Two-sided bounds on the convergence rate of two-level methods

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TWO-SIDED BOUNDS ON THE CONVERGENCE RATE OF TWO-LEVEL METHODS

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Abstract. In this paper we prove two-sided bounds on the convergence rate of a standard two-level subspace correction method. We then apply these estimates to show that a two-level method with pointwise smoother for variational problem in $H_0^{(\text{curl})}$ does not have optimal convergence rate. This result justifies the conclusion, observed numerically and reported in the literature, that a point relaxation as a smoother does not lead to an optimal multigrid method. In fact, we show that for such problems using a well conditioned smoother will always lead to a method that is not optimal.

1. Introduction

This paper is on deriving lower bounds for two-level methods for special class of nearly singular problems of the form

$$Au = (A_0 + A_1)u = f,$$

where $A_0$ is symmetric and positive semi-definite operator, and $A_1$ is symmetric and positive definite operator. We are interested in is when $A$ corresponds to discretization of a second order partial differential equation (PDE), or system of PDEs, the dimension of the kernel of $A_0$ grows with the size of the discrete problem and $A_1$ corresponds to a lower order term. Such situations arise in many applications. As examples, consider the finite element discretizations for $H(\text{grad})$, $H(\text{div})$ and $H^{(\text{curl})}$ systems as discussed in [11, 1, 2] and stable discretizations of nearly incompressible linear elasticity problem [19, 16]. The Augmented Lagrangian Method (see [10, 4, 9]), when applied to solve the indefinite systems arising from mixed finite element discretizations also results in problems in which the higher order part of the operator has kernel of large dimension.

For many problems, like (1.1), discretized by the finite element method, it is well known that multigrid methods with pointwise smoothers do not converge uniformly with respect to the mesh size. The construction of special smoothers and subspace splittings and the relevant uniform convergence results have been discussed in many works; see, for example, Ewing and Wang [5, 6], Vassilevski and Wang [18], Hiptmair [11], Arnold, Falk and Winther [2, 1], and Xu and Hiptmair [12]. For nearly incompressible elasticity and Stokes equations we point to Schöberl’s work [16].

Our goal in this paper is to show that for such problems when singular part $A_0$ of the operator $A$ has a kernel of large dimension, a well conditioned smoother results in a multigrid method, whose convergence rate deteriorates when the mesh size gets smaller, even for two-level methods. As indicated in a related work on subspace correction methods for nearly singular problems [13] this can be explained by considering the splitting $V = \sum_{k=1}^{J} V_k$. 

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defining a multigrid method and in particular, the kernel $\mathcal{N}(A_0)$, of the singular part of $A$ has to be decomposed in a way that is compatible with the decomposition of $V$.

The rest of the paper is organized as follows. We introduce notation and give some preliminary convergence results in §2. Next, in §3, we consider a general procedure for deriving lower bounds for two-level methods. In §4 we apply this abstract theory to a variational problem in $H_0(\text{curl})$ and prove a lower bound on the convergence rate of geometric multigrid with a pointwise smoother applied to this problem. Conclusions are drawn in §5.

2. TWO LEVEL METHOD AND ITS CONVERGENCE RATE

In this section we prove a two-level estimate, using a result for general subspace correction method [22].

2.1. General subspace correction method. Given a Hilbert space $V$ equipped with a positive definite bilinear form $a(\cdot, \cdot)$ and a a continuous linear form $f(\cdot)$, we consider the following variational problem: Find $u \in V$ such that

$$a(u, v) = f(v), \quad \forall v \in V.$$  

The inner product and the corresponding norm in $V$ are denoted with $(\cdot, \cdot)$ and $\| \cdot \|$. Since $a(\cdot, \cdot)$ is a symmetric and positive definite, it also introduces an inner product and a norm, which we denote with $(\cdot, \cdot)_a$ and $\| \cdot \|_a$.

The successive subspace correction algorithm is as follows.

