimal departure strategies for a simple network, say with one or two intersections?

2 - The Riemann Solver generated by a crosslight. Consider an intersection with two or more incoming roads, regulated by a crosslight. Let $\theta_i \in [0, 1]$ be the fraction of time during which drivers from the i-th incoming road get green light, with $\sum_i \theta_i = 1$. If the period of the crosslight approaches zero (switching on and off faster and faster) what is the behavior of the limiting solutions? For some partial results in this direction, see [19]. Note that, by a rescaling of space and time, studying this limiting behavior is equivalent to keeping the frequency constant and looking at the traffic patterns far away from the intersection.

3 - Avoiding stuck traffic. The existence of Nash equilibria, proved in [8] on a network of roads, relies on the assumption that traffic never gets completely stuck. In other words, every driver reaches destination within finite time. As shown in [13], this is not always the case. Can one determine sufficient conditions that guarantee that traffic never gets stuck?

4 - Dynamic stability of Nash equilibria. Assume that, day after day, each driver can change his own departure time seeking to lower the sum of his departure and arrival cost. As a result, the distribution of departures as well as the overall traffic pattern will change each day. This leads to a dynamical system on the space of all departure distributions, having Nash equilibria as steady states. It is natural to expect that, after several days, the departure distribution should converge to a Nash equilibrium. To study this problem, for a single group of drivers traveling on a single road, two specific models were introduced in [6]. Surprisingly, numerical simulations suggest that the unique Nash equilibrium is unstable, while the orbits approach a chaotic attractor. No theoretical analysis has yet proved or disproved this conjecture.

5 - Nash equilibria with partial information. The existing results on the existence of Nash equilibria assume that every driver has (i) complete information on the system, including the number of other drivers and their cost functions, and (ii) the mathematical ability to process this information, computing the Nash equilibrium. Of course, this is an idealization. In a more realistic model, the only information available to drivers may be the set of times and locations (t, x) where the traffic is congested. Some drivers will simply disregard this information. Others will take it into account, but their route and departure time will be optimal for a much simpler model, not necessarily based on conservation laws. Can one introduce a concept of Nash equilibrium corresponding to this more realistic situation?

6 - Optimizing flow at intersections. In all previous models of traffic flow on a network, the Riemann Solvers that determine the dynamics of flow at an intersection are assumed to be a priori given, constant in time. However, one could regard these Riemann Solvers as control parameters, that can be modified in time (possibly depending on the instantaneous traffic conditions on the entire network), in order to improve overall traffic flow. In this direction, can one define specific optimization problems, and find optimal (or near-optimal) feedback solutions?

Note: optimization problems for conservation laws have been studied by various authors. See for example [10, 18]. The main difficulty in deriving necessary conditions for optimality is the possible emergence of shocks in the solution. As a consequence, the flow is no longer differentiable w.r.t. initial data, in the usual sense. Different notions of "shift differentiability" must be introduced.

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