Non-cooperative Differential Games

A Homotopy Approach

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Differential Games

$$\frac{d}{dt}x(t) = G(x(t),\, u_1(t),\, u_2(t)), \qquad x(0) = y, \qquad u_i(t) \in U_i$$

x = state of the system

u_1, u_2 = controls of the two players

Goal of the i-th player:

$$\text{maximize:} \qquad J_i = \int_0^\infty e^{-\rho t} L_i(x, u_1, u_2)\, dt$$

Nash non-cooperative equilibrium solution, in feedback form

$$u_1 = u_1^*(x) \qquad\qquad u_2 = u_2^*(x)$$

is a non-cooperative equilibrium solution if

- Given the dynamics

$$\dot{x} = G(x,\, u_1,\, u_2^*(x))$$

The assignment $u_1 = u_1^*(x)$ provides an optimal solution to the optimal control problem

$$\text{maximize } J_1 = \int_0^\infty e^{-\rho t} L_1(x, u_1, u_2^*(x))\, dt$$

for every initial state of the system $x(0) = y$.

- Similarly $u_2 = u_2^*(x)$ should provide a solution to the corresponding optimal control problem for the second player, given that $u_1 = u_1^*(x)$

Basic assumptions:

- each player has complete information about the state of the system

- players do not communicate, or make deals with each other

Goal of the analysis:

- provide indications what is the “best” or the “rational” course of action for each player

- predict players’ behavior in real life situations

- set the rules so that the outcome of the game is “favorable” from the higher standpoint of the collectivity.

Is the “Nash non-cooperative equilibrium” an appropriate model ??

PDE approach: study the value functions

$V_i(y)$ = value function for the i-th player
(= expected payoff, if game starts at y)

$$\dot{x} = G_1(x, u_1) + G_2(x, u_2)$$

$$J_i \doteq \int_0^\infty e^{-\rho t}\big[L_{i1}(x, u_1) + L_{i2}(x, u_2)\big]\, dt$$

$$u_i^*(x, \nabla V_i) = \operatorname*{argmax}_{\omega \in U_i} \Big\{\nabla V_i(x) \cdot G_i(x, \omega) + L_{ii}(x, \omega)\Big\}$$ **optimal feedback controls**

Hamilton-Jacobi equations for the value functions

$$\begin{cases} \rho V_1(x) & = & H^{(1)}(x, \nabla V_1, \nabla V_2) \\ \rho V_2(x) & = & H^{(2)}(x, \nabla V_1, \nabla V_2) \end{cases}$$

$$H^{(1)}(x, \xi_1, \xi_2) = \xi_1 \cdot G\big(x,\, u_1^*(x, \xi_1),\, u_2^*(x, \xi_2)\big) + L_1\big(x,\, u_1^*(x, \xi_1),\, u_2^*(x, \xi_2)\big)$$

$$H^{(2)}(x, \xi_1, \xi_2) = \xi_2 \cdot G\big(x, u_1^*(x, \xi_1),\, u_2^*(x, \xi_2)\big) + L_2\big(x,\, u_1^*(x, \xi_1),\, u_2^*(x, \xi_2)\big)$$

Highly nonlinear system of PDE's - HARD !

Zero-sum games

Assume: $L_1(x, u_1, u_2) \ = \ -L_2(x, u_1, u_2)$

Then $V_1(x) \ = \ -V_2(x)$

The value function can be characterized as the unique **viscosity solution** to a scalar H-J equation

$$\rho V(x) = H(x, \nabla V(x))$$

Linear-quadratic differential games

(T.Basar, G.Olsder...)

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2 \qquad\qquad x \in I\!R^n$$

$$J_i = \int_0^\infty e^{-\rho t} \left[x^T P_i x + x^T Q_i u + u_i^T R_i u_i + \mathbf{h}_i^T x + \mathbf{k}_i^T u \right] dt$$

Value functions can be found within the family of quadratic polynomials

$$V_i(x) = x^T M_i x + \mathbf{b}_i^T x + c_i$$

Solve a system of algebraic equations for M_i, $\mathbf{b_i}$, c_i

Beyond Linear-Quadratic Games

Finite horizon problem with terminal cost:

backward Cauchy problem for the value functions

$$\begin{cases} \dfrac{\partial}{\partial t} V_i + \nabla_x V_i \cdot g(x, u_1^*, u_2^*) = L_i(x, u_1^*, u_2^*) \\ V_i(T, x) = \psi_i(x) \end{cases} \qquad i = 1, 2$$

