Two-level preconditioning of discontinuous Galerkin approximations of second order elliptic equations

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1. Introduction

1.1. Preliminaries. Consider a second order elliptic problem on a polygonal domain \( \Omega \subset \mathbb{R}^d, d = 2, 3 \):

\[
-\nabla \cdot (a(x)\nabla u) = f(x) \quad \text{in } \Omega,
\]

\[
u(x) = g_D \quad \text{on } \Gamma_D,
\]

\[
a\nabla u \cdot n = g_N \quad \text{on } \Gamma_N.
\]

Here \( n \) is the unit normal vector to \( \partial \Omega = \Gamma \), pointing outward. The boundary is assumed to be decomposed into two disjoint parts \( \Gamma_D \) and \( \Gamma_N \), \( \Gamma_D \cap \Gamma_N = \emptyset \) and the boundary data \( g_D, g_N \) are smooth. For the formulation below we shall need the existence of the traces of \( u \) and \( a\nabla u \cdot n \) on certain interfaces in \( \Omega \). Thus, the solution \( u \) is assumed to have the required regularity. In order to simplify the exposition we shall assume that \( \Gamma_D \) is a nonempty set and has nonzero \( Rd-1 \)-dimensional measure.

In the last decade various finite element discretization for the above problem that use spaces of discontinuous piece-wise polynomial functions have been focus of recent intensive research (see, e.g. [1], [2], [10], [12], [15], [25]). Among the advantages of such approximations are that they do not need to satisfy Dirichlet boundary conditions, have local conservation properties, work well for quasi-uniform meshes of almost arbitrary shape, simplexes, polygons (e.g. pentagons or hexagons) or meshes with hanging nodes.

In Section 2 we introduce several discontinuous Galerkin methods where their basic properties, such as, stability, optimal error estimates, coercivity, etc., are summarized and briefly discussed. These properties are needed when we study iterative methods for solving the corresponding linear system of algebraic equations. Here we consider just few examples of a large number of discontinuous Galerkin FEM approximations of second order elliptic problems that have been introduced and studied in the last several years (see, e.g. [2, 10]). The methods are given by the equation

\[
\text{find } u_h \in \mathcal{V} \text{ such that } \mathcal{A}(u_h, v) = \mathcal{L}(v), \ \forall v \in \mathcal{V},
\]
where $\mathcal{V}$ is the finite element space, and $\mathcal{A}(\cdot, \cdot)$, $\mathcal{L}(\cdot)$ are properly defined bilinear and linear forms on $\mathcal{V}$. More specifically, in Section 2 we introduce five discontinuous Galerkin finite element methods: the standard interior penalty method (IP) with $\mathcal{A}$-form defined by (2.2), the nonsymmetric interior penalty method (NIPG), $\mathcal{A}$-form defined by (2.4), the symmetric stabilized DG method, $\mathcal{A}$-form defined by (2.5), nonsymmetric stabilized DG method of Ewing, Wang and Yang [15] with $\mathcal{A}$-form defined by (2.6), and a nonsymmetric DG method in a constrained space of Rivière, Wheeler and Girault [25] with $\mathcal{A}$-form defined by (2.7). All methods are stable and have optimal convergence rates in $H^1$-norm.

1.2. Aims and main results. The main goal of this paper is to introduce and study efficient iterative methods for the corresponding algebraic problems. Note that the condition number of the DG FE (discontinuous Galerkin finite element) system grows like $O(h^{-2})$ on a quasiform mesh with mesh-size $h$. Therefore construction of optimal solution methods, i.e. with arithmetic work proportional to the numbers of unknowns, is an important problem from both theoretical and practical points of view.

The work of Gopalakrishnan and Kanschat [18], the first one we are aware of, studied the variable V-cycle multigrid operator as a preconditioner of the symmetric DG system. Under certain weak regularity assumptions on geometrically nested meshes in [18] it was shown that the resulted condition number is bounded independently of $h$. If the number of smoothing iterations $m(k)$ grows geometrically with the number of levels $k$, e.g. $m(k) = 2^{J-k}$, where $J$ is the level of the finest grid, then this method has the complexity of a regular W-cycle. The analysis of the preconditioner is based on the abstract multigrid theory [4] for non-inherited bilinear forms and the estimates for interior penalty finite element method.

Further, Brenner and Zhao [9] studied multigrid methods for solving symmetric DG FE schemes on rectangular meshes and showed that V-cycle, W-cycle, and F-cycle algorithms produce uniform preconditioners for sufficiently enough pre- and post smoothing steps $m(k)$. Namely, for the V-cycle they show that if the solution of the original problem satisfies the a priori estimate $\|u\|_{H^{1+\alpha}} \leq C\|f\|_{H^{-1+\alpha}}$, for $\alpha \in (\frac{1}{2}, 1]$, then there is an integer $m_0 > 0$ independent of the number of levels $J$ such that a certain norm of the multigrid error propagation operator $E_{mg}$ satisfies the estimates $\|E_{mg}\| \leq c/m^k$, $k \geq 1$, $m \geq m_0$. Their analysis is based on certain mesh dependent norms and a proper relationship of the discontinuous finite element spaces to some continuous spaces of higher order polynomials. This result is remarkable with the fact that the multigrid convergence improves with the number of smoothing steps based on smoothness of the solution.

Recently Hemker, Hoffmann and van Raalte [19] have presented a local mode analysis of the multigrid convergence for discontinuous Galerkin systems. The results are obtained by Fourier analysis of the discretized Poisson equation in one space dimension. Two different ways for block-tridiagonal partitioning of the discrete operator and the related block-relaxation MG smoothers are considered. Though limited to 1-D and periodic boundary conditions the conclusion is that the point-wise block partitioning shows a much better convergence than the usual cell-wise block-partitioning.
Finally, Johannsen [20] studied a symmetric Gauss-Seidel smoother for NIPG method and showed that the multigrid method with sufficiently pre- and post-smoothing steps provides a uniform preconditioner for the corresponding algebraic problem.

Our results are summarized in what follows.

We start the discussion with the solution methods for the nonsymmetric discontinuous Galerkin methods (NIPG). Our first finding is formulated in Lemma 2.3. It appeared that the nonsymmetric algebraic problems generated by DG method have the skew-symmetric part which is bounded by the symmetric part. This fact, combined with its coercivity in the norm equivalent to the symmetric part, implies that the problem can be preconditioned by any optimal preconditioner for the symmetric form in a GMRES iterative method (cf. [14]). Thus, our main effort has been in the construction and study of efficient preconditioning methods for the symmetric problems.

All known to us works on solution method of optimal complexity use the corresponding discontinuous Galerkin approximations on geometrically nested finite element meshes and the main constructions and tools of the theory of multigrid methods. In this paper we try to do something different and simpler. Our approach could be viewed as the classical two-level method applied to the linear systems generated by the discontinuous Galerkin combined further with multilevel technique. The idea is first to smooth (using Gauss-Seidel, scaled Jacobi or by other smoothers) and then project onto a “coarser” (or auxiliary) function space defined on the same partition. The latter approach was studied in a somewhat abstract setting in Xu [30], see also [6]. We explore three different possibilities for a choice of the “coarser” space.

