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Perturbation to Noether Symmetry and Noether Adiabatic Invariants of General Discrete Holonomic Systems *

ZHANG Ming-Jiang(张明江), FANG Jian-Hui(方建会)**, LU Kai(路凯), ZHANG Ke-Jun(张克军), LI Yan(李燕)

College of Physics Science and Technology, China University of Petroleum, Dongying 257061

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The perturbation to Noether symmetry and Noether adiabatic invariants of general discrete holonomic systems are studied. First, the discrete Noether exact invariant induced directly from the Noether symmetry of the system without perturbation is given. Secondly, the concept of discrete high-order adiabatic invariant is presented, the criterion of the perturbation to Noether symmetry is established, and the discrete Noether adiabatic invariant induced directly from the perturbation to Noether symmetry is obtained. Lastly, an example is discussed to illustrate the application of the results.

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Noether first revealed the close connection between symmetries and the existence of conserved quantities of mechanical systems in 1918,\cite{1} which is an intriguing mechanical problem. The researches on symmetries and conserved quantities of mechanical systems have not only important mathematical significance, but also a profound physical background. In Refs.\cite{2,3}, symmetries and conserved quantities of constrained mechanical systems have been thoroughly investigated. Recently, researches on symmetries and conserved quantities have been extended to the discrete mechanical systems. Levi et al.\cite{4-6} first extended Lie symmetry to discrete systems, Dorodnitsyn\cite{7} adapted Noether’s theory to discrete Lagrangian system, Shi et al.\cite{8,6} studied Lie symmetry, Mei symmetry and conserved quantities of general discrete holonomic systems, and Fu et al.\cite{10} investigated Noether symmetry of discrete mechanico-electrical systems.

However, even a tiny change in symmetries, which can be called a perturbation, may influence the original symmetries and conserved quantities of mechanical systems. Pioneering in this area, Burgers proposed adiabatic invariants for a special kind of Hamilton system.\cite{11} A classical adiabatic invariant is a certain physical quantity that changes more slowly than some slowly-varying parameters of the system.\cite{12} In fact, the parameter varying very slowly is equivalent to the action of a small disturbance. Perturbation to symmetries and adiabatic invariants play a very important role in the research on the quasi-integrability of mechanical systems. At present, studies in this field have become very active, and many important results have been obtained.\cite{13-22} However, all these studies have focused on the perturbation to symmetries and adiabatic invariants of the continual mechanical systems. Perturbation to symmetries and adiabatic invariants of the discrete mechanical systems has never been studied so far.

In this Letter, based on the concept of the discrete high-order adiabatic invariant, the perturbation to Noether symmetry and Noether adiabatic invariants of the general discrete holonomic system are studied.

For brevity of notation, we consider a one-dimensional discrete mechanical system denoted by its discrete versions of Lagrangian $L_d(t_k,t_{k+1},q_k,q_{k+1})$ and generalized force $Q_d(t_k,t_{k+1},q_k,q_{k+1}) (k = 0,1,\ldots,N−1)$, then $t_k$ and $q_k$ are discrete time and discrete generalized coordinates respectively.

The discrete Euler–Lagrange equation and energy evolution equations of the system are\cite{9}

\begin{equation}
D_3L_d(\phi_k)(t_{k+1}−t_k)+D_4L_d(\phi_{k−1})(t_k−t_{k−1})+Q_d(\phi_k)(t_{k+1}−t_k)=0,\tag{1}
\end{equation}

\begin{equation}
D_1L_d(\phi_k)(t_{k+1}−t_k)+D_2L_d(\phi_{k−1})(t_k−t_{k−1})+L_d(\phi_{k−1})−L_d(\phi_k)−Q_d(\phi_k)(q_{k+1}−q_k)=0,\tag{2}
\end{equation}

where $D_j$ is the partial derivative of the discrete function with respect to the argument $j$ and $\phi_k=(t_k,t_{k+1},q_k,q_{k+1})$ represents the discrete sequence. Equations (1) and (2) can also be written as