Is the Cauchy problem well posed ?

$$\xi_i = V_i' \qquad\qquad \xi_i(T,x) = \psi_i'(x)$$

$$(\xi_i)_t + \Big[\xi_i \cdot g(x,\xi_1,\xi_2) - L_i(x,\xi_1,\xi_2)\Big]_x = 0$$

• In one space dimension, if $V_1' \cdot V_2' > 0$ the system is hyperbolic, and the C.P. is well posed

• In one space dimension, if $V_1' \cdot V_2' < 0$ the system is NOT hyperbolic, and the C.P. is ill posed

• In several space dimensions, generically the system is NOT hyperbolic, and the C.P. is ill posed

A.B - W.Shen, Small BV solutions of hyperbolic non-cooperative differential games, *SIAM J. Control Optim.* **43** (2004), 104-215.

A.B. - W.Shen, Semi-cooperative strategies for differential games, *Intern. J. Game Theory* **32** (2004), 561-593.

A Homotopy Technique

Basic setting (one space dimension)

$$\dot{x} \;=\; G(x, u_1, u_2, \theta) \;=\; G_1(x, u_1, \theta) + \theta\, G_2(x, u_2, \theta)$$

goal of i-th player: minimize $J_i \doteq \int_0^\infty e^{-\rho t} L_i(x, u_1, u_2, \theta)\, dt$

$$L_i = L_{i1}(x, u_1, \theta) + L_{i2}(x, u_2, \theta)$$

$\theta = 0 \quad \Longrightarrow \quad$ myopic strategy: $u_2 \;=\; u_2^\dagger(x) \;=\; \operatorname*{argmin}_{\omega \in U_2} L_{22}(x, \omega)$

Optimal control problem for first player:

minimize: $\displaystyle\int_0^\infty e^{-\rho t} L_i(x, u_1, u_2^\dagger(x))\, dt$

subject to: $\dot{x} = G_1(x, u_1(x))$

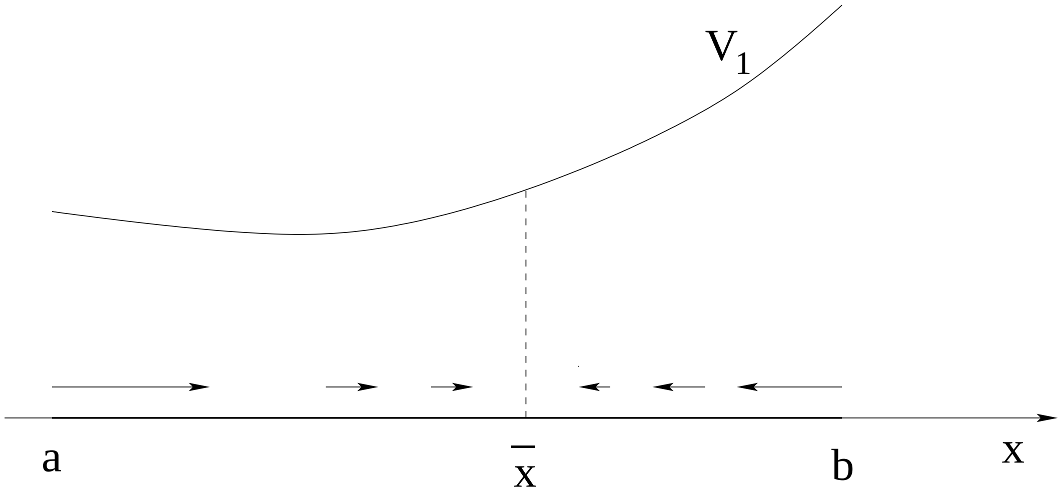
Value function $V_1(x)$ satisfies

$$\rho V_1 = H^{(1)}(x, V_1') \qquad H^{(1)}(x,\xi) = \min_{\omega \in U_1} \left\{ \xi \cdot G_1(x,\omega) + L_1(x,\omega,u_2^\dagger(x)) \right\}$$

Assume: on a fixed interval $I = [a,b]$, for $\theta = 0$ the optimal feedback $u_1^*(x)$ yields a dynamics

$$\dot{x} = G_1(x, u_1^*(x))$$

having a single stable equilibrium point $\bar{x}$



+ smoothness assumptions + transversality assumptions (generically true)

QUESTION : what happens for $\theta > 0$?