In Section 3 we study the classical two-level method (see, e.g. [4, pp. 14-17]), by considering a coarser space $V_0$ that, in general, is not associated with the discontinuous Galerkin method. Let $V$ be the finite element space of discontinuous functions where DG method is defined. Assume that $V_0$ is a proper subspace of $V$ and we can generate an algebraic problem associated with $V_0$. The two level (or two-grid) method is defined in terms of: (a) the corresponding problems on the spaces $V, V_0$, (b) the smoothing operator $R$, and (c) the $L^2$-orthogonal projection $Q : V \mapsto V_0$. As shown in [4, pp. 14-17] under some general assumption, the classical two-grid method is optimally convergent. Our goal is to use $V_0$ which is smaller than $V$ and for which the construction of efficient solution techniques (e.g., multigrid) for the corresponding algebraic problem is well (or better) understood.

We further investigate, in Section 4, three possible choices for $V_0$: (1) $V_0 = V^{CG}$, the space of continuous piecewise linear or bilinear functions over simplexes or quadrilaterals or hexahedral elements; (2) $V_0 = V^{CR}$, the space of Crouzeix-Raviart nonconforming finite elements on simplexes, and (3) $V_0 = V^{CC}$, the space of discontinuous piecewise constant functions on the original partition of the domain. In all cases we introduce proper restriction and prolongation operators between the spaces $V$ and $V_0$. In Theorem 4.3 (see also Lemma 3.3) we show that for globally quasiuniform meshes all three spaces give rise to two-level methods (with a standard smoother) that converge independently of the mesh size $h$. Since we consider nested spaces ($V_0 \subset V$) the analysis is standard, and one can for example use, the results of the general algebraic two-level method developed by Falgout, Vassilevski, and Zikatanov [16]. In Section 5 we present numerical experiments that support
the theoretical results. In the conclusions in Section 6 we summarize the results obtained in this work.

Further, we comment briefly on the interesting case of constructing multilevel preconditioners for the piecewise constant space \( V^{CC} \). It is known that DG method for piece-wise constant functions and general meshes does not provide approximation for the elliptic problem. But the corresponding bilinear form is generated by the penalty of the jumps across the interfaces and produces symmetric and positive definite matrix \( A_P \) (called graph-Laplacian) that can be used to build multigrid of the so-called algebraic multilevel iteration (AMLI) type cycle, in the spirit of [3, 28]. The main ingredients of the analysis that lead to optimal AMLI–cycle type multigrid methods are: (1) weak approximation property of certain projection operator between two consecutive level spaces and (2) tight control of the growth of the energy norm of the projection from level to level. This is a subject of an ongoing research [23]. Another possibility that was explored in [22] is to find an appropriate multi-level hierarchical splitting of the degrees of freedom, to estimate the CBS constant and then apply the constructions from [3, 28]. In general though, one should prefer in practice the multigrid AMLI cycle over the hierarchical one.

2. Discontinuous Galerkin approximation

2.1. Notations. Let \( T \) be a partitioning of \( \Omega \) into finite number of open subdomains (finite elements) \( T \) with boundaries \( \partial T \). We assume that the partition is quasi uniform and regular in the sense of [11]. For each finite element we denote by \( h_T \) its size and further \( h = \max_{T \in T} h_T \). Note that we allow finite elements of polygonal or polyhedral shape, with hanging nodes etc. Let \( F = T_1 \cap T_2 \) be the interface of two adjacent subdomains \( T_1, T_2 \). The set of all such interfaces is denoted by \( E_0 \), note that these interfaces are inside \( \Omega \). Further, \( E_D \) and \( E_N \) will be the faces/edges of finite elements on the boundary \( \Gamma_D \) and \( \Gamma_N \), respectively. Finally, \( E \) will be the set of all faces/edges, \( E = E_0 \cup E_D \cup E_N \). Here we allow finite elements of polygonal or polyhedral shape, with hanging nodes etc. We assume that if \( F \) is an edge or a face of a finite element \( T \in T \) then \( |F| \approx h_T \) for \( d = 2 \) and \( |F|^{1/2} \approx h_T \) for \( d = 3 \). In other words we do not allow very “small” edges or faces.

In association with the partition \( T \) we define the “broken Sobolev space”

\[
H^s(T) = \{ v \in L^2(\Omega) : \forall T \in T, \ v|_T \in H^s(T) \}, \ s \geq 0.
\]

Further, we define the finite element space

\[
V := V(T) := \{ v \in L^2(\Omega) : \forall T \in T, \ v|_T \in P_r(T), \ T \in T \},
\]

where \( P_r \) is the set of polynomials of degree \( r \geq 1 \). Obviously, \( V = \Pi_{T \in T} P_r(T) \) and therefore the approximation properties of \( V \) will follow from the approximation properties of the local \( L^2(T) \)-orthogonal projections.

On \( V(h) := H^2(T) + V \) we define

\[
(a \nabla u, \nabla v)_T := \sum_{T \in T} \int_T a \nabla u, \nabla v \ dx, \quad \langle p, q \rangle_S := \sum_{F \in S} \int_F pq \ ds,
\]

where \( S \) is \( E, E_0 \), or \( E \cup E_D \).

For each \( F = T \cap T' \in E_0 \) we specify one of the subdomains \( T \) or \( T' \) as a “master” side. Then the normal vector \( n \equiv n_F \) to \( F \) is a fixed unit vector pointing outside of the master subdomain \( T \). Now we define the jump \( [v] \) over \( E \) of any function
Finally, we shall use the following bilinear form that defines inner product and norm.

\[
[v]_F := \begin{cases} v|_T - v|_{T'}, & F = \bar{T} \cap \bar{T}', \ i.e. \ F \in \mathcal{E}_0, \\ v|_T, & F = \bar{T} \cap \Gamma_D, \ i.e. \ F \in \mathcal{E} \setminus \mathcal{E}_0. \end{cases}
\]

Further, we shall need the notation for the average value of the traces of \(v \in \mathcal{V}\) on \(F = \bar{T} \cap \bar{T}'\)

\[
\{v\} := \begin{cases} \frac{1}{2}\{v|_T + v|_{T'}\}, & F = \bar{T} \cap \bar{T}', \ i.e. \ F \in \mathcal{E}_0, \\ v|_T, & F = \bar{T} \cap \Gamma_D, \ i.e. \ F \in \mathcal{E} \setminus \mathcal{E}_0 \end{cases}
\]

and the piecewise constant function \(h_\mathcal{E}\) defined on \(\mathcal{E}\) as

\[
h_\mathcal{E} = h_\mathcal{E}(x) = \begin{cases} |F|, & \text{for } x \in F \in \mathcal{E}, \ d = 2 \\ |F|^\frac{2}{d}, & \text{for } x \in F \in \mathcal{E}, \ d = 3. \end{cases}
\]

Finally, we shall use the following bilinear form that defines inner product and norm on \(\mathcal{V}\):

\[
\mathcal{A}(0)(v, w) = (a\nabla v, \nabla w)_T + \kappa \left( h_\mathcal{E}^{-1} [v], [w] \right)_{\mathcal{E}_0 \cup \mathcal{E}_D}, \quad \|v\|_h = \mathcal{A}(0)(v, v)^{\frac{1}{2}}.
\]

