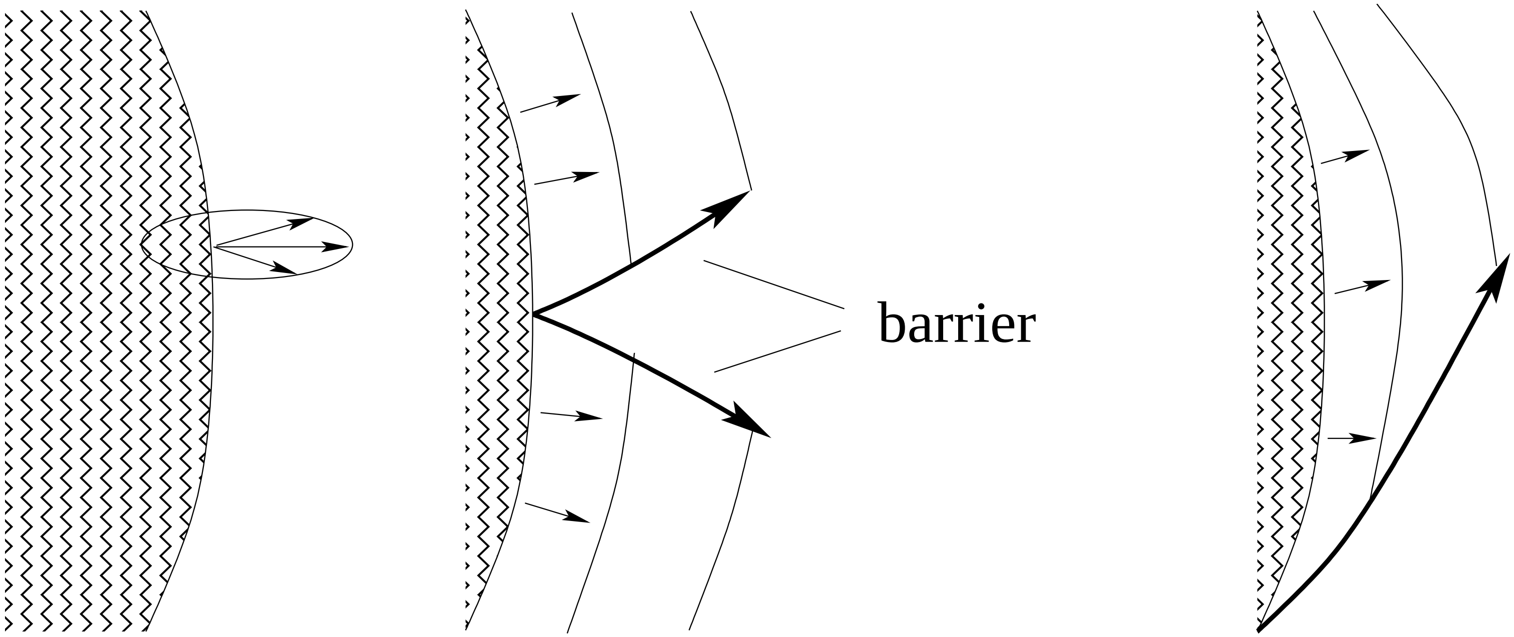
Dynamic Blocking Problems

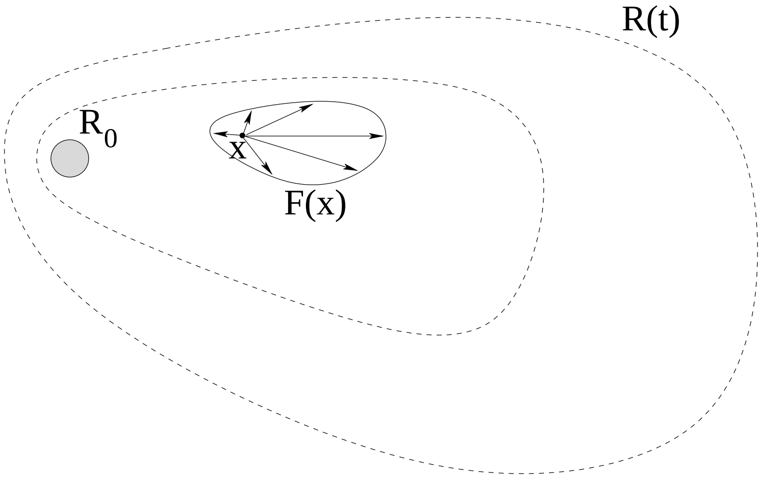
Alberto Bressan

Department of Mathematics, Penn State University

Blocking an advancing wildfire



A Differential Inclusion Model for Fire Propagation



$$R(t) \subset I\!R^2 = \text{set reached by the fire at time } t \geq 0$$

determined as the reachable set by a **differential inclusion**

$$\dot{x} \in F(x) \qquad\qquad x(0) \in R_0 \subset I\!R^2$$

Fire may spread in different directions with different velocities

$$R(t) = \Big\{ x(t)\,; \quad x(\cdot) \ \text{ absolutely continuous}\,,$$

$$x(0) \in R_0\,, \quad \dot{x}(\tau) \in F(x(\tau)) \ \text{ for a.e. } \tau \in [0,t] \Big\}$$