An implicit system of ODE's

$$\rho V_i = H^{(i)}(x, V_1', V_2') \qquad\qquad i = 1, 2$$

$$\begin{aligned} u_i^*(x,\xi_i) &= \operatorname*{argmin}_\omega \left\{ \xi_i \cdot G_i(x,\omega) + L_{ii}(x,\omega) \right\} \\ H^{(i)}(x,\xi_1,\xi_2) &= \xi_i \cdot G(x,u_1^*,u_2^*) + L_i(x,u_1^*,u_2^*) \end{aligned}$$

Set $\xi_i = V_i'$ and differentiate w.r.t. x :

$$\rho \xi_i = H_x^{(i)} + H_{\xi_1}^{(i)} \, \xi_1' + H_{\xi_2}^{(i)} \, \xi_2'$$

Since $H_{\xi_1}^{(1)} = H_{\xi_2}^{(2)} = G$, one obtains

$$\begin{pmatrix} G & H_{\xi_2}^{(1)} \\ H_{\xi_1}^{(2)} & G \end{pmatrix} \begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} = \begin{pmatrix} \rho\xi_1 - H_x^{(1)} \\ \rho\xi_2 - H_x^{(2)} \end{pmatrix}$$

Bifurcation analysis

$$\begin{pmatrix} G & \theta^2\alpha \\ \beta & G \end{pmatrix} \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

If the determinant of the matrix does not vanish, this is equivalent to

$$\begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \frac{1}{G^2 - \theta^2\alpha\beta} \begin{pmatrix} G & -\theta^2\alpha \\ -\beta & G \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

Goal: find regular solutions in a neighborhood of a point $\overline{P} = (\bar{x}, \bar{\xi}, \bar{\eta})$ where $G = \phi = 0$

CASE 1: $\alpha\beta(\overline{P}) < 0 \implies$ for $\theta > 0$ infinitely many solutions exist

CASE 2: $\alpha\beta(\overline{P}) > 0 \quad \Longrightarrow$ under a generic transversality condition, for $\theta > 0$ small a unique solution exists

The implicit system of ODEs
$$\begin{cases} G\,\frac{d\xi}{dx} + \theta^2\alpha\frac{d\eta}{dx} &= \phi \\ \beta\frac{d\xi}{dx} + G\frac{d\eta}{dx} &= \psi \end{cases}$$

can be written as a Pfaffian system:

$$\begin{cases} \omega_1 = -\phi\,dx + G\,d\xi + \theta^2\alpha\,d\eta = 0 \\ \omega_2 = -\psi\,dx + \beta\,d\xi + G\,d\eta = 0 \end{cases}$$

Solutions are obtained by concatenating trajectories of the vector field

$$\mathbf{v}^\theta = \omega_1 \wedge \omega_2 = \begin{pmatrix} G^2 - \theta^2\alpha\beta \\ G\phi - \theta^2\alpha\psi \\ G\psi - \beta\phi \end{pmatrix}$$

Seek: a concatenation of trajectories of $\mathbf{v}^\theta$ providing the graph of a smooth function $x \mapsto (\xi(x), \eta(x))$.