\begin{equation}
D_3L_d(\phi_k)+\frac{t_k−t_{k−1}}{t_{k+1}−t_k}D_4L_d(\phi_{k−1})+Q_d(\phi_k)=0,\tag{3}
\end{equation}

\begin{equation}
D_1L_d(\phi_k)+\frac{t_k−t_{k−1}}{t_{k+1}−t_k}D_2L_d(\phi_{k−1})
+\frac{L_d(\phi_{k−1})−L_d(\phi_k)}{t_{k+1}−t_k}−Q_d(\phi_k)\frac{q_{k+1}−q_k}{t_{k+1}−t_k}=0.\tag{4}
\end{equation}

We introduce the infinitesimal transformations as

\begin{equation}
t_k^* = t_k + \Delta t_k = t_k + \varepsilon \tau_k^0(t_k,q_k),
q_k^* = q_k + \Delta q_k = q_k + \varepsilon \xi_k^0(t_k,q_k),\tag{5}
\end{equation}

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**Email: fangjh@upc.edu.cn

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where $\varepsilon$ is an infinitesimal group parameter, $\tau_k^0$ and $\xi_k^0$ are the discrete infinitesimal generators. The vector field of generators is
\begin{equation}
X_{0,d}^{(0)} = \tau_k^0 \frac{\partial}{\partial t_k} + \xi_k^0 \frac{\partial}{\partial q_k},
\end{equation}
which can be prolonged to the two-point scheme
\begin{equation}
X_{0,d}^{(1)} = X_{0,d}^{(0)} + \tau_{k+1}^0 \frac{\partial}{\partial t_{k+1}} + \xi_{k+1}^0 \frac{\partial}{\partial q_{k+1}}.
\end{equation}
The recursive and derivative operators of discrete systems for any discrete variable or function are represented as
\begin{equation}
R_\pm f(z_k) = f(z_{k\pm 1}),
\end{equation}
\begin{equation}
D_d f(z_k) = \frac{R_+ f(z_{k+1}) - f(z_k)}{t_{k+1} - t_k}.
\end{equation}
As we know, Noether symmetry is an invariance of the Hamiltonian action functional under the infinitesimal transformations. Then the requirement of Noether symmetry of the system gives
\begin{equation}
L_d(\phi_k) D_d(\tau_k^0) + X_{0,d}^{(1)}[L_d(\phi_k)] + Q_d(\phi_k) |\xi_k^0 = 0,
\end{equation}
where $G_{NK}^0 = G_{NK}^0(t_k, q_k)$ is a discrete gauge function.

Criterion 1. For the general discrete holonomic system Eqs. (3) and (4), if a discrete gauge function $G_{NK}^0$ exists such that the infinitesimal generators $\tau_k^0$ and $\xi_k^0$ satisfy Eq. (10), the invariance is Noether symmetry of the system.

Equation (10) is called the discrete Noether identity of the general discrete holonomic system Eqs. (3) and (4).

Proposition 1. For the general discrete holonomic system Eqs. (3) and (4), if the infinitesimal generators $\tau_k^0$ and $\xi_k^0$ of Noether symmetry and discrete gauge function $G_{NK}^0$ satisfy the discrete Noether identity Eq. (10), the Noether symmetry of the system can induce the discrete Noether exact invariant
\begin{equation}
I_{N0,d} = \tau_k^0 R_d L_d(\phi_k) + \xi_k^0 (t_k - t_{k-1}) D_d R_d L_d(\phi_k) + G_{NK}^0 = \text{const.}
\end{equation}

According to the concept of adiabatic invariant, the discrete mechanical system, we have

Definition. If $L_{\varepsilon,d}(t_k, q_{k+1}, q_k, q_{k+1}, \varepsilon)$ is a physical quantity including $\varepsilon$ in which the highest power is $z$ in a discrete mechanical system, and its derivative with respect to discrete time $t_k$ is directly proportional to $\varepsilon^{z+1}$, $L_{\varepsilon,d}$ is called a $z$th order adiabatic invariant of the discrete mechanical system.

