1 - Bijections

A map $f : A \rightarrow B$ is an **injection** if it is one-to-one, i.e. distinct elements $a_1, a_2 \in A$ have distinct images $f(a_1) \neq f(a_2)$. The map $f$ is a **surjection** if it is onto, i.e. every element $b \in B$ is the image of some element of $A$.

We say that a map $f: A \mapsto B$ is a **bijection** if it is one-to-one and onto. If a bijection exists, we regard the two sets $A$ and $B$ as having the same number of elements. This allows us to compare also sets with infinitely many elements.

2 - Mathematical induction

Given a sequence of statements $P_1, P_2, P_3, \ldots$, mathematical induction is a technique for proving that all of the statements are true. Namely, one has to show that

(i) The first statement $P_1$ is true.

(ii) If $P_k$ is true, then also the following statement $P_{k+1}$ is true.

3 - Upper bound, supremum

A set $S \subset \mathbb{R}$ is **bounded above** if there exists a number $u$ such that $u \geq x$ for all $x \in S$. In this case $u$ is called an **upper bound**. The smallest upper bound is called **supremum** and written $\text{sup} \ S$.

**Theorem (completeness of the real numbers).** If a set $S$ is bounded above, then it has a supremum.

To prove that $u = \text{sup} \ S$, one needs to show:

(i) $u \geq x$ for every $x \in S$,

(ii) For every $\varepsilon > 0$, there exists a point $x \in S$ such that $u - \varepsilon < x$.

Notice that (ii) is certainly true if $u \in S$. 

4 - Intervals

An open interval is a set of the form $(a, b) = \{ x \in \mathbb{R}; \ a < x < b \}$. A closed interval is a set of the form $[a, b] = \{ x \in \mathbb{R}; \ a \leq x \leq b \}$. Here one may have $a = -\infty$ or $b = +\infty$. In this case the interval is unbounded.

We say that a sequence of intervals $I_n = [a_n, b_n]$ are **nested** if $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$. This happens if and only if $a_1 \leq a_2 \leq a_3 \leq \cdots$ and $b_1 \geq b_2 \geq b_3 \geq \cdots$

**Theorem (intersection of nested intervals).** Given a sequence of nested intervals $[a_n, b_n]$ which are closed and bounded, their intersection is non-empty. In addition, if their lengths $b_n - a_n$ shrink to zero, then the intersection contains exactly one point.

5 - Sequences

A sequence is a map from $\mathbb{N}$ into $\mathbb{R}$. It is usually denoted as $(x_1, x_2, x_3, \ldots)$, or $(x_n)_{n \geq 1}$. The sequence is **bounded** if all points $x_n$ are contained in a bounded interval $[a, b]$. It is **monotone increasing** if $x_1 \leq x_2 \leq x_3 \leq \cdots$

A sequence can be defined by directly assigning its values: $x_n = f(n)$.

Alternatively, one can define the sequence by induction: (i) fix the initial value $x_1$, and then (ii) give a rule for computing $x_{k+1}$ from the previous value $x_k$.

6 - Limits

We say that the sequence $(x_n)_{n \geq 1}$ **converges to** $x$, and write

$$\lim_{n \to \infty} x_n = x$$

if, for every $\varepsilon > 0$ one can find a number $K(\varepsilon)$ sufficiently large so that

$$x - \varepsilon < x_n < x + \varepsilon$$

(1)

for all $n \geq K(\varepsilon)$.

Intuitively this means that, as $n$ grows large, the numbers $x_n$ become closer and closer to $x$.

To prove that $\lim_{n \to \infty} x_n = x$, one has to study the inequality (1), and show that it is satisfied for all integers $n$ sufficiently large.

7 - Limit theorems

**Theorem (sandwich).** If $(x_n)_{n \geq 1}$, $(y_n)_{n \geq 1}$, and $(z_n)_{n \geq 1}$ are three sequences such that $x_n \leq y_n \leq z_n$ for every $n \geq 1$, and if

$$\lim_{n \to \infty} x_n = x = \lim_{n \to \infty} z_n,$$

then we also have

$$\lim_{n \to \infty} y_n = x.$$
Theorem (sums, products, quotients). If \((x_n)_{n \geq 1}\) and \((y_n)_{n \geq 1}\) are two sequences such that
\[
\lim_{n \to \infty} x_n = x, \quad \lim_{n \to \infty} y_n = y,
\]
and \(c \in \mathbb{R}\) is a real number, then
\[
\lim_{n \to \infty} (x_n + y_n) = x + y, \quad \lim_{n \to \infty} c x_n = c x, \\
\lim_{n \to \infty} (x_n \cdot y_n) = x \cdot y, \quad \lim_{n \to \infty} \frac{x_n}{y_n} = \frac{x}{y} \quad (\text{if } y \neq 0).
\]