Note: along the two surfaces

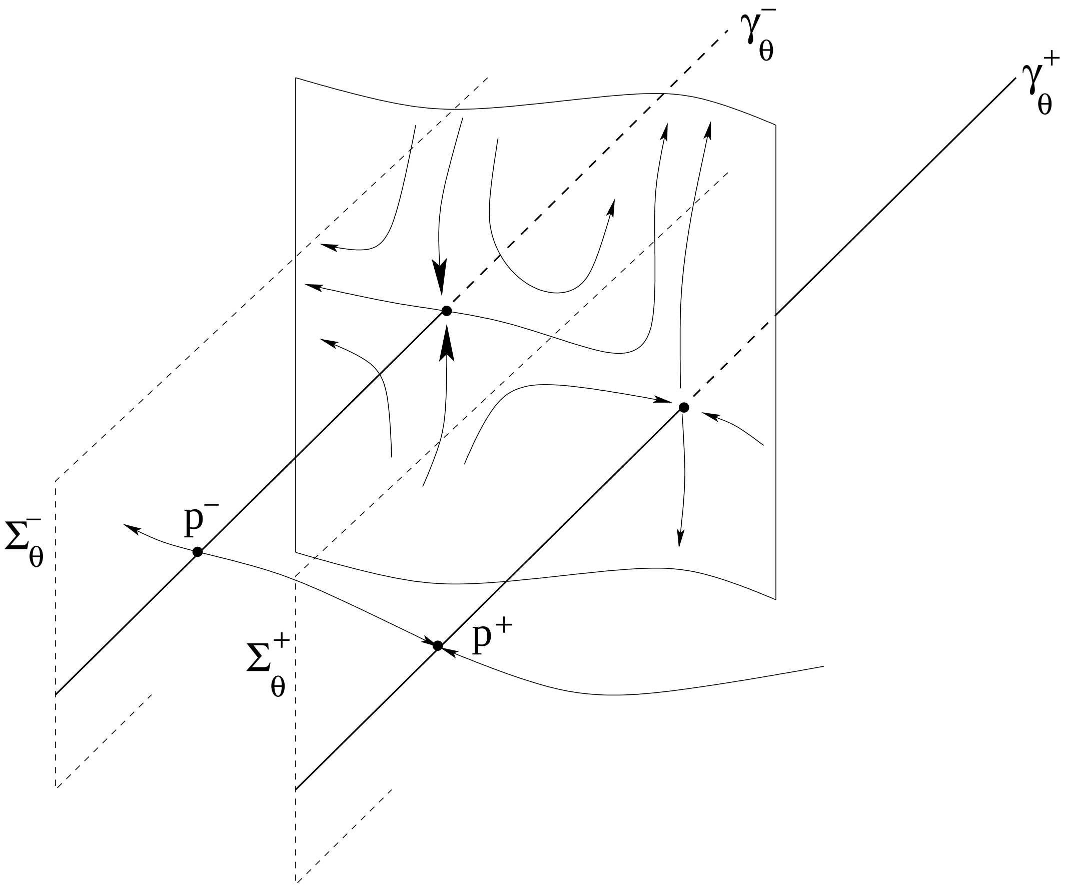
$$\Sigma_\theta^\pm \doteq \Big\{(x,\xi,\eta)\,; \qquad G = \pm\,\theta\sqrt{\alpha\beta}\Big\}$$

the first component of $\mathbf{v}^\theta$ vanishes, hence $\mathbf{v}^\theta$ is vertical.

These surfaces can only be crossed along the two curves where $\mathbf{v}^\theta = 0$

$$\gamma_\theta^\pm \doteq \left\{(x,\xi,\eta)\,; \qquad G = \pm\,\theta\sqrt{\alpha\beta}\,, \qquad \phi = \pm\,\theta\sqrt{\frac{\alpha}{\beta}}\,\psi\right\}$$

Find a heteroclinic orbit joining a point $p^- \in \gamma_\theta^-$ with a point $p^+ \in \gamma_\theta^+$



Under a generic condition, the stable and unstable manifolds intersect transversally, hence a unique heteroclinic orbit exists

Example 1: a linear - quadratic game

$$\dot{x} \;=\; -x + u + \theta v.$$

Payoffs to be maximized:

$$J^u \doteq \int_0^\infty e^{-\rho t}\left(ax - \frac{u^2}{2}\right)\,dt \qquad\qquad J^v \doteq \int_0^\infty e^{-\rho t}\left(bx - \frac{v^2}{2}\right)\,dt \qquad\qquad a, b \neq 0$$

Value functions: $U(x)$, $V(x)$, $\quad \xi = U'$, $\eta = V'$

$$\begin{pmatrix} \xi - x + \theta^2\eta & \theta^2\xi \\ \eta & \xi - x + \theta^2\eta \end{pmatrix} \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} (1+\rho)\xi - a \\ (1+\rho)\eta - b \end{pmatrix}$$

$$G = \xi - x + \theta^2\eta\,, \qquad\qquad \phi = (1+\rho)\xi - a$$

$$G_\eta = \theta^2, \qquad\qquad \phi_\eta = 0$$

When $\theta = 0$, the singular point is $(\bar{x}, \bar{\xi}, \bar{\eta}) = \left(\frac{a}{1+\rho}, \ \frac{a}{1+\rho}, \ \frac{b}{1+\rho} \right)$

$$G = \xi - x + \theta^2 \eta \, , \qquad \alpha = \xi, \qquad \beta = \eta,$$

$$\phi = (1+\rho)\xi - a \, , \qquad \psi = (1+\rho)\eta - b$$

Check: $\alpha\beta \ = \ \frac{ab}{(1+\rho)^2}$

$ab > 0 \quad \Longrightarrow$ unique solution also for $\theta > 0$

$ab < 0 \quad \Longrightarrow$ infinitely many solutions for $\theta > 0$

Example 2: a "sticky price" game

$$\dot{p} \;=\; (b-a)p$$

$p \;=\;$ price $\qquad \begin{cases} a(t) = \text{production rate} \\ b(t) = \text{consumption rate} \end{cases}$