Algorithm 2.1. Let $u^0 \in V$ be given initial guess.

for $\ell = 1, 2, \ldots$

$u_{\ell-1}^0 = u_{\ell-1}$

for $i = 1 : J$

Let $e_i \in V_i$ solve

$$a_i(e_i, v_i) = f(v_i) - a(u_{\ell-1}^{i-1}, v_i) \quad \forall v_i \in V_i \quad (2.1)$$

$$u_{\ell-1}^i = u_{\ell-1}^{i-1} + e_i$$

endfor

$u_\ell = u_J^{\ell-1}$

endfor

We assume that the equations (2.1) are uniquely solvable, that is that the bilinear forms $a_i(\cdot, \cdot)$ satisfy appropriate inf-sup conditions. We note that $a_i(\cdot, \cdot)$ are not required to be symmetric. To state the result from [22], which we will use in the derivation of a two-level estimate, we introduce $T_i : V \mapsto V_i$, whose action $T_i v$ is defined as the unique solution to:

$$a_i(T_i v, v_i) = a(v, v_i), \quad \forall v_i \in V_i \quad (2.2)$$

Relevant here will also be the adjoint (with respect to $a(\cdot, \cdot)$) of $T_i$, denoted with $T_i^*$ and the symmetrization of $T_i$, defined as $T_i = T_i^* + T_i^* T_i$, for $i = 1, \ldots, J$. Even though the general algorithm above can be defined for any decomposition of $V$ in subspaces, in what follows we restrict our considerations on the case when the spaces are nested, namely,

$$V_1 \subset V_2 \subset \cdots \subset V_J = V.$$  

In the special case when the subspace equation (2.1) is solved exactly, we use the notation $P_i$ instead of $T_i$

$$P_i = T_i \text{ if } a_i(\cdot, \cdot) = a(\cdot, \cdot).$$
The action of $P_i$ is defined by solving the following variational problem:

$$a(P_i v, v_i) = a(v, v_i), \quad v \in V, v_i \in V_i.$$  

Note that $P_i$ is an idempotent, that is $P_i^2 = P_i$, and also that $P_i$ is an orthogonal projection in $(\cdot, \cdot)_a$ product. Following classical settings in multigrid theory [3, 20], we write

$$T_i = R_i A_i P_i,$$

where $R_i : V_i \mapsto V_i$ is an isomorphism, defining the action of the smoother, $A_i$ are defined as

$$(A_i v_i, w_i) = a(v_i, w_i).$$

With this notation in hand, we have

$$T_i^\ell = R_i^\ell A_i P_i, \quad \bar T_i = \bar R_i A_i P_i = (R_i + R_i^\ell + R_i A_i R_i) A_i P_i.$$  

Here we have denoted with $R_i^\ell$ the adjoint of $R_i$, $i = 1, \ldots, J$ with respect to the usual inner product $(\cdot, \cdot)$ on $V$. The error propagation operator, whose norm determines the convergence rate of the method is then defined by looking at the error on two consecutive iterations.

$$u - u_i^{\ell-1} = (I - T_i)(u - u_i^{\ell-1}).$$

Clearly we have that

$$u - u^\ell = E(u - u^{\ell-1}) = \ldots = E^\ell(u - u^0)$$

where

$$E = (I - T_j)(I - T_{j-1}) \ldots (I - T_1)$$

The following general result (Theorem 4.2 [22]) will be instrumental in our analysis.

**Theorem 2.2.** Suppose that $V = \sum_{i=1}^J V_i$. Assume also that for all $i = 1, \ldots, J$, $T_i : V_i \mapsto V_i$ are isomorphic and that there exists constant $\omega \in (0, 2)$

$$\|T_i v\|_a^2 \leq \omega(T_i v, v)_a, \quad v \in V.$$

Then the following relation holds,

$$\|E\|_a^2 \equiv \|(I - T_J)(I - T_{J-1}) \ldots (I - T_1)\|^2 = \frac{c_0}{1 + c_0}$$

where

$$c_0 = \sup_{\|v\|_1 = 1} \inf_{\sum_{i=1}^J v_i = v} \sum_{i=1}^J (T_i T_i^{-1} T_i^* w_i, w_i)_a < \infty \text{ with } w_i = \sum_{j=i}^J v_j - T_i^{-1} v_i.$$  

We move on to considering the special case, when $J = 2$.