Confinement Strategies

(A.B., J.Differential Equations, 2007)

Assume: a **controller** can construct a **wall**, i.e. a one-dimensional rectifiable curve γ, which blocks the spreading of the fire.

$$\gamma(t) \subset \mathbb{R}^2 = \text{portion of the wall constructed within time } t$$

$$\sigma = \text{speed at which the wall is constructed}$$

Definition 1. A set valued map $t \mapsto \gamma(t) \subset \mathbb{R}^2$ is an **admissible strategy** if :

(H1) For every $t_1 \leq t_2$ one has $\gamma(t_1) \subseteq \gamma(t_2)$

(H2) Each $\gamma(t)$ is a rectifiable set (possibly not connected). Its length satisfies

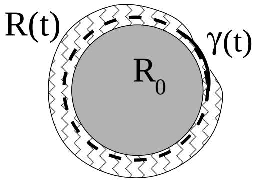
$$m_1(\gamma(t)) \;\leq\; \sigma\, t$$

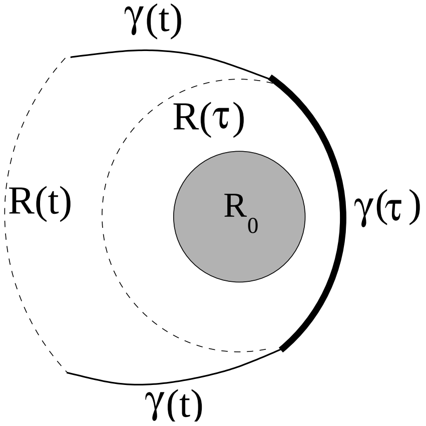
Definition 2. The reachable set determined by the blocking strategy γ is

$$R^\gamma(t) \doteq \Big\{ x(t)\,; \quad x(\cdot) \text{ absolutely continuous}, \quad x(0) \in R_0$$

$$\dot x(\tau) \in F(x(\tau)) \text{ for a.e. } \tau \in [0,t], \qquad x(\tau) \notin \gamma(\tau) \text{ for all } \tau \in [0,t] \Big\}$$

REMARK: Walls must be constructed in **real time !**





An **admissible strategy** is described by a **set-valued function** $t \mapsto \gamma(t) \subset I\!R^2$

$\gamma(t) =$ portion of the wall constructed within time t

Optimal Confinement Strategies

A **cost functional** should take into account

- The value of the region destroyed by the fire.

- The cost of building the wall.

Consider two continuous, non-negative functions $\alpha, \beta : I\!R^2 \mapsto I\!R_+$

$\alpha(x)$ = value of a unit area of land around the point x

$\beta(x)$ = cost of building a unit length of wall near the point x

COST FUNCTIONAL

$$J(\gamma) \doteq \lim_{t\to\infty} \left\{ \int_{R^\gamma(t)} \alpha\, dm_2 + \int_{\gamma(t)} \beta\, dm_1 \right\}$$

MATHEMATICAL PROBLEMS

1. Blocking Problem.

Given an initial set R_0, a multifunction F and a wall construction speed σ, does there exist an admissible strategy $t \mapsto \gamma(t)$ such that **the reachable sets** $R^\gamma(t)$ **remain uniformly bounded** for all $t > 0$?

2. Optimization Problem.

Minimize the cost functional $J(\gamma)$ among all admissible strategies $\gamma \in \mathcal{S}$

(i) **Existence** of an optimal solution

(ii) **Necessary conditions** for the optimality of an admissible strategy $\gamma(\cdot)$

(iii) **Regularity** of the curves $\gamma(t)$ constructed by an optimal strategy

(iv) **Sufficient conditions** for the optimality of a strategy $\gamma(\cdot)$

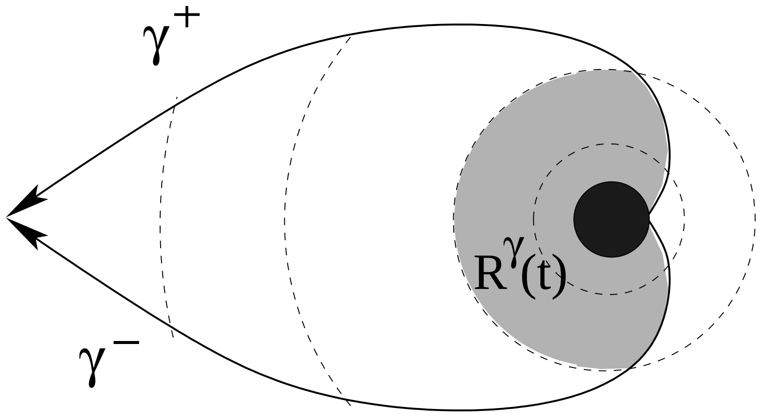
(v) **Numerical computation** of an optimal strategy

