Research Notes on Noncooperative Games and Applications

(*Alberto Bressan, July 2016*)

A basic problem in **optimization theory** is to find the maximum value of a function ϕ over a set X:

$$\max_{x\in X}\ \phi(x)\,. \tag{0.1}$$

One can think of X as the set of possible choices available to an individual, while $\phi(x)$ is his corresponding payoff.

Game theory, on the other hand, is concerned with the more complex situation where two or more individuals, or "players" are present [18]. Each player can choose among his set of available options and seeks to maximize his own payoff. For simplicity, consider the case of two players.

$$\begin{array}{l}\text{Player 1 chooses a strategy } x_1\in X_1 \text{ and seeks to maximize } \phi_1(x_1,x_2).\\ \text{Player 2 chooses a strategy } x_2\in X_2 \text{ and seeks to maximize } \phi_2(x_1,x_2).\end{array} \tag{0.2}$$

Here the catch is that the payoff of each player also depends on the actions of the other player. This is indeed what happens in most economics models, where the profit of one agent also depends on the decisions of the other agents.

In contrast with (0.1), it is clear that the problem (0.2) does not admit an "optimal" solution. Indeed, in general it will not be possible to find a couple $(\bar x_1,\bar x_2)\in X_1\times X_2$ which at the same time maximizes the payoff of the first player and of the second player, so that

$$\phi_1(\bar x_1,\bar x_2) ~=~ \max_{x_1,x_2}\ \phi_1(x_1,x_2)\,,\qquad\qquad \phi_2(\bar x_1,\bar x_2) ~=~ \max_{x_1,x_2}\ \phi_2(x_1,x_2)\,.$$

For this reason, various alternative concepts of solutions have been proposed in the literature. These can be relevant in different situations, depending on the information available to the players and their ability to cooperate.

For example, if the players have no means to talk to each other and do not cooperate, then an appropriate concept of solution is the **Nash equilibrium** [16], defined as a fixed point of the best reply map. In other words, (x_1^*,x_2^*) is a Nash equilibrium if

(i) the value $x_1^*\in X_1$ is the best choice for the first player, in reply to the strategy x_2^* adopted by the second player. Namely

$$\phi_1(x_1^*,x_2^*) ~=~ \max_{x_1\in X_1}\ \phi_1(x_1,x_2^*),$$

(ii) the value $x_2^*\in X_2$ is the best choice for the second player, in reply to the strategy x_1^* adopted by the first player. Namely

$$\phi_2(x_1^*,x_2^*) ~=~ \max_{x_2\in X_2}\ \phi_2(x_1^*,x_2).$$

A different solution concept is relevant when one of the players takes a leading role, and announces his strategy in advance. This leads to the concept of **Stackelberg equilibrium**.

The situation modeled by (0.2) represents a **static game**, i.e. a "one-shot" game. Each player makes one single choice $x_i \in X_i$, and this completely determines the payoffs. In other relevant situations, the game takes place not instantaneously but over a whole interval of time. This leads to the study of **dynamic games**.

We recall that, in the standard model of **control theory** [8, 15], the state of a system is described by a variable $x \in \mathbb{R}^n$. This state evolves in time according to an ODE

$$\frac{d}{dt}x(t) \;=\; f(t, x(t),\, u(t)) \qquad\qquad t \in [0,T]\,. \tag{0.3}$$

Here $t \mapsto u(t) \in U$ is the *control function*, ranging within a set U of admissible control values.

Given an initial condition $x(0) = x_0$, a basic problem in optimal control is to find a control function $u(\cdot)$ which maximizes the payoff

$$J \;=\; \psi(x(T)) - \int_0^T L(t, x(t),\, u(t))\, dt\,.$$

Here ψ is a *terminal payoff*, while L accounts for a *running cost*.

Differential games provide a natural extension of this model to the case where two or more players are present, each with his own payoff [1, 2, 14]. In the case of two players, one thus considers a system whose state $x \in \mathbb{R}^n$ evolves according to the ODE

$$\frac{d}{dt}x(t) \;=\; f(t, x(t),\, u_1(t),\, u_2(t)) \qquad\qquad t \in [0,T]\,. \tag{0.4}$$

Here $t \mapsto u_i(t) \in U_i$, $i = 1, 2$, are the *control functions* implemented by the two players.

In the **finite horizon** problem, for a given initial condition, the goal of the i-th player is

$$\text{maximize:} \qquad J_i \;=\; \psi_i(x(T)) - \int_0^T L_i(t, x(t),\, u_1(t),\, u_2(t))\, dt\,. \tag{0.5}$$

Here $\psi_i(x)$ is the payoff achieved at the terminal time T, while L_i is a running cost.