2.2. **Two-level method.** Since we aim to analyze a two-level method, we set

$$J = 2, T_H = T_1 = P_1 = P_H, \quad V_1 \equiv V_H \subset V_2 =: V.$$  

Here we have used a notation standard in finite element analysis, by introducing $V$ and $V_H$ and using these below instead of $V_1$ and $V_2$. We denote accordingly $P_H = T_H$, that is, we solve exactly the coarse grid problem and then $T_H$ is a projection. We also set $T = T_2$, and define $A : V \mapsto V$ as

$$(Au, v) = a(u, v), \quad \forall u \in V, \forall v \in V.$$
In accordance with the notation introduced in the previous paragraph, we have
\begin{equation}
T = RA, \quad T = RA, \quad T^* = R^t A,
\end{equation}
where \( R : V \rightarrow V \) is an isomorphism. With this notation in hand, we have
\[ w_1 = v_2 := v - v_H, \quad w_2 = (I - T^{-1})(v - v_H). \]
Using the following identity (for a proof, see [22], Lemma 4.9),
\begin{equation}
(T_i^{*+1} - I) \bar{T}_i^{-1} T_i^* (T_i^{-1} - I) = \bar{T}_i^{-1} - I,
\end{equation}
we obtain that
\[ c(v, v_H) = \sum_{i=1}^{J} (\bar{T}_i^{-1} T_i^* v_i, v_i) = \sum_{i=1}^{J} (T_i^{-1} T_i^* (I - T^{-1}) v_i, (I - T^{-1}) v_i) + \|P_H v_2\|^2_a \]
\[ = ((I - T^{-1}) v_2, v_2)_a + \|P_H v_2\|^2_a \]
\[ = ((I - (I + P_H) v) v_2, v_2)_a \]
\[ = ((I - (I - P_H)) (v - v_H), v - v_H)_a \]
\[ = (T^{-1} (v - v_H), v - v_H)_a - \|(I - P_H) v\|^2_a. \]
Using equation (2.7) we obtain:
\begin{equation}
c(v, v_H) = \|v - v_H\|^2_{R^{-1}} - \|(I - P_H) v\|^2_a.
\end{equation}
In order to apply Theorem 2.2 we observe that:
\[ c_0 = \sup_v \frac{1}{\|v\|_a} \inf_{v_H \in V_H} c(v, v_H). \]
Note that only the first term in (2.9) depends on \( v_H \), hence the infimum is attained for \( v_H = \bar{\Pi}_H v \), where \( \bar{\Pi} \) is an \( (\cdot, \cdot)_{R^{-1}} \)-orthogonal projection on \( V_H \). We also have the following implication:
\[ I - \bar{\Pi}_H = (I - \bar{\Pi}_H)(I - P_H), \quad \Rightarrow \quad c_0 = \sup_{v \in V_H} \| (I - \bar{\Pi}_H) v \|^2_{R^{-1}} \sup_{v \in V} \| (I - \bar{\Pi}_H) v \|^2_{R^{-1}}. \]
Hence we have just proved the following result (see also Theorem 4.1 in [8]).