Suppose the general discrete holonomic system Eqs. (3) and (4) is perturbed by small quantity $\varepsilon W_d(t_k, t_{k+1}, q_k, q_{k+1}) = \varepsilon W_d(\phi_k)$, then we have the following total variation of corresponding discrete action functional
\begin{equation}
\Delta S_d + \sum_{k=0}^{N-1} [Q_d(\phi_k + \varepsilon W_d(\phi_k))((t_{k+1} - t_k)\Delta q_k - \sum_{k=0}^{N-1} [Q_d(\phi_k + \varepsilon W_d(\phi_k))((q_{k+1} - q_k)\Delta t_k = 0,
\end{equation}
where
\begin{equation}
S_d = \sum_{k=0}^{N-1} L_d(t_k, t_{k+1}, q_k, q_{k+1})(t_{k+1} - t_k).
\end{equation}
Using the similar deduction in Ref. [8], we obtain the discrete Euler–Lagrange equation and energy evolution equation of the general discrete holonomic system perturbed by small quantity $\varepsilon W_d(\phi_k)$:
\begin{equation} \begin{aligned}
D_3 L_d(\phi_k) + (\overline{t_k - t_{k-1}}) D_d L_d(\phi_k - 1) \\
+ Q_d(\phi_k + \varepsilon W_d(\phi_k)) = 0,
\end{aligned} \end{equation}
\begin{equation} \begin{aligned}
D_1 L_d(\phi_k) + (\overline{t_k - t_{k-1}}) D_d L_d(\phi_k - 1) \\
\sum_{k=0}^{N-1} L_d(t_k, t_{k+1}, q_k, q_{k+1})(t_{k+1} - t_k).
\end{aligned} \end{equation}
Because of the action of $\varepsilon W_d(\phi_k)$, the original symmetries and invariants of the system may vary. Assume that the variation is a small perturbation based on the symmetrical transformations of the system without perturbation, and $\tau_k(t_k, q_k)$ and $\xi_k(t_k, q_k)$ express the generators of infinitesimal transformations after perturbed, then
\begin{equation}
\tau_k = \tau_k^0 + \varepsilon \tau_k^1 + \varepsilon^2 \tau_k^2 + \cdots,
\end{equation}
\begin{equation}
\xi_k = \xi_k^0 + \varepsilon \xi_k^1 + \varepsilon^2 \xi_k^2 + \cdots.
\end{equation}
The infinitesimal transformations become
\begin{equation}
t_k = t_k + \Delta t_k = t_k + \varepsilon \tau_k(t_k, q_k),
\end{equation}
\begin{equation}
q_k = q_k + \Delta q_k = q_k + \varepsilon \xi_k(t_k, q_k).
\end{equation}
According to the Noether symmetry theory, the Noether identity of the general discrete holonomic system perturbed by small quantity $\varepsilon W_d(\phi_k)$ becomes
\begin{equation}
L_d(\phi_k) D_d(\tau_k) + X_{d}^{(1)}[L_d(\phi_k)] + [Q_d(\phi_k + \varepsilon W_d(\phi_k))]
\cdot \left[ \overline{t_k - D_d (q_k) \tau_k} + D_d (G_{NK}) = 0, \right.
\end{equation}
where
\begin{equation}
X_d^{(1)} = X_d^{(0)} + \tau_{k+1} \frac{\partial}{\partial t_{k+1}} + \xi_{k+1} \frac{\partial}{\partial q_{k+1}},
\end{equation}
and $G_{NK} = G_{NK}(t_k, q_k)$ is a gauge function. After perturbed, the gauge function comes into
\begin{equation}
G_{NK} = G_{NK}^0 + \varepsilon G_{NK}^1 + \varepsilon^2 G_{NK}^2 + \cdots.
\end{equation}
Substituting Eqs. (16) into Eq. (19), we have
\[ X_d^{(1)} = \varepsilon^m X_{m,d} \quad (m = 0, 1, \ldots, z), \]  
where
\[ X_{m,d}^{(1)} = \tau_m^k \frac{\partial}{\partial t_k} + \xi_m^k \frac{\partial}{\partial q_k} + \tau_{m+1}^k \frac{\partial}{\partial t_{k+1}} + \xi_{m+1}^k \frac{\partial}{\partial q_{k+1}}. \]  
Substituting Eqs. (16) into Eq. (18), noticing Eqs. (21)–(23), and making the coefficients of \( \varepsilon^m \) equal, we obtain
\[ L_d(\phi_k)D_d(\tau_m^k) + X_{m,d}[L_d(\phi_k)] + Q_d(\phi_k)[\xi_m^k - D_d(q_k)\tau_m^k] + W_d(\phi_k)[\tau_m^k] = 0, \]  
when \( m = 0 \). We note that \( \tau_m^{-1} = \xi_m^{-1} = 0 \) holds, then Eq. (21) turns into Eq. (10).