Blocking the Fire

- Fire propagates in all directions with unit speed: $F(x) = B_1$
- Wall is constructed at speed σ

Theorem *(A.B., J.Differential Equations, 2007)* On the entire plane, the fire can be blocked if $\sigma > 2$, it cannot be blocked if $\sigma < 1$.

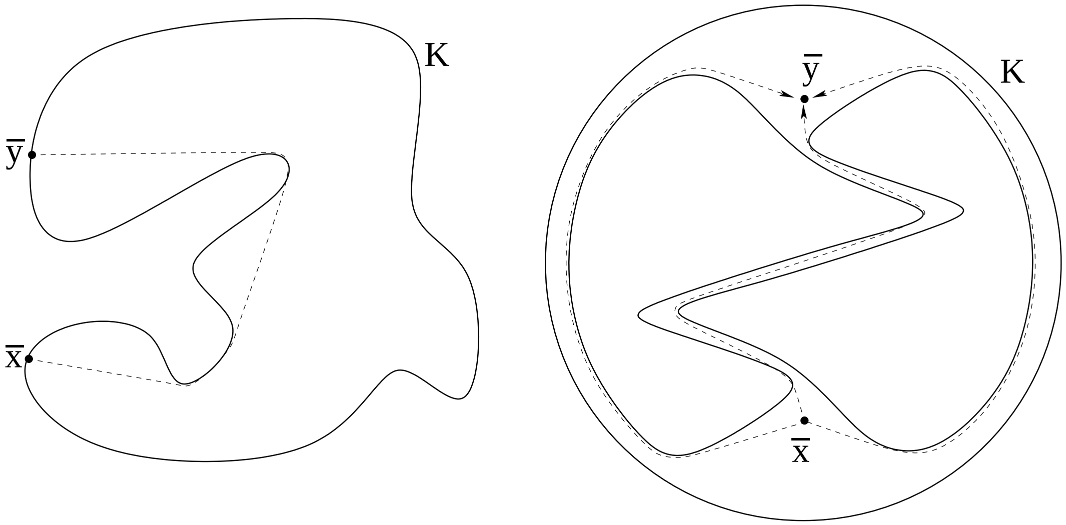
BLOCKING STRATEGY: If $\sigma > 2$, construct two arcs of logarithmic spirals along the edge of the fire



$$\gamma^+(t) \doteq \Big\{(r,\theta)\,;\quad r = e^{\lambda\theta},\quad 1 \le r \le 1+t\Big\}, \qquad \lambda \doteq \frac{1}{\sqrt{\dfrac{\sigma^2}{4} - 1}}$$

Regularity of the distance function

$d_K(x,y) \doteq$ minimum length among all paths $\gamma : [0,1] \mapsto K$ joining x with y



Lemma: *(A.B., T.Wang, J. Math. Anal. Appl. 2009)* If $K \subset \mathbb{R}^2$ is compact, simply connected, then the map $y \mapsto d_K(x,y)$ is $\mathcal{C}^1$ in the interior of $K \setminus \{x\}$.

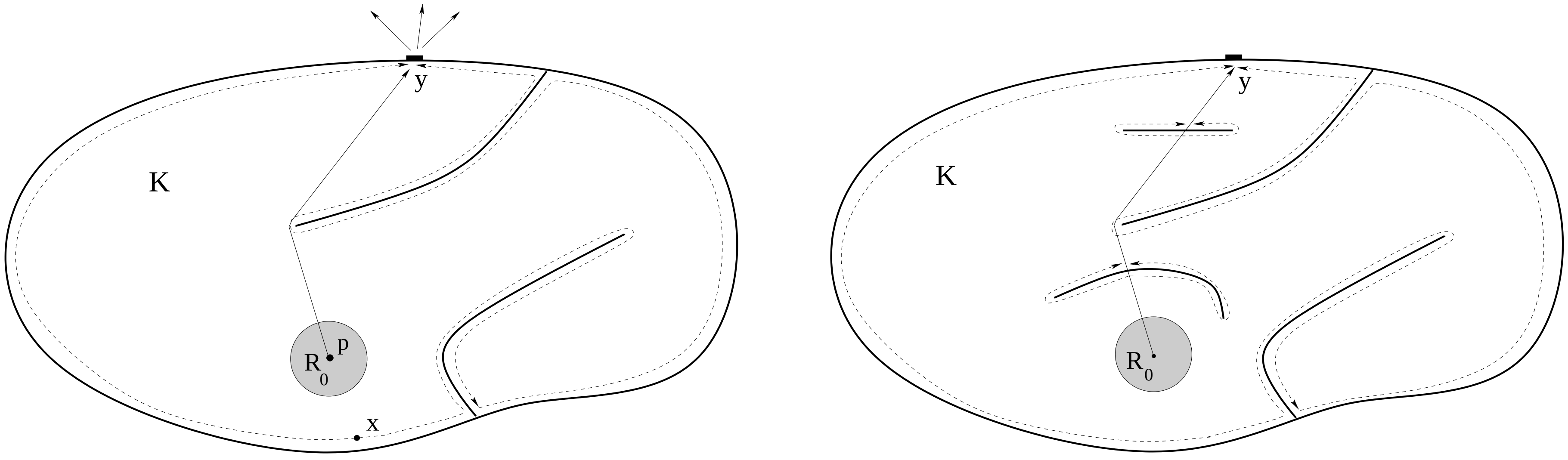
$\sup_{x,y \in K} \; d_K(x,y)$ is attained at boundary points.