Theorem (monotone sequences). If the sequence \((x_n)_{n \geq 1}\) is bounded and monotone increasing, then it has a limit:
\[
\lim_{n \to \infty} x_n = \sup \{x_n; \; n \geq 1\}.
\]

Basic Problems

- Construct a bijection between two infinite sets.
- Using unique factorization, prove that certain numbers such as \(\sqrt{5}\) are irrational.
- Work out proofs using mathematical induction.
- Decide whether a set \(S \subset \mathbb{R}\) is bounded or not. Find its supremum. Given a set \(S\) and a number \(u\), prove that \(u = \sup S\).
- Given a sequence \((x_n)_{n \geq 1}\), check if it converges or not.
- Prove that \(\lim_{n \to \infty} x_n = x\), using the definition of limit, or the basic theorems about limits.
- Study a sequence \((x_n)_{n \geq 1}\) defined inductively: \(x_{k+1} = f(x_k)\). Check if it converges and find its limit.
8 - Convergence criteria

The following theorems guarantee that a sequence \((x_n)_{n \geq 1}\) converges, even if we do not know precisely what the limit is.

- A sequence \((x_n)_{n \geq 1}\) is **bounded** if there exists a number \(M\) large enough so that \(x_n \in [-M, M]\) for all \(n\).

**Theorem (monotone convergence).** Assume that the sequence is increasing, so that \(x_1 \leq x_2 \leq x_3 \leq \cdots\) Then the sequence converges to some limit \(x\) if and only if it is bounded. In this case \(\lim_{n \to \infty} x_n = \sup \{x_n; \ n \geq 1\}\).

- A sequence \((x_n)_{n \geq 1}\) is a **Cauchy sequence** if, for every \(\varepsilon > 0\) one can find an integer \(H(\varepsilon)\) large enough so that \(|x_n - x_m| < \varepsilon\) for all \(n, m > H(\varepsilon)\). Intuitively this means that, when \(m, n \to \infty\), the numbers \(x_m, x_n\) get closer and closer to each other.

**Theorem (Cauchy criterion).** A sequence \((x_n)_{n \geq 1}\) converges to some limit \(x\) if and only if it is a Cauchy sequence.

**Theorem (Bolzano - Weierstrass).** If the sequence \((x_n)_{n \geq 1}\) is bounded, then one can select integer numbers \(n_1 < n_2 < n_3 < \cdots\), such that the subsequence \(x_{n_1}, x_{n_2}, x_{n_3}, \ldots\) converges to some limit.

9 - Divergent sequences

We say that the sequence \((x_n)_{n \geq 1}\) tends to \(+\infty\), and write \(\lim_{n \to \infty} x_n = +\infty\), if for every (arbitrarily large) \(\alpha \in \mathbb{R}\) there exists a number \(K(\alpha)\) such that \(x_n > \alpha\) for every integer \(n \geq K(\alpha)\).

**Theorem (unbounded monotone sequences).** If the sequence \((x_n)_{n \geq 1}\) is monotone increasing and unbounded, then \(\lim_{n \to \infty} x_n = +\infty\).

**Theorem (comparison).** If \(x_n \leq y_n\) for every \(n\), and if \(\lim_{n \to \infty} x_n = +\infty\), then we also have \(\lim_{n \to \infty} y_n = +\infty\).
10 - Series

Given a sequence \((x_n)_{n \geq 1}\), we consider the infinite series

\[
\sum_{n=1}^{\infty} x_n = x_1 + x_2 + x_3 + \cdots
\]

The corresponding sequence of partial sums is defined as

\[
\begin{align*}
  s_1 &= x_1, \\
  s_2 &= x_1 + x_2, \\
  \vdots \\
  s_k &= x_1 + x_2 + \cdots + x_k, \\
  \vdots
\end{align*}
\]

If the sequence of partial sums \(s_k\) has a limit, we say that the series is convergent. We then define

\[
\sum_{n=1}^{\infty} x_n = \lim_{k \to \infty} s_k.
\]

The following theorems guarantee that a series converges.

