Tractions, balances, and boundary conditions for nonsimple materials

Eliot Fried
Department of Mechanical and Aerospace Engineering
Washington University in St. Louis
St. Louis, MO 63130-4899, USA

Abstract

In two monumental works, Toupin (1962, 1964) derived general balance equations and associated traction boundary conditions for an *elastic* body whose strain energy depends on *first and second gradients of the deformation*. Toupin's derivation is based on a virtual work principle asserting that the variation of the total elastic energy be equal to the virtual work exerted on the body by tractions and body forces. A central consequence of Toupin's work is the observation that Cauchy's hypothesis — that the surface traction at a point $\mathbf{x}$ on a surface $\mathcal{S}$ depend on $\mathcal{S}$ through its normal field at $\mathbf{x}$ — is not valid in a theory involving second gradients of the deformation, because in Toupin's theory the traction depends also on the curvature of $\mathcal{S}$ at $\mathbf{x}$.

Unfortunately, because it assumes that the material is elastic and the body is in equilibrium, Toupin's *derivation* of the balance equations and associated traction boundary conditions cannot be applied within a general dynamical framework that includes dissipation.

Our goal here is to derive Toupin's results within a framework that is *independent of constitutive equations.* To do so we use a *nonstandard* form of the principle of virtual power (Gurtin 2001). Conventional versions of this principle are formulated for the body B as a whole rather than for control volumes and as such generally involve particular boundary conditions applied to the boundary ∂B of B. Such formulations allow for a weak statement of the basic force balances and when combined with constitutive equations result in weak statements of the resulting boundary-value problems. *Here the principle of virtual power is used instead as a basic tool in determining the structure of the tractions and of the local force balances.* As such, conditions on ∂B play a role no different from those on the boundary of any control volume. Basic to this view is the premise, central to all of continuum mechanics, that any basic law for the body should hold also for all subregions of the body.

References

- Gurtin, M. E., 2001. A gradient theory of single-crystal viscoplasticity that accounts for geometrically necessary dislocations, *J. Mech. Phys. Solids* **84**, 809–819.
- Toupin, R. A., 1962. Elastic materials with couple stresses, *Arch. Ration. Mech. Anal.* **11**, 385–414.
- Toupin, R. A., 1964. Theory of elasticity with couple stress, *Arch. Ration. Mech. Anal.* **17**, 85–112.