Solution Key

MATH110 **Quiz4** **Section040**

1 (4 pts) Let

$$f(x) = \begin{cases} x & \text{if } x \le 1 \\ ax^2 + b & \text{if } x > 1. \end{cases}$$

Find the values of a and b so that f is continuous and has a derivative at $x = 1$.

Solution. Since f is continuous at $x = 1$, the limit value of f exists as $x \to 1$, say, $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x)$. Evaluating the right limit and the left limit yields

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (ax^2 + b) = a \cdot 1^2 + b = a + b,$$

and

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1.$$

Thus, we have $a+b = 1$. Also, since f has a derivative at $x = 1$, the following equation holds true:

$$\lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h}.$$

Differentiating f defined piecewisely, we get

$$f'(x) = \begin{cases} 1 & \text{if } x < 1 \\ 2ax & \text{if } x > 1. \end{cases}$$

We obtain the right slope and the left slope at $x = 1$:

$$\lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = 2a \cdot 1 = 2a,$$

and

$$\lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = 1.$$

Whence, $2a = 1$. Solving the system of equations gives $a = \dfrac{1}{2}$ and $b = \dfrac{1}{2}$.

2 (3 pts) Let $f(x) = 2x^2 + 1$.

(a) Find the point on the graph of f where the slope of the tangent line is equal to 12.

(b) Find the equation of the tangent line of part (a).

Solution. For (a). The derivative of f is $f'(x) = 4x$. Geometrically speaking, the derivative $f'(x)$ represent the slope of the tangent line to the curve. Solving $4x = 12$, we obtain $x = 3$. Plugging $x = 3$ in $f(x) = 2x^2 + 1$, we have the function value $f(3) = 19$ at $x = 3$. The point is $(3, 19)$.

For (b). The slope of the tangent line to the curve at the point $(3, 19)$ is 12. By the point-slope formula of the line, we can write out the equation of the tangent line under consideration:

$$y - 19 = 12(x - 3),$$

or simply $y = 12x - 17$.

3(3 pts) Find the derivative of f and evaluate $f'(x)$ at the given value of x.

$$f(x) = \frac{x}{x^4 - 2x^2 - 1}; x = -1.$$

Solution. Using the quotient rule of differentiation, we have

$$\begin{aligned} f'(x) &= \frac{(x^4 - 2x^2 - 1) \cdot 1 - x(4x^3 - 4x)}{(x^4 - 2x^2 - 1)^2} \\ &= \frac{x^4 - 2x^2 - 1 - 4x^4 + 4x^2}{(x^4 - 2x^2 - 1)^2} \\ &= \frac{-3x^4 + 2x^2 - 1}{(x^4 - 2x^2 - 1)^2} \end{aligned}$$

Plugging $x = -1$ in $f'(x)$ gives

$$f'(-1) = \frac{-3(-1)^4 + 2(-1)^2 - 1}{((-1)^4 - 2(-1)^2 - 1)^2} = \frac{-2}{4} = -\frac{1}{2}.$$