2.2. Discontinuous Galerkin methods. We introduce various bilinear form on \(\mathcal{V} \times \mathcal{V}\):

1. Symmetric interior penalty (IP) form (see, e.g. [1, 2, 21]):

\[
\mathcal{A}^{sym}(u_h, v) \equiv (a\nabla u_h, \nabla v)_T + \kappa \left( h_\mathcal{E}^{-1} [u_h], [v] \right)_{\mathcal{E}_0 \cup \mathcal{E}_D} - \langle \{a\nabla u_h \cdot n\}, \{v\} \rangle_{\mathcal{E}_0 \cup \mathcal{E}_D} - \langle \{u_h\}, \{a\nabla v \cdot n\} \rangle_{\mathcal{E}_0 \cup \mathcal{E}_D};
\]

if \(\kappa\) is sufficiently large then the bilinear form (2.2) is coercive and bounded in \(\mathcal{V}\) equipped with the norm (2.1); this follows easily from the inequality for \(v, w \in \mathcal{V}\) (cf. e.g. [21])

\[
\langle [v], \{a\nabla w \cdot n\} \rangle_{\mathcal{E}_0 \cup \mathcal{E}_D} \leq C \kappa (a\nabla w, \nabla w)_T + \frac{\kappa}{4} \left( h_\mathcal{E}^{-1} [v], [v] \right)_{\mathcal{E}_0 \cup \mathcal{E}_D};
\]

with a constant \(C > 0\) independent of the mesh size \(h\);

2. Nonsymmetric interior penalty (NIPG) form (see, e.g. [25]):

\[
\mathcal{A}^{nsp}(u_h, v) \equiv (a\nabla u_h, \nabla v)_T + \kappa \left( h_\mathcal{E}^{-1} [u_h], [v] \right)_{\mathcal{E}_0 \cup \mathcal{E}_D} - \langle \{a\nabla u_h \cdot n\}, \{v\} \rangle_{\mathcal{E}_0 \cup \mathcal{E}_D} + \langle \{u_h\}, \{a\nabla v \cdot n\} \rangle_{\mathcal{E}_0 \cup \mathcal{E}_D};
\]

this bilinear form is coercive in \(\mathcal{V}\) equipped with the norm (2.1) for any \(\kappa > 0\);

3. Nonsymmetric stabilized form (see, Ewing, Wang, and Yang [15]):

\[
\mathcal{A}^{new}(u_h, v) \equiv (a\nabla u_h, \nabla v)_T + \kappa \left( h_\mathcal{E}^{-1} [u_h], [v] \right)_{\mathcal{E}_0 \cup \mathcal{E}_D} - \langle \{a\nabla u_h \cdot n\}, \{v\} \rangle_{\mathcal{E}_0 \cup \mathcal{E}_D} + \langle \{u_h\}, \{a\nabla v \cdot n\} \rangle_{\mathcal{E}_0 \cup \mathcal{E}_D} + \frac{1}{4} \kappa^{-1} \left( h_\mathcal{E} [a\nabla u_h \cdot n], [a\nabla v \cdot n] \right)_{\mathcal{E}_0},
\]

which is coercive in \(\mathcal{V}\) equipped with the norm (2.1) for any \(\kappa > 0\);

4. Symmetric stabilized form (see, Ewing, Wang, and Yang [15]):

\[
\mathcal{A}^{new}(u_h, v) \equiv (a\nabla u_h, \nabla v)_T + \kappa \left( h_\mathcal{E}^{-1} [u_h], [v] \right)_{\mathcal{E}_0 \cup \mathcal{E}_D} - \langle \{a\nabla u_h \cdot n\}, \{v\} \rangle_{\mathcal{E}_0 \cup \mathcal{E}_D} + \langle \{u_h\}, \{a\nabla v \cdot n\} \rangle_{\mathcal{E}_0 \cup \mathcal{E}_D} + \frac{1}{4} \kappa^{-1} \left( h_\mathcal{E} [a\nabla u_h \cdot n], [a\nabla v \cdot n] \right)_{\mathcal{E}_0},
\]
which is coercive for sufficiently large $\kappa$; indeed, using (2.3) and the inequality

$$h_F \| a \nabla v \cdot n \|^2_{L^2(F)} \leq C \int_T a \nabla v \cdot \nabla v \, dx, \quad v \in V$$

we easily conclude that for $\kappa > 0$ sufficiently large the last three terms are absorbed by the first and second terms and the form is coercive in the norm (2.1).

(5) Nonsymmetric form on a constrained space (see, e.g. [25]):

\begin{equation}
(2.7)
V^* = \{ v \in V : \int_F [v]ds = 0, \forall F \in \mathcal{E}_0 \}.
\end{equation}

Then $A_{ncg}(u_h, v) = A^{ns}(u_h, v)$ for $u_h, v \in V^*$. If $r = 1$ then $V^*$ is the space of Crouzeix-Raviart nonconforming finite elements.

We summarize the properties of the bilinear forms in the following lemma:

**Lemma 2.1.** Assume that the finite element partition $T$ is regular and locally quasi uniform. Then the nonsymmetric bilinear forms $A^{ns}$ and $A^{ncg}$ are coercive and bounded in $V$ equipped with the norm (2.1) for any $\kappa > 0$. Similarly, the symmetric bilinear forms $A^{sym}$ and $A^{scw}$ are coercive and bounded in $V$ equipped with the norm (2.1) for $\kappa > 0$ sufficiently large.

Now we introduce the discontinuous Galerkin finite element method for (1.1):

Find $u_h \in V$ such that

\begin{equation}
(2.8)
A(u_h, v) = L(v), \quad \forall v \in V,
\end{equation}

with $A(\cdot, \cdot)$ one of the forms defined by (2.2), (2.4), (2.6), or (2.5) and

\begin{equation}
(2.9)
L(v) = \int_{\Omega} fvdx + \int_{\Gamma_N} g_N v ds - \int_{\Gamma_D} g_D a \nabla v \cdot n ds + \kappa \int_{\Gamma_D} h^{-1}_E g_D v ds.
\end{equation}

Restricting the above equations to the constraint space and choosing $A(\cdot, \cdot)$ to be defined by (2.7) we get another DG method that has been studied by Rivi`ere, Wheeler, and Girault in [25].

The following lemma is an easy corollary of the fact that the bilinear form $A(\cdot, \cdot)$ is coercive and bounded in $V$ in the norm (2.1) (see, e.g. [2, 15, 21, 25]):

**Theorem 2.2.** Assume that $f \in L^2(\Omega)$, $g_N \in L^2(\Gamma_N)$, $g_D \in L^2(\Gamma_D)$ and the conditions of Lemma 2.1 hold. Then the problem (2.8) has unique solution $u_h \in V$.

If the solution $u \in H^s(\Omega)$, $2 \leq s \leq r + 1$, then all proposed methods have optimal order of convergence and the following estimate is valid

$$\|u - u_h\|_H \leq Ch^{s-1}\|u\|_{H^s(\Omega)}.$$

Further, the DG methods with symmetric forms (2.2) and (2.5) and the DG method with nonsymmetric forms on the constrained space $V^*$ have optimal convergence in $L^2(\Omega)$-norm, namely, for $u \in H^s(\Omega)$, $2 \leq s \leq r + 1$

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^{s}\|u\|_{H^s(\Omega)}.$$

Finally, the nonsymmetric DG methods with forms defined by (2.4) and (2.6) have the same convergence rate in $L^2$-norm as in $\|\cdot\|_{H^s}$-norm.

