Solution Key

MATH110 **Quiz4** **Section009**

1 (4 pts) Let

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ ax + b & \text{if } x > 1. \end{cases}$$

Find the values of a and b so that f is continuous and has a derivative at $x = 1$.

Solution. Since f is continuous at $x = 1$, the limit value of f also exists as $x \to 1$. This implies that $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$, namely,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (ax + b) = a + b,$$

and

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1^2 = 1.$$

Thus, we have $a + b = 1$. Also, since f has a derivative at $x = 1$,

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h}.$$

Evaluating the right slope and the left slope at $x = 1$ yields

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = 2 \cdot 1 = 2,$$

and

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = a.$$

Thus we get $a = 2$. Solving the system of the equations in terms of a and b gives $a = 2$ and $b = -1$. □

2 (3 pts) Find the slope and an equation of the tangent line to the graph of the function f at the specified point.

$$f(x) = x^4 - 3x^3 + 2x^2 - x + 1; (1, 0).$$

Solution. Let us find the derivative $f'(x)$:

$$f'(x) = 4x^{4-1} - 3 \cdot 3x^{3-1} + 2 \cdot 2x^{2-1} - 1 = 4x^3 - 9x^2 + 4x - 1.$$

The slope of the tangent line to the curve at the point $(1, 0)$ is $f'(1) = 4 \cdot 1^3 - 9 \cdot 1^2 + 4 \cdot 1 - 1 = -2$. Using the point-slope formula of the line, we have

$$y - 0 = -2(x - 1),$$

or $y = -2x + 2$. □

3(3 pts) Find the derivative of f and evaluate $f'(x)$ at the given value of x.

$$f(x) = \frac{x}{x^4 - 2x^2 - 1}; x = -1.$$

Solution. Let us find the derivative $f'(x)$:

$$\begin{aligned} f'(x) &= \frac{(x^4 - 2x^2 - 1)(1) - x(4x^3 - 4x)}{(x^4 - 2x^2 - 1)^2} \\ &= \frac{x^4 - 2x^2 - 1 - 4x^4 + 4x^2}{(x^4 - 2x^2 - 1)^2} \\ &= \frac{-3x^4 + 2x^2 - 1}{(x^4 - 2x^2 - 1)^2}. \end{aligned}$$

Evaluating $f'(1)$ gives

$$f'(1) = \frac{-3 \cdot 1^4 + 2 \cdot 1^2 - 1}{(1^4 - 2 \cdot 1^2 - 1)^2} = \frac{-2}{4} = -\frac{1}{2}.$$