**Theorem (comparison).** Assume \(0 \leq x_n \leq y_n\) for every \(n\).
- If the series \(\sum_{n=1}^{\infty} y_n\) converges, then the series \(\sum_{n=1}^{\infty} x_n\) converges as well.
- If the series \(\sum_{n=1}^{\infty} x_n\) diverges, then the series \(\sum_{n=1}^{\infty} y_n\) diverges as well.

To use the above theorem, it is useful to keep in mind that:

- the series \(\sum_{n=1}^{\infty} \frac{1}{n^p}\) converges if \(p > 1\), diverges if \(p \leq 1\).
- the series \(\sum_{n=0}^{\infty} a^n\) converges if \(|a| < 1\), diverges if \(|a| \geq 1\).

One should also remember the formula for the partial sums

\[
1 + a + a^2 + \cdots + a^k = \frac{1 - a^{k+1}}{1 - a}
\]

hence

\[
\lim_{k \to \infty} (1 + a + a^2 + \cdots + a^k) = \lim_{k \to \infty} \frac{1 - a^{k+1}}{1 - a} = \frac{1}{1 - a} \quad \text{if } |a| < 1.
\]

**Theorem (ratio test).** Assume that

\[
\lim_{n \to \infty} \frac{|x_{n+1}|}{|x_n|} = L < 1.
\]
Then the series $\sum_{n=1}^{\infty} x_n$ converges.

Intuitively, the series $\sum x_n$ converges if the terms $x_n$ become smaller and smaller (i.e. approach zero) quickly enough.

10 - Limits of functions

Definition of limit: Consider a function $f : A \rightarrow \mathbb{R}$. We say that

$$\lim_{x \to c} f(x) = L$$

if, for every $\varepsilon > 0$ one can find $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{for all} \quad x \in A \quad \text{such that} \quad |x - c| < \delta, \quad x \neq c.$$

Theorem (sequential criterion). One has $\lim_{x \to c} f(x) = L$ if and only if, for every sequence $x_n$ converging to $c$, the sequence $f(x_n)$ converges to $L$.

Theorem (properties of limits). Assume that

$$\lim_{x \to c} f(x) = L, \quad \lim_{x \to c} g(x) = M.$$

Then

$$\lim_{x \to c} (f(x) + g(x)) = L + M, \quad \lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M, \quad \lim_{x \to c} a f(x) = aL.$$

If $M \neq 0$, then also

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Theorem (comparison). Assume that $f, g, h$ are three functions defined on the same domain $A$, with $f(x) \leq g(x) \leq h(x)$ for all $x \in A$. If $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$, then we also have $\lim_{x \to c} g(x) = L$.

11 - Continuous functions

Definition of continuous function: A function $f : A \rightarrow \mathbb{R}$ is continuous at a point $c \in A$ if $\lim_{x \to c} f(x) = f(c)$. This means that, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon \quad \text{for all} \quad x \in A \quad \text{such that} \quad |x - c| < \delta.$$

We say that a function $f : A \rightarrow \mathbb{R}$ is continuous if $f$ is continuous at every point of its domain $A$.

Examples of continuous functions are: $f(x) = a$ (constant function), $f(x) = x$, $f(x) = \sin x$, $f(x) = \cos x$, $f(x) = \sqrt{x}$.

Theorem (more continuous functions). Let $f, g$ be continuous functions, defined on the same domain $A$. Then the functions $f + g$, $f \cdot g$, $a f$ are also continuous. Moreover, the quotient function $h(x) = f(x)/g(x)$ is continuous at every point $x$ where $g(x) \neq 0$.  


Theorem (composition of continuous functions). If \( f : A \rightarrow \mathbb{R} \) and \( g : B \rightarrow \mathbb{R} \) are continuous functions, with \( f(A) \subseteq B \), then the composed map \( h(x) = g(f(x)) \) is also continuous.

12 - Continuous functions on an interval

Theorem. Let \([a, b]\) be a closed interval, and let \( f : [a, b] \rightarrow \mathbb{R} \) be a continuous function. Then there exists points \( x_{\ast} \) and \( x^{\ast} \) where \( f \) attains its minimum and its maximum values. Namely

\[
f(x_{\ast}) = m = \inf_{x \in [a, b]} f(x), \quad f(x^{\ast}) = M = \inf_{x \in [a, b]} f(x).
\]

Moreover, the image \( f([a, b]) \) is precisely the closed interval \([m, M]\). In other words, \( f \) attains all the intermediate values between the minimum and the maximum.