For the construction of preconditioners of the nonsymmetric problems we shall need the following property of the method (2.8) when $A(\cdot, \cdot)$ is nonsymmetric form (see, e.g. [20]):
Lemma 2.3. There is a constant \( C > 0 \), independent of \( h \), such that the skew-symmetric part of \( A(\cdot, \cdot) \) defined as

\[
B(w, v) = \frac{1}{2} (A(w, v) - A(v, w)) = -\{\{a \nabla w \cdot n\}, [w]\}_{\mathcal{E}_0 \cup \mathcal{E}_D} + \{\{a \nabla v \cdot n\}, [v]\}_{\mathcal{E}_0 \cup \mathcal{E}_D}
\]

satisfies the inequality

\[
|B(w, v)| \leq C \sqrt{\kappa} \|w\|_h \|v\|_h, \quad \forall w, v \in V.
\]

Since \( V^* \subset V \) the above inequality of valid for \( w, v \in V^* \) as well.

Here the constant \( C \) depends only on the shape regularity of the mesh. As noted in [20] for \( d = 2, k = 1, a \equiv 1 \), and \( T \) uniform square mesh \( C = 2^{\frac{1}{4}} \).

Remark 2.4. The above Lemma 2.3, gives rise to the boundedness,

\[
A(w, v) \leq \gamma_2 \left( A^{(0)}(w, w) \right)^{\frac{1}{2}} \left( A^{(0)}(v, v) \right)^{\frac{1}{2}}.
\]

Here \( A^{(0)} \) is the s.p.d. form (2.1) and \( A \) is any of the above DG forms (symmetric or non–symmetric). We also have the coercivity of \( A \) in the norm \( \| \cdot \|_h \) (cf. Lemma 2.1),

\[
\gamma_1 (A^{(0)}v, v) \leq A(v, v).
\]

Then a result in [14] implies that any of the studied DG forms (non–symmetric or symmetric) \( A \) can be preconditioned by any optimal order preconditioner \( B^{(0)} \) for the operator \( A^{(0)} \) in a GMRES iterative method, with a rate of convergence bounded by

\[
\left( 1 - \frac{\gamma_2^2}{\gamma_1^2} \kappa_0^{-2} \right)^{\frac{1}{2}},
\]

where \( \kappa_0 \) bounds the condition number of \( B^{(0)}^{-1} A^{(0)} \). It is clear that in the case of \( A \) being s.p.d. sharper bounds are possible. Also, when \( A \) is s.p.d., a direct approach can be taken (see Section 3), where the form \( A^{(0)} \) is only used in the analysis and not in the actual construction of the preconditioner for \( A \).

Based on Remark 2.4, at least in theory, one can focus on the task of constructing optimal order preconditioners for the simple interior penalty form \( A^{(0)} \), a case studied previously in the literature. For example, additive Schwarz methods for \( A^{(0)} \) are found in [26], whereas optimal order MG methods for \( A^{(0)} \) were analyzed in [18].

Remark 2.5. We note that \( A^{(0)} \) for the case of piecewise constant spaces reduces to a “cell–centered” bilinear form. In that case, optimal order MG methods were analyzed in [5] for the very special case of uniformly refined orthogonal grids. The purpose of our paper is to extend the use of piecewise constant spaces to more general (non–orthogonal, triangular and quadrilateral), but still uniformly refined grids.

Remark 2.6. We finally mention that the abstract setting of [30] or [6] allows in principle, to extend the two–level approach of the present paper (all but the piecewise constant spaces), to handle more general unstructured quasiform DG discretizations using as auxiliary space a standard f.e. discretization space on a
auxiliary uniformly refined mesh. This approach, however has not been pursued in the present paper.

2.3. Smoothed aggregation AMG method. The case of general unstructured meshes can be handled, for example, by the smoothed aggregation AMG, as long as a weak approximation property holds. More precisely, to establish uniform convergence (or design uniform preconditioner), the following estimate is needed:

\[ \| u - I_H u \|_0 \leq C H \| u \|_h, \]

where \( I_H \) is some (piecewise constant) interpolant of \( u \) on a triangulation \( T_H \) of domains (aggregates of fine–grid elements) with diameter \( O(H) \). A feasible way to prove the above estimate is to let \( I_H = Q_H \tilde{Q}_h u \). Here, \( Q_H \) is the standard \( L_2 \)–projection onto the space of piecewise constants w.r.t. to the elements (subdomains) of \( T_H \), whereas \( \tilde{Q}_h \) is a properly defined projection onto a space of \( H^1 \)–conforming functions associated with the fine–grid triangulation \( T_h \) that satisfies the property

\[ h^{-2} \| u - \tilde{Q}_h u \|_0^2 + \| Q_h u \|_1^2 \leq C \| u \|_h^2. \]

Such estimates are found for example, in [26]. Then, by the triangle inequality, we obtain

\[ \| u - I_H u \|_0 \leq \| u - \tilde{Q}_h u \|_0 + \| (I - Q_H)\tilde{Q}_h u \|_0 \]

\[ \leq C h \| u \|_h + C H \| \tilde{Q}_h u \|_1 \]

\[ \leq C(h + H) \| u \|_h. \]

Then, a proper application of the convergence proof in [27] will lead to a convergence factor of the respective \( V \)–cycle smoothed aggregation AMG bounded by

\[ 1 - C \frac{1}{\ell^3}, \]

with \( \ell = \log \frac{H_\ell}{h} \), where \( H_\ell \) bounds the diameter of the \( \ell \)th (coarsest) level aggregates.

3. The classical two–level scheme applied to symmetric DG problems

Further on we shall discuss the solution of the DG system defined by (2.8). In order to simplify the exposition further we shall skip the index \( h \) in the notation for the DG solution, namely instead of \( u_h \) we shall use \( u \). Then following [4] we shall consider a two level iteration for the algebraic problem, find \( u \in V \) such that

\[ A(u, \phi) = L(\phi) \text{ for all } \phi \in V \]

and the projections:

\[ P_A : V \mapsto V_0 \text{ such that } A(P_A v, \phi) = A(v, \phi) \text{ for all } \phi \in V_0, \]

where \( A(u, \phi) = \mathcal{A}(u, \phi) \) is symmetric and positive definite in \( V \). In this case the bilinear form defines a norm

\[ \| v \| = \mathcal{A}(v, v)^{1/2}. \]

Here we have in mind (2.8), where \( \mathcal{A}(u, \phi) \) is defined by (2.2) or (2.5). The form \( \mathcal{A} \) defines a linear operator \( A : V \mapsto V \) by

\[ (Av, \phi) = A(v, \phi) \text{ for all } v, \phi \in V, \]

which is symmetric and positive definite in \( \langle \cdot, \cdot \rangle \), the standard \( L^2 \)-inner product on \( V \). Since \( V \) is finite dimensional, then \( A(v, v) \leq \lambda (v, v) \) for all \( v \in V \) where \( \lambda > 0 \) is the largest eigenvalue of the linear operator \( A \).

Next, for \( V_0 \subset V \) we define the linear operator: \( A_0 : V_0 \mapsto V_0 \)

\[ (A_0 v, \phi) = \mathcal{A}(v, \phi) \text{ for all } v, \phi \in V_0, \]

and the projections:
Obviously, the operator $A_0$ is symmetric and positive definite in $(\cdot, \cdot)$ and $QA = A_0 P_A$. Finally, we assume that we are given the operator $R : V \mapsto V_0$ and its adjoint $R^T$ in the $L^2$-inner product.