**Lemma 2.3.** The following relation gives the convergence rate of a two-level method:
\begin{equation}
\|E\|^2_a = 1 - \frac{1}{K}, \quad \text{where} \quad K = \sup_{v \in V} \| (I - \bar{\Pi}_H) v \|^2_{R^{-1}}.
\end{equation}

3. **General two-sided estimates and inequalities related to Gauss-Seidel smoother**

We restrict further our considerations to the the case, when \( V \) is finite dimensional and is spanned by \( \{\phi_i\}_{i=1}^n \) and define the action of \( A \) via its action on the basis:
\[ (A \phi_i, \phi_j) := a(\phi_i, \phi_j). \]
We will also need the diagonal of \( A \),
\[ (D \phi_i, \phi_i) := a(\phi_i, \phi_i), \quad \text{and} \quad (D \phi_i, \phi_j) := 0, \quad i \neq j. \]
and the strict lower triangle of $A$,

$$(Lu, v) := -a(\phi_i, \phi_j), \quad i > j \quad \text{and} \quad (Lu, v) := 0, \quad i \leq j.$$  

Further, the “maximum number of non-zeroes per row” in $A$ is denoted with $n_z$ and defined as follows:

$$n_z := \max_i \#\{j, \text{ such that, } a(\phi_i, \phi_j) \neq 0\}.$$  

Here $\#\{\cdot\}$ stays for cardinality (number of elements) in a finite set.

3.1. **General two-sided estimates.** We now give two-sided estimates of the energy norm of the error transfer operator $E$ for the two-level method. As we have already pointed out,

$$E = (I - P_H)(I - RA),$$  

where $P_H : V \mapsto V_H$ is the $a(\cdot, \cdot)$ orthogonal projection on $V_H$. The exact convergence rate for a two-level method is given by (see also [8]). The equation 2.10 can be rewritten in an equivalent form:

$$\|E\|_2^2 = 1 - \frac{1}{K} = 1 - \frac{1}{\sup_{v \in V_h} k(v)},$$  

where

$$k(v) = \frac{\|v\|_2^2}{\|v\|_a^2}.$$  

We remind here that $\Pi_H$ was defined as the $(R^{-1}, \cdot)$ orthogonal projection on $V_H$. We now apply the results from the previous section, together with the identity for the convergence rate of the two-level method, to obtain two-sided estimate on first on $k(v)$ for all $v \in V$ then on $K$ and $\|E\|_2^2$. The estimates that we give, depend on the eigenvalues $\lambda_{\min}(R^{-1})$ and $\lambda_{\max}(R^{-1})$, defined as follows:

$$\lambda_{\min}(R^{-1}) = \inf_{v \in V_h} \frac{(\bar{R}^{-1}v, v)}{\|v\|_2^2}, \quad \lambda_{\max}(R^{-1}) = \sup_{v \in V_h} \frac{(\bar{R}^{-1}v, v)}{\|v\|_2^2}.$$  

We also define the $(\cdot, \cdot)$ orthogonal projection $Q_H : V \mapsto V_H$ by

$$(Q_H u, v) = (u, v) \quad \forall u \in V, \forall v \in V.$$  

We have the following theorem, which is the main abstract result in this paper.

**Theorem 3.1.** Let $0 \neq v \in V$ be arbitrary. Then the following estimates hold:

$$\lambda_{\min}(R^{-1}) \frac{\|(I - Q_H)v\|_2^2}{\|v\|_a^2} \leq k(v) \leq \lambda_{\max}(R^{-1}) \frac{\|(I - Q_H)v\|_2^2}{\|v\|_a^2}. \tag{3.1}$$

**Proof.** We first note that $(R^{-1}, \cdot)$ is the closest element to $v \in V$ in $\|\cdot\|_{R^{-1}}$ norm, and $Q_H v$ is the closest element to $v \in V$ in $\|\cdot\|$ norm. Hence we have:

$$\|(I - Q_H)v\|_2^2 \leq \|(I - \bar{\Pi}_H)v\|_2^2 \leq \frac{1}{\lambda_{\min}(R^{-1})} \|(I - \bar{\Pi}_H)v\|_{R^{-1}}^2.$$  

Thus,

$$\lambda_{\min}(R^{-1}) \frac{\|(I - Q_H)v\|_2^2}{\|v\|_a^2} \leq k(v).$$
On the other hand,
\[ \|(I - \Pi_H)v\|_{R^{-1}} \leq \|(I - Q_H)v\|_{R^{-1}} \leq \lambda_{\text{max}}(\bar{R}^{-1})\|(I - Q_H)v\|. \]

Hence,
\[ k(v) \leq \lambda_{\text{max}}(\bar{R}^{-1})\frac{\|(I - Q_H)v\|^2}{\|v\|^2}. \]

The proof is then easily completed by combining these estimates. \hfill \Box

We point out here, that if \( \bar{R} \) is well conditioned, then the estimates above give tight and reliable bounds on the convergence rate, and in fact they justify the use of the measures of the quality of coarse spaces introduced in [7].