**Criterion 2.** For the general discrete holonomic system Eqs. (3) and (4) perturbed by small quantity \( \varepsilon \) \( W_d(\phi_k) \), if the discrete gauge function \( G_{m,k}^m \) exists such that the infinitesimal generators \( \tau_m^k \) and \( \xi_m^k \) satisfy Eq. (24), the corresponding variety of Noether symmetry is called the perturbation to Noether symmetry.

Equation (24) can be called the criterion equation of the perturbation to Noether symmetry of general discrete holonomic system Eqs. (3) and (4).

**Proposition 2.** For the general discrete holonomic system Eqs. (3) and (4) perturbed by small quantity \( \varepsilon \) \( W_d(\phi_k) \), if the infinitesimal generators \( \tau_m^k \) and \( \xi_m^k \) and discrete gauge function \( G_{m,k}^m \) satisfy the criterion Eq. (24), the perturbation to Noether symmetry of the system can induce a \( z \)th order discrete adiabatic invariant
\[ I_{N,z,d} = \varepsilon^m \left\{ \tau_m^k R_{-L_d(\phi_k)} + \tau_m^k (t_k - t_k) D_k[R_{-L_d(\phi_k)}] + \xi_m^k (t_k - t_k) D_k[R_{-L_d(\phi_k)}] + G_{m,k}^m \right\}, \]
\( (m = 0, 1, \ldots, z). \)

**Proof:** Using representations Eqs. (8), (9) and (23), the expansion of Eq. (24) gives
\[ L_d(\phi_k)D_d(\tau_m^k) + X_{m,d}[L_d(\phi_k)] + Q_d(\phi_k)[\xi_m^k - D_d(q_k)\tau_m^k] + W_d(\phi_k)[\tau_m^k] = 0, \]
\[ \left. \begin{array}{c}
\tau_m^k \frac{\partial L_d(\phi_k)}{\partial t_k} + \tau_m^k \frac{t_k - t_k - 1 \partial L_d(\phi_k)}{t_k - 1} + \xi_m^k \frac{\partial L_d(\phi_k)}{\partial q_k} \\
+ \tau_m^k \frac{\partial L_d(\phi_k)}{\partial \eta_k} + \tau_m^k \frac{t_k - t_k - 1 \partial L_d(\phi_k)}{t_k - 1} + \xi_m^k \frac{\partial L_d(\phi_k)}{\partial \eta_k} \\
+ \tau_m^k \frac{\partial L_d(\phi_k)}{\partial t_{k+1}} + \tau_m^k \frac{t_k - t_k - 1 \partial L_d(\phi_k)}{t_k - 1} + \xi_m^k \frac{\partial L_d(\phi_k)}{\partial \eta_{k+1}} \\
+ \tau_m^k \frac{\partial L_d(\phi_k)}{\partial \eta_{k+1}} + \tau_m^k \frac{t_k - t_k - 1 \partial L_d(\phi_k)}{t_k - 1} + \xi_m^k \frac{\partial L_d(\phi_k)}{\partial \eta_{k+1}} \\
+ \tau_m^k \frac{t_k - t_k - 1 \partial L_d(\phi_k)}{t_k - 1} + \xi_m^k \frac{\partial L_d(\phi_k)}{\partial \eta_{k+1}} - \tau_m^k \frac{L_d(\phi_k)}{t_k - 1} - t_k \right\} = 0. \]
where \( \mathcal{C} \), Eq. (10) of the system without perturbation, we have an abatic invariant. We also suppose that the generators \( k \) and \( \mathcal{L} \) are linear, i.e.