No strategy can block the fire if $\sigma \leq 1$

$y =$ position of "last brick of the wall"

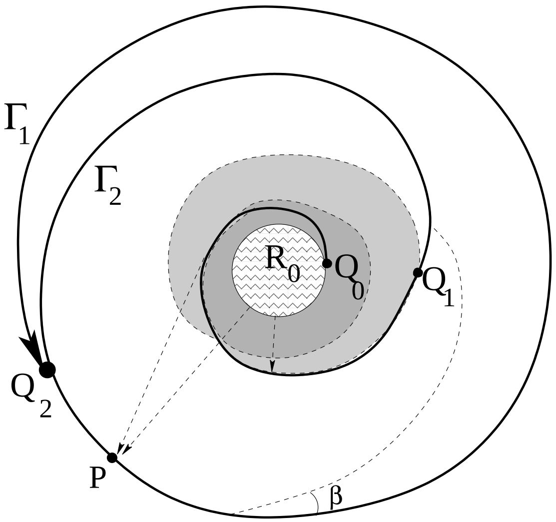
$$d_K(p,y) \;\leq\; \max_{x\in\partial K} d_K(x,y) \;\leq\; \frac{1}{2}\Big(2\,m_1(\partial K)\Big)$$

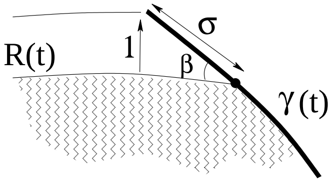
Fire reaches y before time $T < d_K(p,y) - \frac{r}{2}$



When can the fire be blocked ?

Conjecture: Assume the fire propagates with speed 1 in all directions. On the entire plane the fire can be blocked if and only if $\sigma > 2$





$$\sigma \sin\beta = 1$$

Single spiral strategy: curve closes on itself if and only if $\sigma > \sigma^\dagger = 2.614430844\ldots$ (M. Burago, 2006)

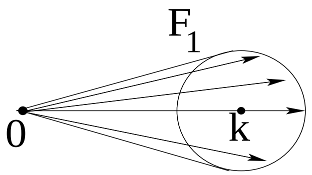
Theorem. Restricted to a half plane, the fire can be blocked if and only if $\sigma > 1$ (A.B. - T.Wang, *J.Math Anal.Appl.* 2009)

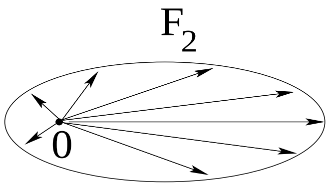
Non-isotropic fire propagation

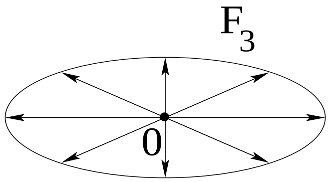
(A.B., M. Burago, A. Friend, and J. Jou, *Analysis and Applications*, 2008)

$$F = \{(r\cos\theta\,,\ r\sin\theta)\,;\quad 0 \le r \le \rho(\theta)\}\,,$$

$$\rho(-\theta) = \rho(\theta)\,, \qquad\qquad 0 \le \rho(\theta') \le \rho(\theta) \qquad\qquad \text{for all}\ \ 0 \le \theta \le \theta' \le \pi\,.$$







Theorem. If the wall construction speed satisfies

$$\sigma\ >\ [\text{vertical width of } F]\ =\ 2\max_{\theta\in[0,\pi]}\ \rho(\theta)\sin\theta$$

then, for every bounded initial set R_0, a blocking strategy exists

Existence of Optimal Strategies

Fire propagation: $\dot{x} \in F(x)$ $x(0) \in R_0$

Wall constraint: $\int_{\gamma(t)} \psi \, dm_1 \leq t$ $(1/\psi(x) =$ construction speed at $x)$

Minimize: $J(\gamma) = \left\{ \int_{R^\gamma(t)} \alpha \, dm_2 + \int_{\gamma(t)} \beta \, dm_1 \right\}$

Assumptions:

(A1) The initial set R_0 is open and bounded. Its boundary satisfies $m_2(\partial R_0) = 0$.