3.2. Inequalities for Gauss-Seidel smoother. We first show that when \( \bar{R} \) corresponds to symmetrization of point Gauss-Seidel smoother, then \( \bar{R}^{-1} \) is spectrally equivalent to the “diagonal” of \( A \), that is \( \bar{R} \approx D \). More general choices of smoothers are considered in [17] and Xu [21]. The proof that we present here uses that \( R \) is defined by forward Gauss-Seidel method, that is \( R = (D - L)^{-1} \). But similar estimates hold for all pointwise relaxations.

Lemma 3.2. Let \( 0 \neq v \in V \). Then the following inequalities hold:
\[ (\bar{R}^{-1}v, v) \leq C(n_z)(Dv, v), \quad (\bar{R}^{-1}v, v) \geq \frac{1}{4}(Dv, v), \]
where the constant \( C \) depends on \( n_z \).

Proof. We first observe that
\[ \bar{R}^{-1} = A + LD^{-1}L^t = (D - L)D^{-1}(D - L)^t \]
and that \( D = R^{-1} + R^{-t} - A \). The first inequality then follows from the Schwarz inequality and the fact that for a fixed \( i \), there are finite number of \( j \), (bounded by \( n_z \)) and such that \( (A\phi_i, \phi_j) \) could be nonzero.

To prove the second inequality, we first use the fact that \( D \) and \( A \) are SPD, to get that for \( v \in V_h \):
\[ ((D - L)v, v) = \frac{1}{2}((A + D)v, v) \geq \frac{1}{2}(Dv, v). \]

Taking \( v = (D - L)^{-1}w \) we have for all \( w \in V_h \):
\[ \frac{1}{2}(D(D - L)^{-1}w, (D - L)^{-1}w) \leq ((D - L)^{-1}w, w) = (D(D - L)^{-1}w, D^{-1}w) \]

Since \( D \) is SPD we apply the Schwarz inequality to the right side of the above inequality and use that \( \bar{R} = (D - L)^{-1}D(D - L)^{-t} \), to get that
\[ \frac{1}{2}(\bar{R}w, w) \leq (\bar{R}w, w)^{1/2}(D^{-1}w, w)^{1/2}. \]

This can be rewritten in an equivalent form as
\[ (\bar{R}w, w) \leq 4(D^{-1}w, w), \]
and finally
\[ (\bar{R}^{-1}v, v) \geq \frac{1}{4}(Dv, v), \quad \forall v \in V_h. \]

\hfill \Box
4. Example: Variational Problem in $H_0(\text{curl})$ and Its Discretization

Let $\Omega$ be a bounded polyhedral domain in $\mathbb{R}^3$. We consider two-level method applied to the discretization of the following variational problem: Find $u \in H_0(\text{curl})$, such that

$$a(u, v) = (\text{curl} u, \text{curl} v) + (u, v) = (f, v), \quad \forall v \in H_0(\text{curl}) ,$$

where $(\cdot, \cdot)$ is the standard $L_2(\Omega)$ inner product, corresponding to the inner product on $V$ introduced in the abstract estimates in the previous sections. The space $H_0(\text{curl})$ is the subspace of $H(\text{curl})$ whose elements have zero tangential traces on the boundary. We discretize this problem using the lowest order Nedelec finite elements on a quasi-uniform partition of $\Omega$ with $d$-dimensional simplexes and denote the underlying finite element space with with $V_h$. The lowest order Nedelec finite element space $V_h$ consists of functions whose restrictions on every element are of the form: $a + (b \wedge x)$, for some given vectors $a \in \mathbb{R}^3$ and $b \in \mathbb{R}^3$. A basis in $V$ is given by the functions:

$$\varphi_e(x)|_T = \lambda_{i,T} \nabla \lambda_{j,T} - \lambda_{j,T} \nabla \lambda_{i,T}, \quad e = (i, j), \quad i < j.$$

where $\lambda_{k,T}$ is the $k$-th barycentric coordinate in the simplex $T \in T_h$. Accordingly, the degrees of freedom (the dual basis) for the Nedelec space are the integrals of the tangential components of the vector fields along the edges of the triangulation:

$$(4.3) \quad \sigma_e(u) = \int_e u \cdot \tau_e.$$ 

Since is no ambiguity in just denoting $e(\cdot)$ instead of $\sigma_e(\cdot)$ and it simplifies the notation a bit, we will use the former notation for the functional defined above. In such case, the elements of $V$ are in $H(\text{curl}, \Omega)$, since their tangential components on every edge of the triangulation are continuous. Thus for every vector field $u \in V_h$, we have

$$u(x) = \sum_e e(u) \varphi_e(x).$$

4.1. Assumptions and auxiliary results. We assume that the triangulation $T_h$ of $\Omega$ has been obtained by a successive regular refinement of a coarser triangulation $T_H$. We denote the corresponding finite element space on $T_H$ with $V_H$. We further assume that the refinement is done in such a way that there exist a vertex $x_0 \in T_h$, such that:

1. $x_0$ a middle point of a coarse grid edge $e_0$;
2. $x_0$ is neither on the boundary, nor a vertex of $T_H$;
3. The piece-wise linear continuous basis function $\varphi_0(x)$ corresponding to this vertex vanishes at all vertices of $T_H$.

In $\mathbb{R}^3$ dimensions, such assumption is indeed satisfied if $T_h$ is obtained by a usual refinement procedure. For example if each tetrahedron $\tau \in T_H$ is divided into 6 tetrahedrons, using the middle points of the edges of $\tau$ (four congruent tetrahedrons to $\tau$ and two in the interior of $\tau$). It is also satisfied if $T_h$ is obtained via other, more general refinement strategies. Same is true for the standard refinement in 2 spatial dimensions.

On $V$ now, we introduce the inner product

$$(u, v)_{0,h} := \sum_e e(u)e(v)\|\varphi_e\|^2.$$
Simple scaling argument shows that the corresponding norm satisfies the following equivalence relation.

\begin{equation}
\gamma_0^{-1} \| u \|^2 \leq \| u \|_{0,h}^2 = \sum_e \| e(u) \|^2 \| \varphi_e \|^2 \leq \gamma_1 \| u \|^2.
\end{equation}

Note that we also have

\begin{equation}
(Du, u) \approx h^{-2} \| u \|^2 \approx h^{-2} \| u \|_{0,h}^2.
\end{equation}

The estimates given in equation (4.5) can be obtained in a similar fashion as the one in (4.4), by mapping to the reference element and using the equivalence of norm on finite dimensional space (of dimension < 6).

4.2. **Nedelec projection and \( L_2 \) projection.** Let \( \Pi_H : V_h \mapsto V_H \) denote the canonical Nedelec interpolation operator [14, 15]. The following well known result holds for gradient vector fields.

**Lemma 4.1.** Consider a vertex \( x_0 \in T_h \) as in assumption (A1), Let \( \varphi_0(x) \) be the corresponding piecewise linear continuous basis function and \( v_0 = \nabla \varphi_0 \). Then \( \Pi_H v_0 = 0 \).

**Proof.** The result follows from the fact that the Nedelec projection and the standard linear interpolation satisfy:

\begin{equation}
\Pi_H v_0 = \Pi_H \nabla \varphi_0 = \nabla \phi_0, \quad \phi_0 = (\phi_0)_I,
\end{equation}

where \((\cdot)_I\) stays for the standard nodal interpolation by piece-wise linear continuous functions. Since \((\phi)_I = 0\), for all \( \phi \) that vanish at the vertices of \( T_H \), the right side of (4.6) vanishes and the proof is complete. \( \square \)

Next lemma is a stability result, which holds if the ratio \( H/h \) is bounded independently of \( h \).