Theorem. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a continuous function. Then \( f \) is uniformly continuous, in the sense that, given \( \varepsilon > 0 \), one can find \( \delta > 0 \) such that

\[
|f(x) - f(y)| < \varepsilon \quad \text{for all} \quad x, y \in [a, b] \quad \text{such that} \quad |x - y| < \delta.
\]

Example. We say that a function \( f : A \rightarrow \mathbb{R} \) is Lipschitz continuous if there exists a constant \( K \) such that

\[
|f(x) - f(y)| \leq K|x - y| \quad \text{for all} \quad x, y \in A.
\]

In this case, the function \( f \) is uniformly continuous.

13 - The derivative

Let \( f \) be a function defined in a neighborhood of a point \( c \). The derivative of \( f \) at \( c \) is

\[
f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}
\]

provided that the above limit exists. In this case we say that \( f \) is differentiable at \( c \).

Theorem. If \( f \) is differentiable at \( c \), then \( f \) is continuous at \( c \). However, a function may be continuous but not differentiable.

Differentiation rules: If \( f, g \) are differentiable at the point \( c \), and \( \alpha \) is any number, then

\[
(\alpha f)'(c) = \alpha f'(c), \quad (f + g)'(c) = f'(c) + g'(c)
\]

product rule: \((f \cdot g)(c) = f'(c)g(c) + f(c)g'(c)\),

quotient rule: \((\frac{f}{g})'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g^2(c)} \quad \text{(if } g(c) \neq 0)\),

chain rule: \((g \circ f)'(c) = g'(f(c)) f'(c)\).

Here \((g \circ f)(x) = g(f(x))\) is the composed mapping.
Assume that \( f, g \) are inverse of each other, so that \( y = f(x) \) implies \( x = g(y) \). Differentiating the equality \( g(f(x)) = x \) using the chain rule we obtain

\[
g'(y)f'(x) = 1, \quad \text{hence} \quad g'(y) = \frac{1}{f'(x)}
\]

where the points \( x, y \) are related by \( y = f(x), x = g(y) \).

14 - The mean value theorem

If a differentiable function \( f : [a, b] \mapsto \mathbb{R} \) attains a local maximum (or a local minimum) at some interior point \( c \), with \( a < c < b \), then its derivative satisfies \( f'(c) = 0 \).

Theorem (Rolle). If a differentiable function \( f : [a, b] \mapsto \mathbb{R} \) satisfies \( f(a) = f(b) \), then there exists a point \( a < c < b \) such that \( f'(c) = 0 \).

Mean value theorem. If \( f : [a, b] \mapsto \mathbb{R} \) is a differentiable function, then there exists a point \( c \in [a, b] \) such that

\[
\frac{f(b) - f(a)}{b - a} = f'(c)
\]

[slope of secant line] = [slope of tangent line at the point \( c \)]

Consequences of the mean value theorem:

- If \( f'(x) = 0 \) for all \( x \in [a, b] \), then \( f \) is constant.
- If \( f'(x) = g'(x) \) for all \( x \in [a, b] \), then \( f - g \) is constant, hence there exists a number \( C \) such that \( f(x) = g(x) + C \) for all \( x \).
- If \( f'(x) \geq 0 \) for all \( x \in [a, b] \), then the function \( f \) is increasing. That means: if \( x < y \) then \( f(x) \leq f(y) \).
- If \( f'(x) > 0 \) for all \( x \in [a, b] \), then the function \( f \) is strictly increasing. That means: if \( x < y \) then \( f(x) < f(y) \).
- If \( f : [a, b] \mapsto \mathbb{R} \) is differentiable and \( f'(x) \geq 0 \) for \( x < c \) and \( f'(x) < 0 \), then \( f \) attains its maximum at the point \( c \).
Basic Problems

• Check if a sequence is convergent, or properly divergent, using the definition or a comparison method.

• Decide if a series converges. Explicitly compute its sum, in some special cases (like a geometric series).

• Prove that a function has a limit, using the definition or applying various limit theorems.

• Check if a function is continuous at a given point, using the definition or a continuity theorem.