**Algorithm 3.1 (Two level algorithm).** (see, e.g. [4])

(0) Let $u_0$ be given

For $u_i$ “approximating” $u$, the solution of $Au = f$, define $u_{i+1}$ as follows:

1. $x_1 = u_i - R(Au_i - f)$ (smooth)
2. $x_2 = x_1 - A_0^{-1} Q(Ax_1 - f)$ (correct)
3. $u_{i+1} = x_2 - R^T(Ax_2 - f)$ (smooth).

To study the convergence of this algorithm we set $e_i = u - u_i$ and derive the following relation (see, e.g. [4])

$$e_{i+1} = (I - R^T A)(I - P_A)(I - RA)e_i \equiv E_{tg} e_i.$$ 

Thus

$$|||e_{i+1}||| \leq |||E_{tg}||| |||e_i|||,$$

where $|||E_{tg}|||$ is the operator norm of $E_{tg}$. Note that $E_{tg}$ is symmetric in $A(\cdot, \cdot)$.

Now following [4] we consider the case when $R = \lambda^{-1} I$, where $I$ is the identity operator on $V$. For this method the following assumption plays an important role:

**Assumption 3.2.** There is a constant $C_0 \geq 1$, independent of $h$, such that

$$\inf_{z \in V_0} |||v - z|||_0 = |||v - Qv|||_0 \leq C_0 \lambda^{-\frac{1}{2}} |||v|||, \quad \forall v \in V.$$  

The following lemma from [4, pages 16–19] shows that the convergence of the two-grid algorithm 3.1 with $R = \lambda^{-1} I$ is a consequence of the assumption 3.2:

**Lemma 3.3.** Let the Assumption 3.2 be satisfied. Then $|||E_{tg}||| \leq 1 - \frac{1}{C_0} = \delta < 1$ and the two-level algorithm 3.1 converges independently of $h$.

Now we shall demonstrate this abstract result when two-level method is applied to the problem (2.8) with a symmetric bilinear form $A(\cdot, \cdot)$ defined by (2.2) or (2.5) for a particular case of finite element space that satisfies the following assumptions:

**Assumption 3.4.** Let $V_0 = V^{CG}$, where $V^{CG}$ is the finite elements space of continuous piecewise linear or bilinear functions, be well defined on the partition $T$ and $V^{CG} \subset V$.

**Assumption 3.5.** The mesh $T$ is globally quasiuniform and therefore there is a constant $C > 0$ such that $\lambda = \sup_{v \in V} A(v, v)/(v, v) \leq Ch^{-2}$.

These two assumptions are quite restrictive and reduce the DG method to globally quasiuniform simplicial or hexahedral meshes. However, this setting is useful from a theoretical point of view and allows to put in a prospective our work we present in the next Section.

Under the Assumption 3.4 we can define the linear map $\bar{Q} : V \mapsto V^{CG}$ constructed by certain averaging process described in the next section. Further, the operator $A_0$ is defined as above by $(A_0 v, \phi) = A(v, \phi), \quad \forall v, \phi \in V^{CG}$. Since the
form $\mathcal{A}(\cdot, \cdot)$ is coercive in $\mathcal{V}$ and $Q$ is an $L^2$-orthogonal projection, obviously the Assumption 3.2 will be satisfied if we show that
\[ \|v - \bar{Q}v\|_0 \leq C h \|v\|, \quad \forall v \in \mathcal{V}. \]
This will be shown in the next section by computing both quantities in terms of the degrees of freedom of $\mathcal{V}$. However, for the sake of completeness we give a shorter proof of (3.2) based on duality, restricted to elliptic problems that have full regularity. Let $\psi \in H_0^1(\Omega)$ be the solution of the equation
\[ -\nabla \cdot (a \nabla \psi) = v - P_A v. \]
Assume that $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ and
\[ (3.3) \quad \|\psi\|_{H^2(\Omega)} \leq C \|v - P_A v\|_0. \]
Now multiply the above equation by $v - P_A v$, integrate over $\Omega$, and use the fact that the traces of $\psi$ and $a \nabla \psi \cdot n$ are uniquely determined across the interfaces $\mathcal{E}$ and $\psi$ vanishes on $\partial \Omega$. Thus
\[ \|v - P_A v\|_0^2 = (a \nabla \psi, \nabla(v - P_A v))_T - (a \nabla \psi \cdot n, [v - P_A v])_{\mathcal{E}} = \mathcal{A}(\psi, v - P_A v). \]
Now using the symmetry of $\mathcal{A}(\cdot, \cdot)$ and since by definition $\mathcal{A}(\psi^*, v - P_A v) = 0$ for $\psi^* \in \mathcal{V}^{CG}$, we get
\[ \|v - P_A v\|_0^2 = \mathcal{A}(\psi, v - P_A v) = \mathcal{A}(\psi - \psi^*, v - P_A v), \quad \forall \psi^* \in \mathcal{V}^{CG}. \]
Now we take $\psi^* \in \mathcal{V}^{CG}$ to be the standard nodal Lagrange interpolant so that
\[ \mathcal{A}(\psi - \psi^*, v - P_A v) \leq C h \|v - P_A v\|_0. \]
Then apply Schwarz inequality and use the stability of $P_A$ in $\|\cdot\|$-norm to get
\[ \mathcal{A}(\psi - \psi^*, v - P_A v) \leq \mathcal{A}(\psi - \psi^*, \psi - \psi^*)^{\frac{1}{2}} \mathcal{A}(v - P_A v, v - P_A v)^{\frac{1}{2}} \leq C h \|v - P_A v\|_0 \|v\|. \]
Thus, $\|v - P_A v\|_0 \leq C h \|v\|$ and the Assumption 3.5 concludes the proof.

**Remark 3.6.** Similar to the above approach has been extended to the multilevel case in [18]. Since the problems under consideration are elliptic, it is typical situation that the standard smoothers used give rise to well-conditioned (w.r.t. the current level mesh-size) smoothing operators, the analysis of the two–level scheme based on the simple Richardson smoothing, in principle, covers the general case. By standard smoothers, here we mean, ones that utilize local operations, such as Gauss–Seidel, Jacobi, or slightly more generally, based on overlapping Schwarz methods with small domains.

4. Convergence of the Two Level Method

In this Section we verify Assumption 3.2 for three particular choices of the auxiliary “coarse” space $\mathcal{V}_0$. We present here a theorem that furnishes the uniform convergence of three types of two level methods, which differ by the choice of the coarse space. The first method corresponds to a coarse space using continuous piecewise linear functions, $\mathcal{V}^{CG}$, second, we take the space of piece wise constants, $\mathcal{V}^{CC}$, and the last is the space of Crouzeix-Raviart nonconforming finite elements, $\mathcal{V}^{CR}$. It is clear that all these are proper subspaces of $\mathcal{V}$. To simplify the exposition we consider the space $\mathcal{V}$ of piece-wise linear functions. For higher order polynomials we can do one step element-wise projection onto the discontinuous linear functions and then apply the considerations described below.
The result that we show below, are established using somewhat more algebraic technique, and in this section we introduce the necessary notation for this change. We target a specific situation of hierarchical meshes, which will be further assumed (in order to define multilevel methods) obtained by successive steps of refinement of an initial coarse triangulation. To simplify the presentation we utilize a mesh-independent notation introduced below.