\[
\tau^0_k(t_k, q_k) = C_1 t_k + C_2 q_k + C_3,
\]

where \( C_1, \ldots, C_6 \) are constants.

Substituting Eqs. (32)–(37) into Noether identity (38), we have

\[
\begin{align*}
- \frac{1}{2} \left( \frac{q_{k+1} - q_k}{t_{k+1} - t_k} \right)^2 D_d(\tau_k^0) + & \frac{q_{k+1} - q_k}{t_{k+1} - t_k} D_d(\xi_k^0) \\
- \frac{q_{k+1} - q_k}{t_{k+1} - t_k} \xi_k^0 + & \left( \frac{q_{k+1} - q_k}{t_{k+1} - t_k} \right)^2 \tau_k^0 \\
+ & D_d(G_{NK}^0) = 0.
\end{align*}
\]

It follows that when

\[
\tau_k^0(t_k, q_k) = 0, \quad \xi_k^0(t_k, q_k) = C_6,
\]

the function

\[
G_{NK}^0 = C_6 q_k,
\]

satisfies the Noether identity Eq. (10). According to proposition 1, the general discrete holonomic system has the following discrete Noether exact invariant

\[
I_{N0,d} = C_6 \left( \frac{q_k - q_{k-1}}{t_k - t_{k-1}} + q_k \right) = \text{const.}
\]

Secondly, we seek the first order discrete Noether adiabatic invariant. We also suppose that the generators \( \tau_k^1 \) and \( \xi_k^1 \) are linear, i.e.

\[
\tau_k^1(t_k, q_k) = C_7 t_k + C_8 q_k + C_9,
\]

where \( C_7, \ldots, C_6 \) are constants.

Substituting Eqs. (32)–(37), (42) and (43) into the criterion equation (24) of the perturbation to Noether symmetry of the system, we have

\[
\begin{align*}
- \frac{1}{2} \left( \frac{q_{k+1} - q_k}{t_{k+1} - t_k} \right)^2 D_d(\tau_k^1) + & \frac{q_{k+1} - q_k}{t_{k+1} - t_k} D_d(\xi_k^1) \\
- \frac{q_{k+1} - q_k}{t_{k+1} - t_k} \xi_k^1 + & \left( \frac{q_{k+1} - q_k}{t_{k+1} - t_k} \right)^2 \tau_k^1 \\
+ & D_d(G_{NK}^1) = 0.
\end{align*}
\]

It follows that when

\[
\tau_k^1(t_k, q_k) = 0, \quad \xi_k^1(t_k, q_k) = C_{12},
\]

the function

\[
G_{NK}^1 = C_6 + C_{12} q_k,
\]

satisfies the criterion equation (24) of the perturbation to Noether symmetry. According to proposition 2, the general discrete holonomic system has the following first order discrete Noether adiabatic invariant

\[
I_{N1,d} = C_6 \left( \frac{q_k - q_{k-1}}{t_k - t_{k-1}} + q_k \right) + C_{12} \left( \frac{q_k - q_{k-1}}{t_k - t_{k-1}} + (C_6 + C_{12}) q_k \right).
\]

Further we can obtain more discrete high-order adiabatic invariants.

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