$$J^{prod} = \int_0^\infty e^{-\rho t}\Big[p(t)b(t) - c(a(t))\Big]\,dt$$

$$J^{cons} = \int_0^\infty e^{-\rho t}\Big[\phi(b(t)) - p(t)b(t)\Big]\,dt$$

$c(a) =$ production cost, $\qquad \phi(b) =$ utility for the consumer

Optimal feedback strategies

$V(p)$ = value function for the producer
$W(p)$ = value function for the consumer

$$a^*(p,V') \;=\; \arg\max_a \Big\{V'\cdot(-pa)-c(a)\Big\}$$

$$b^*(p,W') \;=\; \arg\max_b \Big\{W'\cdot pb+\phi(b)-p\,b\Big\}$$

Myopic strategy for the consumer

$$b^\dagger(p)=\arg\max_b \Big\{\phi(b)-p\,b\Big\}$$

A small coalition of consumers

- a fraction $\theta \in [0,1]$ of all consumers join together and play a long-term strategy

- the remaining fraction $(1-\theta)$ consists of individual consumers who adopt the myopic strategy

$$\dot{p} = p\left[(1-\theta)b^{\dagger}(p) + \theta b - a\right]$$

$$J^{\text{prod}} = \int_0^{\infty} e^{-\rho t}\left[(1-\theta)b^{\dagger}(p) + \theta p\, b - c(a)\right]\, dt$$

$$J^{\text{cons}} = \int_0^{\infty} e^{-\rho t}\left[\psi(b) - pb\right]\, dt$$

$V(p)$ = value function for the producer

$W(p)$ = value function for the strategic consumer group

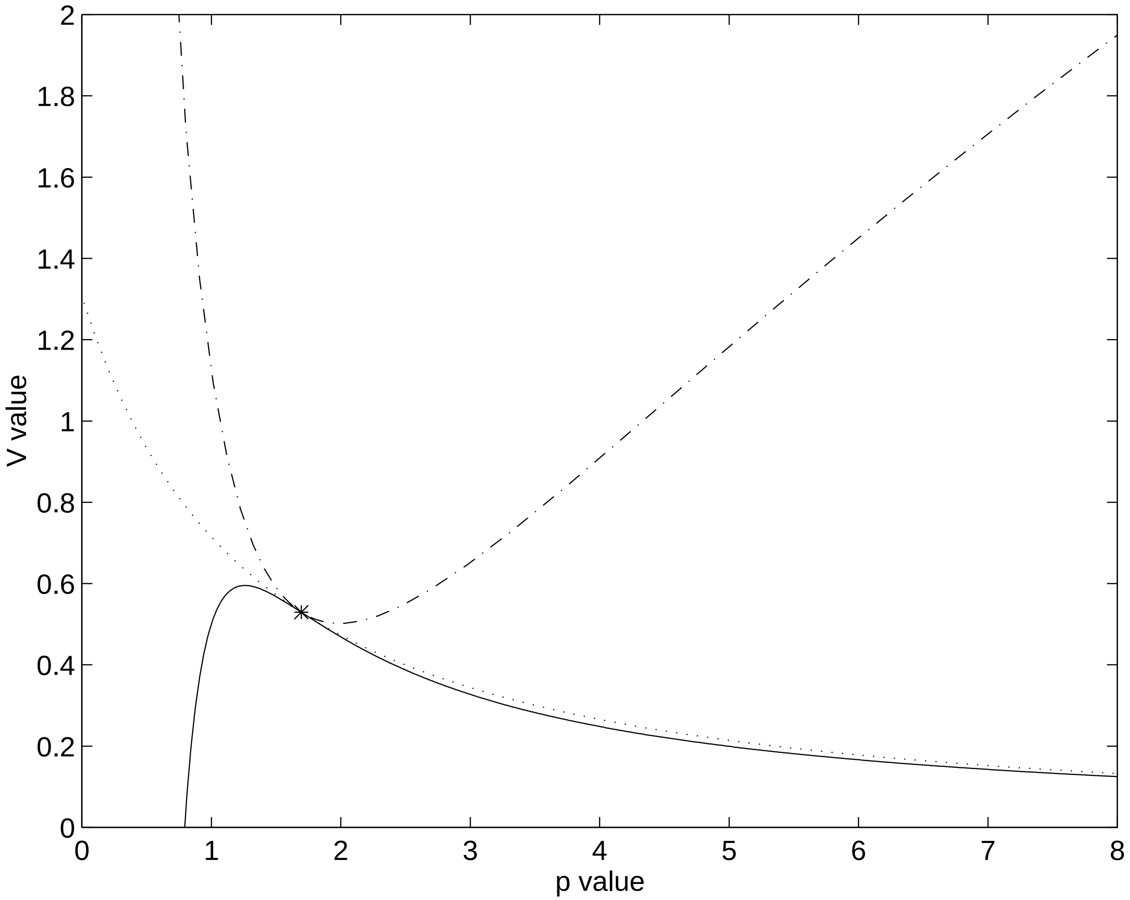
$\theta = 0 \qquad \Longrightarrow \qquad$ optimal control problem for the producer