4.1. Some simplifications. There are several simplifications that we shall make in order to construct a uniform with respect to \( h \) preconditioner for the bilinear form \( A(\cdot, \cdot) \). We first observe that \( A^{(0)}(\cdot, \cdot) \) is spectrally equivalent to \( A(\cdot, \cdot) \). Hence we only need to precondition \( A^{(0)}(\cdot, \cdot) \). Next, let \( \tilde{A}_P(\cdot, \cdot) \) denote the part of \( A^{(0)}(\cdot, \cdot) \) corresponding to the penalty term, namely

\[
\tilde{A}_P(u, v) := \kappa \sum_{F \in E \cup F_D} h_F^{-1} \int_F [u] [v] \, ds.
\]

The following lemma shows that we can replace \( \tilde{A}_P(\cdot, \cdot) \) with \( A_P(\cdot, \cdot) \), defined as

\[
A_P(u, v) := \kappa \sum_{F \in E \cup F_D} h_F^{-1} \int_F ( [u] [v] )_I \, ds,
\]

where \((w)_I\) denotes the piecewise linear continuous interpolant of a function \( w \) over the interface \( F \) (edge in 2-D or face in 3-D). These are linear function defined by the values at the endpoints of the edge or the vertices of the face, as shown on Figure 1.

![Figure 1. Degrees of freedom on the interfaces for the continuous interpolant over an edge \( F \) in 2-D (left) and 3-D (right)](image)

The following general result will be later used (only for \( d=1,2 \))

**Lemma 4.1.** Let \( u \) be a linear function on a simplex \( T \) in \( \mathbb{R}^d \). Then

\[
\frac{1}{d+2} \int_T (u^2)_I \, dx \leq \int_T u^2 \, dx \leq \int_T (u^2)_I \, dx,
\]

where \((u)_I\) denotes the linear nodal value interpolant on \( T \).

**Proof.** Note that the right hand side is

\[
\int_T (u^2)_I \, dx = \frac{|T|}{d+1} \sum_{i=1}^{d+1} u_i^2.
\]
On the other hand
\[ \int_T u^2 \, dx = \frac{|T|}{d+1} \sum_{i,j=1}^{d+1} m_{ij} u_i u_j, \]
where \( m_{ij} \) are the elements of the proper mass (Gram) matrix for the canonical simplex. One easily sees that this matrix is nonsingular and has eigenvalues independent on \( h \). This concludes the proof. \( \square \)

From this lemma it follows that the bilinear form \((a \nabla u, \nabla v)_T + A_P(u, v)\) is spectrally equivalent to \( A(\cdot, \cdot) \) and as a consequence, a uniform preconditioner for the former will also be a uniform preconditioner for the latter. Hence in what follows, we only consider the bilinear form \((a \nabla u, \nabla v)_T + A_P(u, v)\) and for convenience and without loss of any generality, we denote

\[ (4.1) \quad A(u, v) := (a \nabla u, \nabla v)_T + A_P(u, v). \]

Although such notation seems in conflict with the definition of \( A(\cdot, \cdot) \) that we had in the discrete problem (2.8), by the discussions above the constructing a uniform preconditioner for (4.1) is equivalent to constructing preconditioner for the form given by (2.8).

4.2. Degrees of freedom and algebraic notation. Let us denote with \( V (V_0) \) the space of vectors of degrees of freedom of the space \( V (V_0) \). Once basis in \( V (V_0) \) is fixed, which we take to be the standard nodal basis, \( V \) can be identified with \( \mathbb{R}^{(d+1)N_T} \) for a simplicial partition and linear functions, where \( N_T \) is the number of simplices in the partition. Thus, when discussing the two level method here, we use the standard \( l_2 \) inner product for elements on \( V \) and \( V_0 \), namely, if \( v, w \in V \) are considered as a column vectors with components the degrees of freedom in \( V \) then \((v, w)_{l_2} = w^T v\). This will generate various scaled “mass” and “stiffness” matrices by the relations:

\[ (4.2) \quad (A_D u, v)_{l_2} = h^{2-d}(a \nabla u_h, \nabla v)_T, \quad (A_P u, v)_{l_2} = h^{2-d} A_P(u, v)_{l_2}, \quad A = A_D + A_P. \]

Note that such scaling gives matrices with entries independent on \( h \) (only depending on the angles in the finite element mesh).

4.3. On the convergence analysis of the two-level method. The two-level preconditioner that we focus on is as described by (3.1) with \( R \) replaced by \( M^{-1} \). The convergence rate is then measured in an operator norm corresponding to the vector norm \( \| v \|^2 = (A v, v)_{l_2}, \) i.e., the norm induced by \( A \) on \( V \). Further we shall need a notation for \( \pi_A : V \mapsto V_0 \), the \( A \)-orthogonal projectors of \( u \in V \) in \( l_2 \) norms, i.e.

\[ (4.3) \quad \pi_A : V \mapsto V_0 : \quad (A[\pi_A u], v)_{l_2} = A(u, v), \quad \forall v \in V_0. \]

As shown in [16], the convergence rate of the two-level algorithm algorithm 3.1 in \( V \) is given by

\[ (4.4) \quad \| E_{tg} \| \equiv \| (I - M^{-T} A)(I - \pi_A)(I - M^{-1} A) \| = 1 - 1/K, \]
where $\pi_A$ is the $A$-orthogonal projection on the coarse space defined by (4.3), and $K > 1$ is the following quantity (see [16]):

$$K = \sup_{v \in V} \frac{||(I - \pi_M)v||_M^2}{||v||^2}.$$  

(4.5)

The operator $\tilde{M}$ is the positive definite symmetrization of $M$ and has the form $\tilde{M} = M^T(M + M^T - A)^{-1}M$. Following a standard notation, we have denoted the $\tilde{M}$-orthogonal projection on the coarse space with $\pi_{\tilde{M}}$. An explicit form of this projection is $\pi_{\tilde{M}} := P(PTMP)^{-1}PT\tilde{M}$, where $P$ is the matrix representing the coordinates of the coarse space canonical basis (e.g. the nodal basis for the piecewise linear continuous elements) in terms of the discontinuous Galerkin finite element canonical basis. For example, if $M$ corresponds to forward Gauss-Seidel method, then $\tilde{M} = A + L^TD^{-1}L$, where $L$ is the strict lower triangle of $A$ and $D$ is the diagonal of $A$.

4.4. On the estimate of $K$. With these new notation let us first point out some observations related to the quantities given in the algorithm (3.1). From the scaling used in (4.2) above to define the bilinear forms on $\ell_2$ we have: since $\tilde{M}$ is sparse, we have that $\lambda = \mathcal{O}(1)$ and $\tilde{\lambda} = \mathcal{O}(h^2)$, where $\tilde{\lambda}$ is the minimal eigenvalue of $A$; For smoothers $M$, that correspond to scaled Jacobi, Gauss-Seidel, Schwarz methods with small subdomains or Richardson method, we have that

$$\|v\|_{\tilde{M}} \lesssim \|v\|_{\ell_2}, \quad \text{with} \quad \|v\|_{\tilde{M}}^2 := (\tilde{M}v, v)_{\ell_2}.$$  

(4.6)

To obtain an estimate $K$ for any choice of coarse space, let $\tilde{Q}$ be a projection operator from $V$ on the coarse space $V_0$. From the fact that for any given $v \in V$, $\pi_{\tilde{M}}v$ is the closest element to $v$ from $V_0$ in the $\|\cdot\|_{\tilde{M}}$ norm, and also equation (4.6), we get the following simple estimate on $K$:

$$K \sup_{v \in V} \frac{||(I - \pi_{\tilde{M}})v||_{\tilde{M}}^2}{||v||^2_{\ell_2}} \leq \sup_{v \in V} \frac{||(I - \tilde{Q})v||_{\tilde{M}}^2}{||v||^2} \lesssim \sup_{v \in V} \frac{||(I - \tilde{Q})v||_{\ell_2}^2}{||v||^2}.$$  

A uniform convergence result (thus providing a uniform preconditioner) will then follow if we define appropriate $\tilde{Q}$, and prove estimates of the form

$$||(I - \tilde{Q})v||_{\ell_2}^2 \lesssim \|v\|^2, \quad \forall v \in V,$$  

(4.7)

for all choices of $V_0$ – piecewise linear and continuous elements, piecewise constant elements, and Crouzeix–Raviart elements.