In alternative, one can consider the **infinite horizon** problem, where the game goes on for all times $t > 0$, and each player seeks to minimize a running cost, exponentially discounted in time:

$$\text{minimize:} \qquad J_i \;=\; \int_0^{+\infty} e^{-\gamma t}\, L_i(t, x(t),\, u_1(t),\, u_2(t))\, dt\,. \tag{0.6}$$

As in the case of one-shot games, various concepts of solution (Nash, Stackelberg, ...) can be considered. In addition, different types of strategies are available to players, depending on the amount of information they have. Namely, one may consider cases where

- The state x of the system is known only at the initial time $t = 0$.
- The players can observe the current state of the system $x(t)$ at each time $t \in [0,T]$.

- Instead of the deterministic ODE (0.4), the evolution of the system in is governed by stochastic differential equation. The initial state $x(0)$ is a random variable, with a given probability distribution.

- The functions ψ_i, L_i, determining the payoffs for each player, are not precisely known.

For example, agents buying and selling in a stock market have only partial information about the value of their assets, and hence on their eventual payoff.

If players can observe the current state of the system, then they can adopt *feedback strategies* $u_i = u_i(t, x)$. In other cases, they can only implement *open-loop* strategies $u_i = u_i(t)$ depending exclusively on time.

This leads to a rich variety of mathematical models, which largely remain yet to be studied. An interesting (and difficult!) problem arises when different players have access to different amount of information. In this case, the better informed agent seeks to make the most of his advantage, while less informed competitors will try to gain information by carefully observing the actions of the leading player.

An important class of differential games, currently receiving much attention, are those involving infinitely many players. In this case, no player can single-handedly affect the state $x(t)$ of system. The time evolution of the system is only determined by the average combined actions of all players.

Our present research on game theory and applications is presently focused in two main directions:

- General theory of non-cooperative differential games.

- Analysis of specific models arising in economics and finance.

1 General theory of non-cooperative differential games

To understand whether the non-cooperative game described at (0.4)-(0.5) has equilibrium solutions in feedback form, a basic approach calls for the study of the **value functions** V_i. Here $V_i(\tau, \bar{x})$ is the expected payoff for the i-th player, if the game were to start at time τ from the initial state $\bar{x}$.

Under suitable regularity conditions, it can be shown that these functions V_i satisfy a system of Hamilton-Jacobi equations. By studying these PDEs, one can understand whether the game has a solution, and how to compute the optimal strategies for the various players.

In the case of a game in finite time horizon (0.4)-(0.5), the system of H-J equations takes the form

$$\begin{cases} \partial_t V_1 &= H^{(1)}(x, \nabla V_1, \nabla V_2)\,, \\ \partial_t V_2 &= H^{(2)}(x, \nabla V_1, \nabla V_2)\,, \end{cases} \qquad \begin{cases} V_1(T, x) &= \psi_1(x), \\ V_2(T, x) &= \psi_2(x). \end{cases} \tag{1.7}$$

This leads to a backward Cauchy problem (with terminal conditions at $t = T$), for a highly nonlinear first order system of PDEs. The solution $(V_1(t,x), V_2(t,x))$ should be found for $t \in [0,T]$, $x \in \mathbb{R}^n$.

On the other hand, the game in infinite time horizon (0.4)-(0.6) leads to a system of the form

$$\begin{cases} \gamma V_1 &= H^{(1)}(x, \nabla V_1, \nabla V_2)\,, \\ \\ \gamma V_2 &= H^{(2)}(x, \nabla V_1, \nabla V_2)\,, \end{cases} \tag{1.8}$$

Notice that this is not a Cauchy problem. The solution $(V_1(x), V_2(x))$ should be found for $x \in \mathbb{R}^n$.

In general, the equations (1.7) or (1.8) are not covered by the standard theory of PDEs, and are very difficult to solve. In one space dimension, solutions to (1.7) with small initial data have been constructed in [9]. On the other hand, the recent analysis in [2, 10] shows that, in dimension $n \geq 2$, the Cauchy problem (1.7) is *ill posed*: solutions either do not exist, or are very sensitive to small changes in the data [10, 2]. In connection with Stackelberg equilibria, these issues have been recently studied also in [11].

In the special case where the system has linear dynamics, for example

$$\dot{x} \;=\; Ax + B_1 u_1 + B_2 u_2,$$

and the cost (or payoff) functions are quadratic polynomials w.r.t. the variables x, u_1, u_2, an explicit solution can be found, where the value functions V_i are quadratic polynomials of the variable $x = (x_1, \ldots, x_n)$.

Given a general nonlinear game, by a Taylor expansion one can (at least locally) approximate the dynamics by a linear one, and the cost function by a quadratic one. A natural question is whether the explicit solution found for the linear-quadratic approximation is close to the solution of the original nonlinear game.

For the problem in finite time horizon, the results in [10, 2] show that the answer is largely negative. For games with infinite time horizon, some positive results have been proved in [**?**] in the case of one space dimension. The multidimensional case, where $x \in \mathbb{R}^n$ with $n \geq 2$, has not yet been investigated.