(A2) The multifunction F is Lipschitz continuous w.r.t. the Hausdorff distance. For each $x \in \mathbb{R}^2$ the set $F(x)$ is compact, convex, and contains a ball of radius $\rho_0 > 0$ centered at the origin.

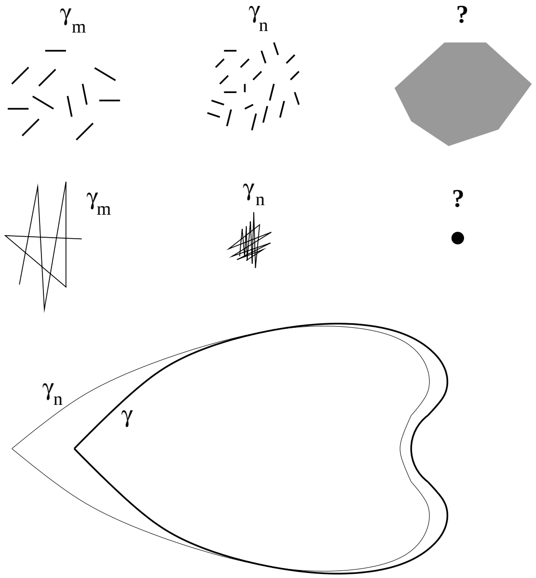
(A3) For every $x \in \mathbb{R}^2$ one has $\alpha(x) \geq 0$, $\beta(x) \geq 0$, and $\psi(x) \geq \psi_0 > 0$. α is locally integrable, while β and ψ are both lower semicontinuous.

Theorem *(A.B. - C. De Lellis, Comm. Pure Appl. Math. 2008).*

Assume (A1)-(A3), and $\inf_{\gamma \in \mathcal{S}} J(\gamma) < \infty$.

Then the minimization problem admits an optimal solution γ^*.

Direct method: Consider a minimizing sequence of strategies $\gamma_n(\cdot)$
Define the optimal strategy γ^* as a suitable limit



STEP 1: Replace the $\gamma_n(\cdot)$ by **complete strategies**

$$\gamma_n^c \doteq \gamma_n(t) \cup \left\{x \in I\!R^2\,; \quad \theta^*(\gamma_n(t), x) > 0\right\}$$

$$\theta^*(E, x) \doteq \limsup_{r \downarrow 0} \frac{m_1(B(x,r) \cap E)}{2r}$$

STEP 2: For each rational time τ, order the connected components of $\gamma_n(\tau)$ (which are **compact**) according to decreasing length:

$$\gamma_n(\tau) = \gamma_{n,1} \cup \gamma_{n,2} \cup \gamma_{n,3} \cup \cdots$$

Taking a subsequence, as $n \to \infty$ for every $\tau \in \mathbf{Q}$ we have

$$\gamma_{n,i}(\tau) \to \gamma_i(\tau) \qquad \text{in the Hausdorff distance}$$

The optimal strategy is $\gamma^c(\tau) \doteq$ completion of $\gamma(\tau) \doteq \bigcup_{i \geq 1} \gamma_i(\tau)$.

$$\gamma(t) \;=\; \bigcap_{\tau > t,\ \tau \in \mathbf{Q}} \gamma(\tau) \qquad\qquad t \in I\!R$$

Necessary conditions for optimality

(A.B., J.Differential Equations, 2007)

$$\text{minimize} \qquad \int_{R^\gamma(T)} \alpha(x)\, dx$$

$$\text{subject to:} \qquad m_1(\gamma(t)) \leq \sigma t \qquad \text{for all } t \geq 0$$

GOAL: derive a set of ODE's describing the walls built by an optimal strategy

DIFFERENT TYPES OF ARCS

(F) Free arc:, constructed away from the fire front

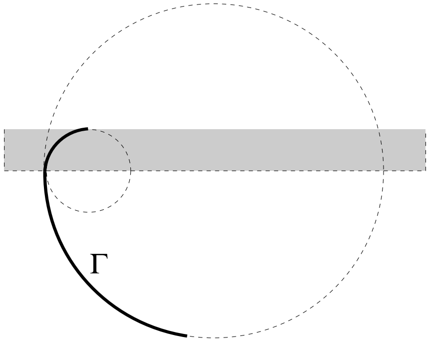
(B) Boundary arc: constructed along the advancing fire front