**Lemma 4.2.** Then there exists constant \( \beta > 0 \) independent of \( h \), such that for all \( v_h \in V_h \) the following inequality holds:

\begin{equation}
\| \Pi_H v_h \|^2 + \| (I - \Pi_H) v_h \|^2 \leq \beta^{-1} \| v_h \|^2.
\end{equation}

**Proof.** It is obvious, that the statement of the lemma will follow if we prove the inequality:

\begin{equation}
\| \Pi_H v_h \|^2 \leq \beta_1 \| v_h \|^2,
\end{equation}

since by Schwarz inequality, we can then take for example \( \beta^{-1} = 2 + 3\beta_1 \).

To prove (4.8), we fix \( v_h \in V_H \) and set \( w_H = \Pi_H v_h \). We then have:

\[
w_H = \sum_{e_H \in E_H} e_H(v_h) \varphi_{e_H}(x),
\]

and hence for a fixed edge \( e_h \in \mathcal{E}_h \), the corresponding functional is

\[
e_h(w_H) = \sum_{e_H \in E_H} e_H(v_h)e_h(\varphi_{e_H}).
\]

Consider now a fixed \( e_h \in \mathcal{E}_h \). Let \( s(e_h) \) denote the set of edges from \( \mathcal{E}_H \) that intersect the support of \( \varphi_{e_H}(x) \), \( m(e_h) \) is the cardinality of \( s(e_h) \), and \( m_0 \) is the maximum \( m(e_h) \), that is,

\[
s(e_h) = \{ e_H \in \mathcal{E}_H \mid e_h(\varphi_{e_H}) \neq 0 \}, \quad m(e_h) = \#s(e_h), \quad m_0 = \max_{e_h \in \mathcal{E}_H} m(e_h).
\]
Below, for simplicity, we write $\sum_{e_H}, \sum_{e'_H}$ and $\sum_{e_h}$ instead of $\sum_{e_H \in E_H}, \sum_{e'_H \in E_H}$ and $\sum_{e_h \in E_h}$ respectively. For a fixed edge $e_h$ we have:

$$[e_h(w_H)]^2 = \left[\sum_{e_H} e_H(v_h)e_h(\varphi_{eH})\right]^2 = \left[\sum_{e_H \in s(e_h)} e_H(v_h)e_h(\varphi_{eH})\right]^2 \leq m(e_h) \sum_{e_H \in s(e_h)} [e_H(v_h)e_h(\varphi_{eH})]^2 \leq m_0 \sum_{e_H \in s(e_h)} [e_H(v_h)e_h(\varphi_{eH})]^2.$$.

Applying the Schwarz inequality again gives

$$[e_h(w_H)]^2 \leq m_0 \sum_{e_H \in s(e_h)} [e_H(v_h)]^2 \sum_{e_H \in s(e_h)} [e_h(\varphi_{eH})]^2.$$.

Multiplying by $\|\varphi_{eH}\|^2$ and summing over all edges $e_h$ gives

$$\|w\|^2 \leq \gamma_0 \sum_{e_h \in E_h} [e_h(w_H)]^2\|\varphi_{eH}\|^2 \leq \gamma_0 \beta_2 \sum_{e_h} [e_h(v_h)]^2\|\varphi_{eH}\|^2 \leq \beta_1 \|w\|^2,$$

where the constants $\beta_2$ and $\beta_1$ depend on $m_0$ and the ratio $H/h$ and the equivalence constants from (4.4) $\gamma_0$ and $\gamma_1$.

We now prove a result, showing that shows that $(I - Q_H)$ is a bounding operator on the null space of $\Pi_H$.