2 A model of the limit order book in a stock market

Among several applications of game theory to finance, we consider the competitive bidding for a random incoming order, in a stock market. In this specific model, we assume that an external buyer asks for a random amount $X > 0$ of a certain asset (say, a particular stock traded on the stock market). This external agent will buy the amount X at the lowest available price, as long as this price does not exceed a given upper bound $\overline{P}$. One or more sellers offer various quantities of this asset at different prices, competing to fulfill the incoming order, whose size is not known a priori.

Having observed the prices asked by his competitors, each seller must determine an optimal strategy, maximizing his expected payoff. Of course, when other sellers are present, asking a higher price for an asset reduces the probability of selling it.

This model was introduced in [3] and further studied in [4, 12, 13]. Sellers may differ from each other in various respects:

- Each agent assigns a different probability distribution to the random variable X, based on his own beliefs. An optimistic seller expects a large incoming order, which will fill most of the outstanding bids. A pessimistic seller will expect a small order, filling only the lowest priced bids. In the following, we denote by

$$\psi_i(s) \;=\; Prob.\{X > s\} \tag{2.9}$$

  the probability distribution assigned by the i-th seller to the random variable X.

- The i-th agent has a total amount q_i of assets to offer for sale, and assigns a different fundamental value p_i to these assets. By selling a unit amount at price p, he would thus achieves a profit $p - p_i$.

For this game-theoretical model it is natural to seek a Nash equilibrium, where each player optimizes his own expected payoff, given the strategies adopted by all other players. Three natural questions arise:

- Does a Nash equilibrium exist?
- Is it unique?
- How can it be computed?

The results proved in [3, 4] show that, for the existence of a Nash equilibrium, the crucial assumption is that the probability distributions ψ_i be log-convex, i.e.

$$(\ln \psi_i(s))'' \;\geq\; 0 \quad \text{for all } s > 0\,.$$

In several cases, one can show that this equilibrium solution is unique and can be determined by solving a two-point boundary value problem for a system of ODEs with discontinuous right hand side [3, 12].

We remark that all the above models are 'static", in the sense that they are concerned with a single incoming order.

As a possible further development, one may study how the limit order book evolves in time, given a stream of incoming orders. A particularly interesting issue is how to quantify the advantage of fast w.r.t. slow traders. In other words, assume that some traders can change their strategy more quickly than others, in response to incoming information (for example, new buying orders).

- Can one quantify the difference in the expected profit of fast compared with slow traders?

- What should be the best reply for the slow traders?

3 Game theoretical models of debt and bankruptcy

Consider a borrower who need to repay a debt. If the interest rate α charged on the loan is constant in time, the total amount of debt $x(t)$ varies in time according to the ODE

$$\dot{x}(t) \;=\; \alpha x(t) - u(t) \tag{3.10}$$

where $u(t)$ is the rate at which repayments are made. We regard x as the state of the system, while the function $u(\cdot)$ is the control. This leads to a standard problem of optimal control in infinite time horizon:

$$\text{minimize:} \qquad \int_0^{+\infty} e^{-rt} L(u(t))\, dt\,, \tag{3.11}$$

subject to the dynamics (3.10), and some constraint on the maximum value of the debt, say

$$x(t) \;\leq\; x^{max}$$

(otherwise the trivial strategy $u(t) \equiv 0$ would be optimal). Notice that γ is a exponential discount rate, while the function $L(u)$ describes a running cost.

If there is no risk of default, the lenders's investment is secure, and under perfect competition they will charge and interest rate equal to the discount rate: $\alpha = \gamma$. However, assume that there is a positive probability that in the future the debtor may go bankrupt. In this case, the lenders will charge a higher interest rate, to make up for the expected loss of part of their capital if bankruptcy occurs. The evolution equation (3.10) is thus replaced by

$$\dot{x}(t) \;=\; \alpha(t)x(t) - u(t), \tag{3.12}$$

where $\alpha(t)$ now depends on the bankruptcy risk at all future times $\tau > t$.

It is natural to assume that, at any given time, the instantaneous bankruptcy risk depends on the size of the debt itself, reflecting lack of confidence by investors. This leads to a highly non-standard problem of optimal control, since the time derivative $\dot{x}(t)$ depends not only on the control $u(t)$, but also on the entire future evolution of the debt.

For this models of this type, several questions arise:

- Given an initial condition $x(0) = x_0$ and an open-loop control $t \mapsto u(t)$, does the ODE (3.12) have a (unique) solution?
- Is there an optimal solution $t \mapsto (x(t), u(t))$, which minimizes the total cost (3.11) to the borrower?
- Can one determine this optimal solution by deriving a set of necessary conditions for optimality?
- Does there exist an optimal strategy in feedback form: $u = u(x)$?

Some results in this direction have been obtained in [7, 5].

An interesting new model of debt and bankruptcy, where the uncertainty comes not from the bankruptcy risk but from the random evolution of the borrower's income, has been recently introduced in [17].

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