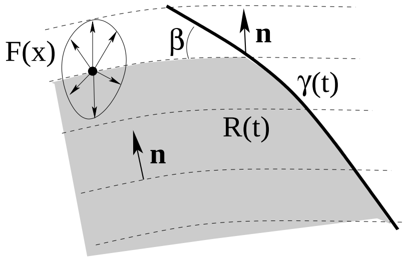
Optimality conditions

1. A free arc Γ. The curvature must be proportional to the local value of the land

$$r(s) = \text{radius of curvature} \qquad \alpha = \text{land value}$$

$$r(s) \cdot \alpha(\Gamma(s)) = \text{const.}$$

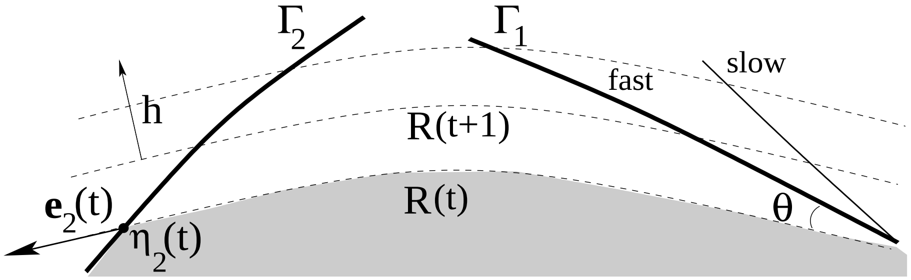




2. A single boundary arc Γ. The wall is constructed at maximum speed σ, always remaining at the edge of the burned set

$$\sigma \sin\beta \;=\; \max_{y\in F(x)} \mathbf{n}\cdot y$$

3. Two or more boundary arcs: $\Gamma_1, \ldots, \Gamma_\nu$, constructed simultaneously for $t \in [a, b]$

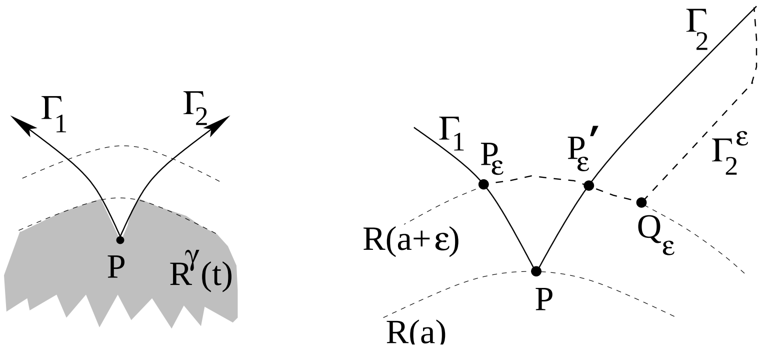


Sum of construction speeds $\leq \sigma$

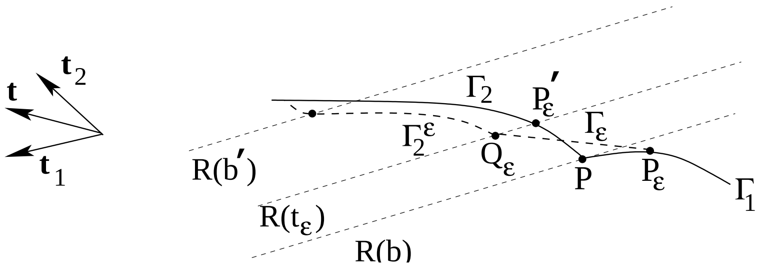
At which speed should each wall be constructed ?

(Solution is found applying Pontryagin's Maximum Principle)