4.5. Definition of projection operators for different cases. In this Subsection we define appropriate projections $\tilde{Q}$ that satisfy inequality (4.7). It is well known that the bilinear form that corresponds to $A$ can be written as

$$(Au, v)_{\ell_2} = (\tilde{A}_D u, v)_{\ell_2} + (\tilde{A}_P u, v)_{\ell_2}, \quad (\tilde{A}_D u, v)_{\ell_2} = \sum_e a_e \delta_{e,T} u \delta_{e,T} v,$$  

(4.8)

where as before $A_P$ is the part corresponding to the penalty term, and the summation in the first part of the above expression is taken over all edges in the graph representing the stiffness matrix $A$. Here, $\delta_{e,T} u = u_{i,T} - u_{j,T}$ denote the difference between the components of the vector of degrees of freedom, $u$. To be more precise, $i$ and $j$ are vertices from the triangulation, forming the edge $e$ and $u_{i,T}$ and $u_{j,T}$ are the corresponding degrees of freedom (see Figure 2). According to the discussion at the beginning of Subsection 4.4 we assume that the coefficients $a_e$ in the above
bilinear form do not depend on the mesh size $h$. It is also easy to see that the following equivalence relation holds:

$$\langle Du, u \rangle_{L^2} = h^{2-d} \sum_{T \in T} a|\nabla u|^2 \approx \sum_{T \in T} \sum_{i,j=1}^{d+1} (u_i,T - u_j,T)^2. \quad (4.9)$$

where “$\approx$” means that the two bilinear forms are equivalent, with constants that may depend on the coefficient $a$, the shape (angles) of $T$, and $d$, but are independent on $h$. The proof of such relation is obtained, for example, by mapping to the canonical simplex $T$ and then using that both left side and the right side are norms on the factor space $P_1(T)/\mathbb{R}$.

In establishing estimates like $(4.7)$ we will need some simple notions from graph theory. For each vertex $i$ of the triangulation $T$, we define $G(i) = (V(i), E(i))$ to be the graph, whose vertexes $V(i)$ are the degrees of freedom corresponding to this vertex in every simplex $T$ that contains $i$ (see Fig. 3), namely

$$V(i) = \{ (i, T) : T \supset i \}.$$  

Note that set of vertices $V(i)$ can be identified with the set of simplices that have $i$ as a vertex. With this identification, we say that a pair of simplexes $(T^+, T^-)$ is an element of $E(i)$ (that is a forms an edge in $G(i)$) if and only if the intersection of $T^+$ and $T^-$ is a $(d-1)$ dimensional face. In another word, $G(i)$ will correspond to a “cycle” in two dimensions and to a 3-regular planar graph (i.e. planar graph, such that all its vertices have valence exactly 3) in three spatial dimensions. The cardinality of $V(i)$ we denote with $m_i$. One main assumption that we shall make is that $m := \max_{1 \leq i \leq n} m_i$ is bounded independently of the mesh size $h$. This assumption clearly holds true for shape regular meshes (as long as the angles in each simplex are bounded). A simple result that we need from graph theory is an estimate related to the second eigenvalue of the graph Laplacian of a graph $V(i)$ for fixed $i$: (see for example Fiedler [17])

$$\sum_{(s,T) \in V(i)} (u_s,T - \frac{1}{m_i} \sum_{(k,T) \in V(i)} u_k,T)^2 \leq \frac{2\mu}{\Phi^2} \sum_{e \in E(i)} (\delta_e u)^2; \quad (4.10)$$

where $\Phi$ is the isoperimetric characteristic of the underlying graph and $\mu$ is the maximum vertex valence in the graph. A simple estimate on $\Phi$ (see Fiedler [17]) gives that $\Phi \geq \frac{2}{m}$. There are more refined estimates, but we will not need them for our considerations.

The projection operators $\tilde{Q}$ in the various cases are then defined as follows:
For piecewise linear continuous elements, we denote the projection with $\tilde{Q}_1$. We need to define the values of $\tilde{Q}_1$ at the vertices of the triangulation. For every such vertex $i$, we set:

$$(\tilde{Q}_1v)_i = \frac{1}{m_i} \sum_{(k,T) \in V(i)} v_{k,T}.$$ 

For piecewise constant coarse space, we define a value for $\tilde{Q}_0v$ for each simplex of the triangulation by

$$(\tilde{Q}_0v)_T = \frac{1}{d+1} \sum_{k \in T} v_{k,T}.$$ 

Finally, for the Crouzeix-Raviart coarse space, we take as a projection, the $\ell_2$-orthogonal projection on the Crouzeix-Raviart degrees of freedom, denoted with $\tilde{Q}_{CR}$ and defined as $Q_{CR} := P_{CR}(P_{CR}^T P_{CR})^{-1} P_{CR}^T$, where $P_{CR}$ is the coefficient matrix representing the canonical basis of the Crouzeix-Raviart space by the discontinuous space.

The corresponding estimate (4.7) for $\tilde{Q}$ in all cases follows directly from (4.10). We summarize this observation in the following lemma

**Lemma 4.2.** The following weak approximation properties hold (cf. (4.7)):

$$\|u - \tilde{Q}u\|_{L^2} \lesssim \|u\|.$$ 

where $\tilde{Q}$ is any of the projections $\tilde{Q}_1$, $\tilde{Q}_0$ or $\tilde{Q}_{CR}$.

**Proof.** For $\tilde{Q}_1$ we have

$$\|u - \tilde{Q}_1u\|_{L^2}^2 = \sum_T \sum_{i \in T} (u_{i,T} - (\tilde{Q}_1u)_{i,T})^2$$

$$\leq \sum_i m_i^2 \sum_{e \in E(i)} (\delta_e u)^2$$

$$\leq m^2 (A_P u, u) \lesssim \|u\|^2.$$ 

Here in the second and third line the summation in $i$ is over all vertices of the partition $T$. 
Next, for $\tilde{Q}_0$ we apply the same argument, but sum over all elements
\[
\|u - \tilde{Q}_0 u\|_{l_2}^2 = \sum_T \sum_{i \in T} (u_{i,T} - (\tilde{Q}_0 u)_{T})^2 \lesssim \sum_{T \in T} (d+1) \sum_{i,j=1}^{d+1} (u_{i,T} - u_{j,T})^2 \lesssim \|u\|_{l_2}^2,
\]
where in the last inequality, we have used (4.9).

The proof of the estimate (4.7) for the Crouzeix-Raviart elements, follows from a standard argument, first observing that $\tilde{Q}_1 = \tilde{Q}_{CR} \tilde{Q}_1$, because the piecewise linear elements are subspace of Crouzeix-Raviart space. We then have
\[
\|(I - \tilde{Q}_{CR}) u\|_{l_2}^2 \leq \|(I - \tilde{Q}_{CR})(I - \tilde{Q}_1) u\|_{l_2}^2 \leq \|(I - \tilde{Q}_1) u\|_{l_2}^2 \leq \|u\|_{l_2}^2.
\]
and this completes the proof.