Example: $c(a) = \dfrac{a^2}{2}, \qquad \phi(b) = 2\sqrt{b}$

myopic strategy for the consumer: $b^\dagger(p) = \dfrac{1}{p^2}$

The value function for the producer satisfies

$$\rho V \;=\; \frac{1 + V'(p)}{p} + \frac{[p\,V'(p)]^2}{2}$$



choose the branch corresponding to a stable dynamics: $\dot{p}(p - p_0) \leq 0$

Bifurcation Problem

$$\begin{cases} \rho V &= \dfrac{(pV')^2}{2} + (1+V')\left[\dfrac{1-\theta}{p} + \dfrac{\theta}{(1-\theta W')^2 p}\right] \\ \\ \rho W &= p^2 V' W' + \dfrac{1-\theta}{p} W' + \dfrac{1-\theta W'}{(1-\theta W')^2 p} \end{cases}$$

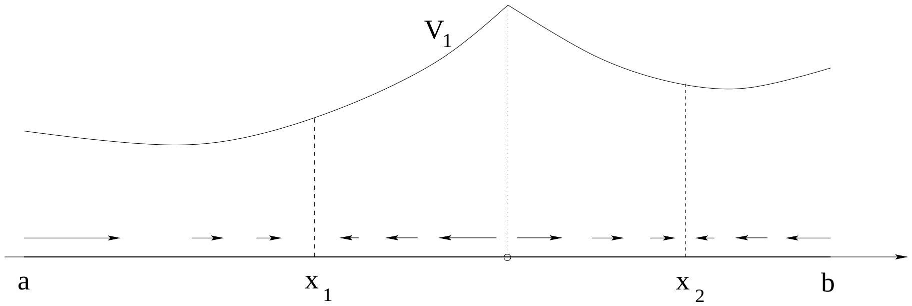
$$\xi \doteq V' \qquad\qquad \eta = W' < 0$$

$\alpha\beta \;=\; \dfrac{2(\bar{p}^3 - 1)\bar{\eta}}{\bar{p}^2} \;<\; 0 \quad \Longrightarrow$ infinitely many solutions for $\theta > 0$

Open questions:

- If infinitely many solutions exist, can we select a “best one” ?

- What happens in the generic one-dimensional case, when the optimal feedback has jumps ?



- Can a branch of solutions be continued for all $\theta \in [0,1]$? How do we recognize a critical value $\bar{\theta}$ where a further bifurcation occurs ?

- Study bifurcation problems in two or more space variables.

Price - Inventory game: state of the system described by (p, I).

Best reply map: $\mathcal{B}(u_1,u_2)=(\widehat{u}_1,\widehat{u}_2)$

$\widehat{u}_1(\cdot)$ = best reply of player 1, to strategy $u_2(\cdot)$ of second player

$\widehat{u}_2(\cdot)$ = best reply of player 2, to strategy $u_1(\cdot)$ of first player

The couple of strategies (u_1^*,u_2^*) is a Nash equilibrium solution iff it provides a fixed point of $\mathcal{B}$

$$\mathcal{B}(u_1^*,u_2^*)=(u_1^*,u_2^*)$$

- For a fixed $\theta>0$ small, does the sequence of iterates

$$\mathcal{B}(u_1^0,u_2^0),\quad \mathcal{B}^2(u_1^0,u_2^0),\quad \mathcal{B}^3(u_1^0,u_2^0),\quad \ldots \qquad \text{converge ?}$$

Here the initial guess is (u_1^0,u_2^0), i.e. the solution for $\theta=0$.