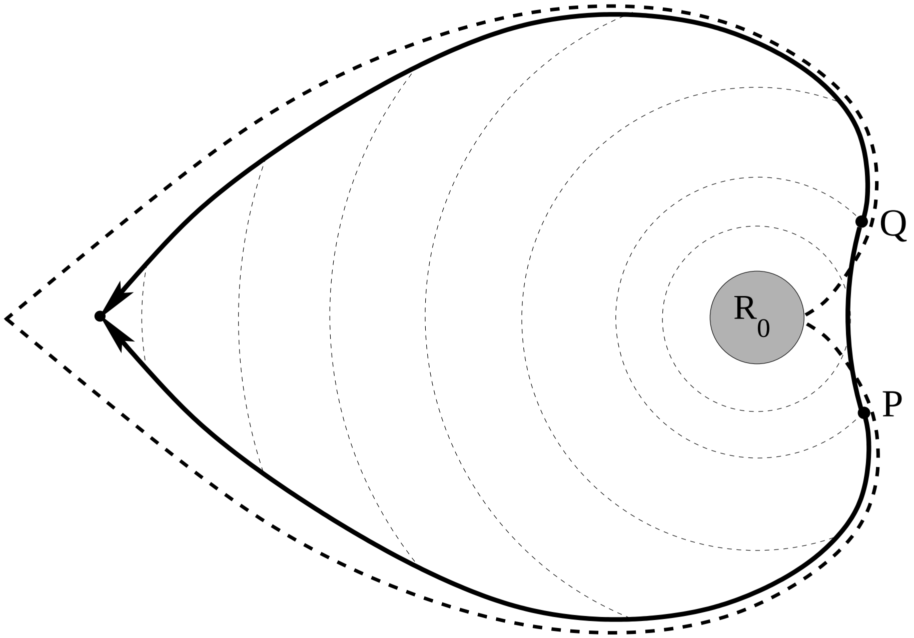
Junctions between different arcs



Two boundary arcs originating at the same point are not optimal



Non-parallel junctions between a free arc and a boundary arc are not optimal



circle + two spirals

(is better than two spirals only)

Equivalent Formulation

(A.B. - T. Wang, *Control Optim. Calc. Var.* 2009)

$$\text{(strategy)} \qquad t \mapsto \gamma(t) \qquad \longleftrightarrow \qquad \Gamma \qquad \text{(single wall)}$$

$$\gamma(\cdot) \qquad \longrightarrow \qquad \Gamma \doteq \left(\bigcup_{t>0} \gamma(t)\right) \setminus \left(\bigcup_{t>0} R^{\gamma}(t)\right) \qquad \text{(useful walls)}$$

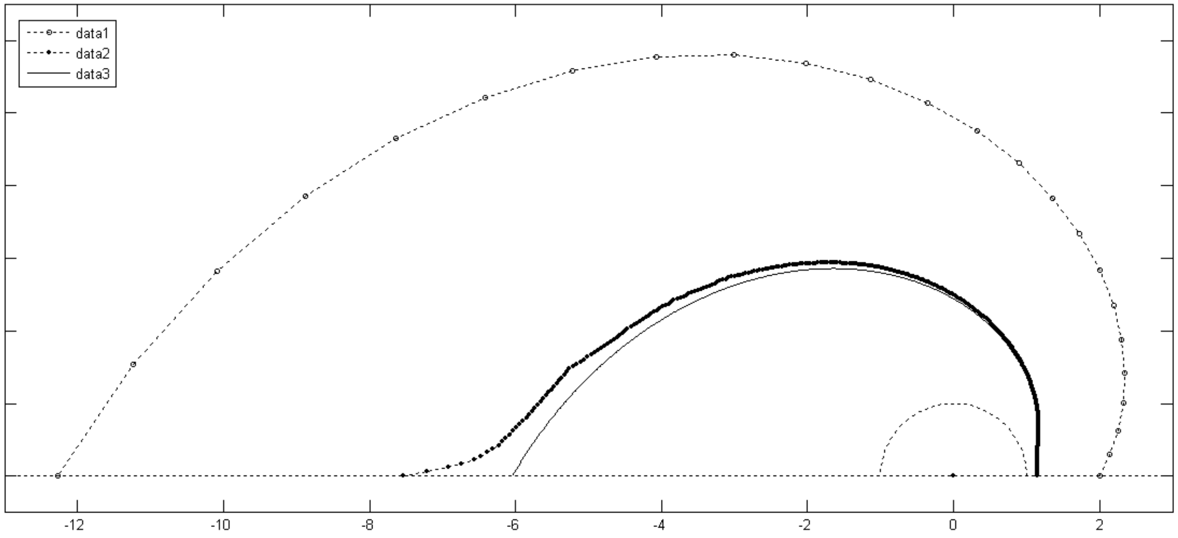
$$\Gamma \qquad \longrightarrow \qquad \gamma(t) \doteq \Gamma \cap \overline{R^{\Gamma}(t)} \qquad \text{(walls touched by the fire within time } t\text{)}$$

- Blocking Problems and Optimization Problems can be reformulated in terms of one single rectifiable set Γ

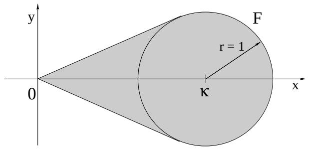
Numerical Experiments

(A.B. - T. Wang, *Control Optim. Calc. Var.* 2009)

1. The isotropic case. $F(x) = R_0 = B_1$ (unit disc), $\sigma = 4$. Minimize: total burned area.

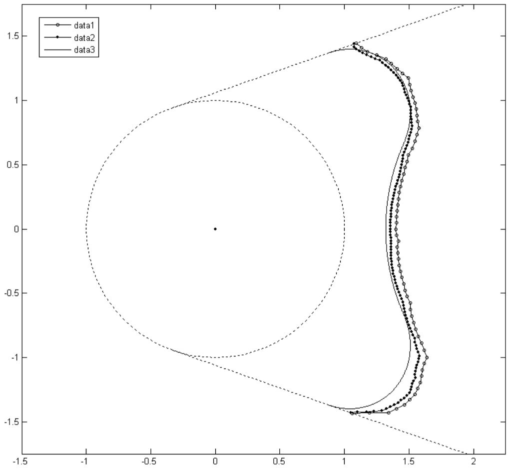


2. A non-isotropic case.



$$F = \Big\{ (\lambda x, \lambda y)\,; \quad (x-3)^2 + y^2 \le 1, \quad \lambda \in [0,1] \Big\}$$

Choose: $\sigma = 4.1$, $R_0 =$ unit disc.