**Lemma 4.3.** Let $v_0 \in V$ be such that $\Pi_Hv_0 = 0$. Then $c$

$$\beta\|v_0\|^2 \leq \|(I - Q_H)v_0\|^2,$$

where $\beta$ is the same as in Lemma 4.2.

**Proof.** Denote $w_0 = (I - Q_H)v_0$. From Lemma 4.2 it follows that

$$\beta[\|\Pi_Hw_0\|^2 + \|(I - \Pi_H)w_0\|^2] \leq \|w_0\|^2.$$.

Further, from $\Pi_HQ_H = Q_H$ and $\Pi_Hv_0 = 0$ (Lemma 4.1), we have that

$$(I - \Pi_H)(I - Q_H) = (I - \Pi_H), \quad (I - \Pi_H)w_0 = (I - \Pi_H)v_0 = v_0.$$.

Hence,

$$\|(I - Q_H)v_0\|^2 = \|w_0\|^2 \geq \beta[\|\Pi_Hw_0\|^2 + \|(I - \Pi_H)w_0\|^2] = \beta[\|\Pi_H - Q_H\|v_0\|^2 + \|v_0\|^2] \geq \beta\|v_0\|^2,$$

and the proof is completed. \qed

**Theorem 4.4.** There exists constant $c$ independent on $h$, such that the following estimate holds for the convergence rate $\|E\|_{a}^2$ of a two-level method with coarse space $V_H$ and pointwise Gauss-Seidel as a smoother:

$$1 - ch^2 \leq \|E\|_{a}^2.$$
Proof. From the Lemma 3.2 and (4.5) we have the following spectral equivalence relations:

\[(\overline{R}^{-1}v, v) \approx (Dv, v) \approx h^{-2}\|v\|^2.\]  

(4.10)

Taking supremum on both sides the inequalities in (3.1), from (4.10) we obtain

\[h^{-2} \sup_{v \in V_h} \frac{\|(I - Q_H)v\|^2}{\|v\|^2_a} \lesssim K \lesssim h^{-2} \sup_{v \in V_h} \frac{\|(I - Q_H)v\|^2}{\|v\|^2_a}.\]  

(4.11)

Substituting any \(v \in V\) in the left side of (4.11) will give us lower bound.

Consider now a vertex \(x_0 \in T_h\), that is not a vertex of \(T_H\) (such vertex exists according to our assumption). Let \(\varphi_0(x)\) be the corresponding piecewise linear continuous basis function and \(v_0 = \nabla \varphi_0\). By Lemma 4.1, for \(v_0 = \nabla \varphi_0\), we have \(\Pi_H v_0 = 0\). Hence we may conclude that

\[h^{-2} \leq c_1 h^{-2} \frac{\|v_0\|^2}{\|v_0\|^2_0} = h^{-2} \frac{\|(I - Q_H)v_0\|^2}{\|v_0\|^2_a} \leq cK,\]

where \(c_1\) is independent of \(h\). The relation (2.10) then gives the desired bound below for \(\|E\|^2_a\).

\[\Box\]

5. Conclusions

We would like to point out that such analysis can be carried out not only for Nedelec elements, but also for problems in \(H(\text{div})\), anisotropic problems, and other instances of nearly singular problems. One important conclusion that can be drawn is that if the problem (1.1) corresponds to finite element discretization of a second order PDE, or a system of PDEs, and the dimension of the kernel of the singular part \(A_0\) grows with the problem size, then multigrid method has chance to work only with block smoother as a relaxation.

Another set of applications of the two-sided estimates given here is in constructing algebraic multigrid methods, since such estimates show that for pointwise smoother, the quality of the coarse grid space can be measured by looking at \(\|(I - Q_H)v\|/\|v\|_a\) instead of looking at \(\|(I - \Pi_H)v\|/\|v\|_a\). The former quantity, as we have shown it is in some equivalent to the latter and also it is much easier to work with, because a minimizing coarse space for \(\|(I - Q_H)v\|/\|v\|_a\) is known.

References


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