The uniform convergence of the two level method is then direct consequence of the Lemma 4.2 and is stated in the following theorem.

**Theorem 4.3.** The two level method with Gauss-Seidel as a smoother and coarse space $Y^{CC}$ is uniformly convergent with respect to the number of degrees of freedom. The same result is true for the coarse spaces $Y^{CG}$ and $Y^{CR}$ under the additional assumption 3.4.

5. Numerical Experiments

We present two test problems. The first is the Poisson equation with homogeneous Dirichlet boundary data: $-\Delta u = 1$, in $\Omega = (0,1)^3$, $u = 0$, on $\partial\Omega$. The second example is an elliptic PDE with piece-wise constant coefficient: $-\nabla \cdot (a \nabla u) = 1$, in $\Omega = (0,1)^3 \setminus [0.5,1)^3$, $u = 0$, on $\partial\Omega$, the coefficient $a$ has jumps (a 3-D chess board pattern) as follows: $a = 1$, in $(I_1 \times I_1 \times I_1) \cup (I_2 \times I_2 \times I_1) \cup (I_1 \times I_2 \times I_2) \cup (I_2 \times I_1 \times I_2)$ and $a = \epsilon$, in the other parts of $\Omega$, where $I_1 = [0,0.5]$ and $I_2 = [0.5,1]$. For both test examples we have used a coarse tetrahedral mesh which is uniformly refined to form a sequence of meshes. The discretization is the symmetric interior penalty (IP) method using linear and quadratic elements. In both test problems the value of the penalty term is $\kappa = 10$ (cf. (2.2)) for linear, and $\kappa = 20$ for quadratic finite elements.

We have tested two preconditioners:

1. two level method with the “coarse” level being the continuous piecewise polynomial space on the same mesh. Both levels use polynomials of the same degree.
2. multilevel method for the solution of the coarse grid problem, thus obtaining a multilevel preconditioner for the DG space. Both preconditioners use one pre- and post-smoothing step with symmetric Gauss-Seidel smoother.

The intergrid transfer operators are the standard ones, because we have a sequence of uniformly refined grids.

The numerical results are summarized in Tables 1, 2, and 3, where we report the number of iterations/average reduction factor (in PCG) for the two level (tl) and the multigrid (mg) preconditioners. For the first set of experiments, we present results for both linear and quadratic finite elements in Tables 1 and 2, respectively. For the second test we only present results for the linear finite elements.
Table 1. Numerical results for the Poisson problem for linear FE. The total number of degrees of freedom for the DG method is $n_h$ and $n_{c,h}$ is the number of degrees of freedom for the “coarse” space of continuous FE.

<table>
<thead>
<tr>
<th>Level</th>
<th>$n_h/n_{c,h}$</th>
<th>$n_h/n_{c,h}$</th>
<th>$n_h/n_{c,h}$</th>
<th>$n_h/n_{c,h}$</th>
<th>$n_h/n_{c,h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>384/35</td>
<td>3,072/189</td>
<td>24,576/1,241</td>
<td>196,608/9,009</td>
<td>1,572,864/68,705</td>
</tr>
<tr>
<td>tl</td>
<td>11 / 0.1584</td>
<td>13 / 0.2307</td>
<td>13 / 0.2304</td>
<td>13 / 0.2238</td>
<td>12 / 0.2094</td>
</tr>
<tr>
<td>mg</td>
<td>11 / 0.1604</td>
<td>13 / 0.2322</td>
<td>13 / 0.2365</td>
<td>13 / 0.2337</td>
<td>13 / 0.2252</td>
</tr>
</tbody>
</table>

Table 2. Numerical results for the Poisson problem for quadratic FE. The total number of degrees of freedom for the DG method is $n_h$ and $n_{c,h}$ is the number of degrees of freedom for the “coarse” space of continuous FE.

<table>
<thead>
<tr>
<th>Level</th>
<th>$n_h/n_{c,h}$</th>
<th>$n_h/n_{c,h}$</th>
<th>$n_h/n_{c,h}$</th>
<th>$n_h/n_{c,h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>960/189</td>
<td>7,680/1,241</td>
<td>61,440/9,009</td>
<td>491,520/68,705</td>
</tr>
<tr>
<td>tl</td>
<td>9 / 0.1237</td>
<td>10 / 0.1478</td>
<td>10 / 0.1425</td>
<td>9 / 0.1267</td>
</tr>
<tr>
<td>mg</td>
<td>10 / 0.1414</td>
<td>11 / 0.1729</td>
<td>11 / 0.1725</td>
<td>11 / 0.1655</td>
</tr>
</tbody>
</table>

The results shown in Table 3 are for $\epsilon = 1$ (no jumps), $\epsilon = 0.1,0.01$, and $\epsilon = 0.001$. We have used the same discretization settings as for the model Poisson problem on the unit cube. Note that in this second set, we have a non-convex domain, and also the coefficients have jumps up to several orders of magnitude. Additional assumption on the coarse meshes in this case is that the coarse grid element boundaries match with the interfaces, where the coefficient has jumps. Table 3 gives the average convergence factor and number of PCG iterations for the two level and multilevel methods respectively.

The evident conclusion from all the tables, is that all tested preconditioners lead to optimal order methods; the total number of iterations and resulting convergence factors stay fairly insensitive with respect to the mesh sizes (or number of refinement levels).

6. Conclusions

We have presented a unified two–level scheme that leads to uniformly convergent two level methods for DG finite element discretizations. Our examples of auxiliary coarse spaces included continuous or Crouzeix–Raviart finite elements, as well as piecewise constants. A stable multilevel hierarchy in the piecewise linear continuous space can be obtained in a standard way (and this was the hierarchy that we used in our numerical experiments). For the Crouzeix–Raviart spaces, as it is well–known, the interior penalty term with the jumps (denoted here with $(A_{P\cdot}, \cdot)$) may be omitted. There are a number of MG methods devised for the space $V^{CR}$, cf., e.g., [4], [8], [7], [29]). These methods, in conjunction with the two–level technique presented in this paper provide optimal multilevel preconditioners as well.
From practical point of view, however, using piece-wise constants as an auxiliary coarse space is the most appealing case, since it is the simplest one for applying our methods to more general, unstructured, meshes, and its matrix representation is the graph Laplacian. Of course in this case some additional challenges in the analysis need to be overcome, since a simple piecewise constant interpolation is generally unstable and does not lead to uniform MG $V$–cycle convergence.

As mentioned previously, preconditioning of cell-centered finite difference approximations, resulted from mixed finite element methods on uniform rectangular meshes was studied in [5] for nested refinements. In this case the piece-wise constants have the needed approximation property and the iteration methods were studied in the general framework of multigrid method [4]. We also outlined one possible approach (based on the smoothed aggregation AMG method) in Remark 2.5. In the case of hierarchy of uniformly refined meshes, generalized $W$–cycle MG, the so–called AMLI–cycle (cf. [28]) needs to be employed, and we can prove a uniform convergence result in this case as well (see [23]). The picture will be more complete if such result can be extended to handle grids that are not necessarily obtained by uniform refinement. This however is a topic of a current research.

**References**


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