Analytic solution: A.Friend (2007). Numerical solution: T.Wang (2008)

Some open problems

1 (Isotropic blocking problem). On the whole plane, assume:
- fire propagates with unit speed in all directions
- wall construction speed is $\sigma \leq 2$.

Prove that NO blocking strategy exists.

2 (Existence of optimal strategies). Determine whether an optimal strategy exists, in the general case where the velocity sets satisfy $0 \in F(x)$ but without assuming $B(0, \rho) \subset F(x)$.

3 (Regularity). If the initial set R_0 has a smooth boundary and the cost functions are smooth, what is the regularity of an optimal strategy ?
Does it produce a finite number of piecewise $\mathcal{C}^1$ arcs ?
Where can corners occur ?

4 (Regularity). Prove or disprove: For the isotropic problem, if R_0 is convex, the wall constructed by an area-minimizing strategy is connected.

5 (Sufficient conditions). Prove that the “circle + two spirals” strategy is optimal for the isotropic problem.

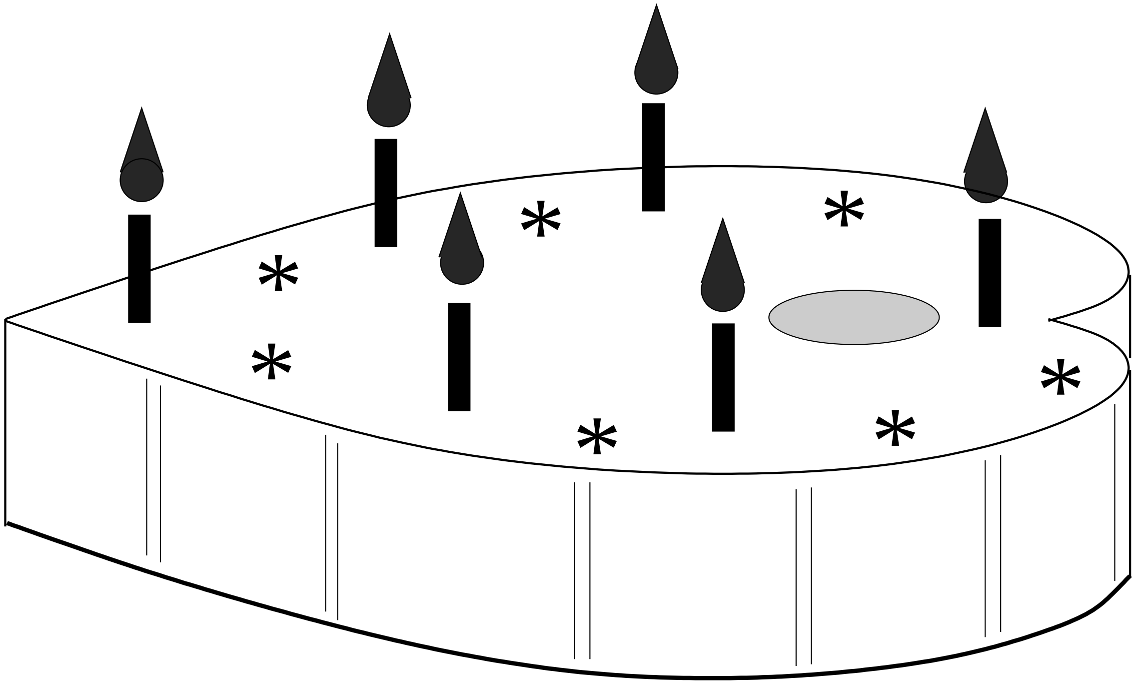
6 (Two fires) Find the optimal strategy for the isotropic problem, when R_0 is the union of two discs.

7 (General necessary conditions for optimality)
For an optimal strategy $\gamma^*(\cdot)$ there exists a global Lagrange multiplier $V(t)$ corresponding to the “instantaneous value of time”. Is it true that:

- The map $t \mapsto V(t)$ is non-increasing, and remains constant during time intervals when free arcs are being constructed ?

- The radius of curvature of the walls can be bounded below in terms of the value of time ?

8 Numerical algorithms to compute optimal strategies
(*Tao Wang)*



Happy Anniversary

Francis !!