• Prove that an equation \( f(x) = 0 \), with \( f \) continuous, has a solution in a suitable interval \([a, b]\).

• Compute the derivative of a function \( f \), using various differentiation rules.

• Prove that a function is increasing, or decreasing, on a given interval, using the mean value theorem. Find points of local maximum and of local minimum.

• Establish an inequality of the form \( f(x) \leq g(x) \) for \( x \in [a, b] \), applying the mean value theorem.
Math 401 - Introduction to Real Analysis

Additional Topics for the Final Exam - Review

15 - L’Hospital’s Rule

This method is useful to compute the limit of a quotient: \( \lim_{x \to a^+} \frac{f(x)}{g(x)} \), when it has the indeterminate form \( \frac{0}{0} \), or \( \frac{\infty}{\infty} \).

**Theorem.** Let the functions \( f, g \) be differentiable on the open interval \((a, b)\), with \( g’(x) \neq 0 \) for every \( x \). Assume that either \( \lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0 \), or that \( \lim_{x \to a^+} g(x) = \pm \infty \).

\[
\text{If } \lim_{x \to a^+} \frac{f'(x)}{g'(x)} \text{ exists, then } \lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}.
\]

16 - Taylor formula

The higher order derivatives of a function \( f \) at a point \( x \) are denoted as

\[
f'(x), \quad f''(x), \quad f'''(x), \ldots, f^{(n)}(x), \ldots
\]

Given a point \( x_0 \), and an integer \( n \geq 1 \), the polynomial of degree \( n \) that best approximates \( f \) in a neighborhood of the point \( x_0 \) is

\[
P_n(x) = f(x_0) + f'(x_0) \frac{(x - x_0)}{1!} + f''(x_0) \frac{(x - x_0)^2}{2!} + f'''(x_0) \frac{(x - x_0)^3}{3!} + \cdots + f^{(n)}(x_0) \frac{(x - x_0)^n}{n!}.
\]

At the special point \( x_0 \), the polynomial \( P_n \) has the same value and the same derivatives as \( f \), up to order \( n \):

\[
P_n(x_0) = f(x_0), \quad P'_n(x_0) = f'(x_0), \quad \ldots, \quad P^{(n)}_n(x_0) = f^{(n)}(x_0).
\]

If \( f \) is \( n + 1 \) times differentiable, the error in the approximation can be expressed as

\[
f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1} \quad \text{for some } c \in [x_0, x].
\]
17 - The Riemann integral

Let \( f \) be a function defined on an interval \([a, b]\). By a \textbf{partition} \( P \) of \([a, b]\) we mean a finite set of points \( a = x_0 < x_1 < x_2 < \cdots < x_n = b \). The \textbf{upper} and \textbf{lower Riemann sums} corresponding to the partition \( P \) are defined respectively as

\[
S^+(f, P) = \sum_{i=1}^{n} (x_i - x_{i-1}) M_i \quad M_i = \sup \{ f(x) ; \ x \in [x_{i-1}, x_i] \},
\]

\[
S^-(f, P) = \sum_{i=1}^{n} (x_i - x_{i-1}) m_i \quad m_i = \inf \{ f(x) ; \ x \in [x_{i-1}, x_i] \}.
\]

By choosing a point \( t_i \in [x_{i-1}, x_i] \) inside each interval of the partition \( P \) we obtain a \textbf{tagged partition} \( \hat{P} \). The \textbf{Riemann sum} corresponding to the tagged partition \( \hat{P} \) is defined as

\[
S(f, \hat{P}) = \sum_{i=1}^{n} (x_i - x_{i-1}) f(t_i).
\]

Clearly we have

\[
S^-(f, P) \leq S(f, \hat{P}) \leq S^+(f, P)
\]

for every choice of the points \( t_1, \ldots, t_n \).

The \textbf{mesh} of the partition \( P \) is defined as the maximum length of the intervals \([x_{i-1}, x_i]\) and denoted as \( \|P\| \). Choosing partitions whose mesh becomes smaller and smaller, we expect that all the corresponding Riemann sums will approach a certain number \( L \). If this happens, we say that \( L \) is the Riemann integral of \( f \) on the interval \([a, b]\).

More precisely, we say that \( f \) is Riemann integrable on \([a, b]\) and

\[
\int_a^b f(x) \, dx = L
\]

if, for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that, for every tagged partition \( \hat{P} \) with mesh \( \leq \delta \) we have

\[
|S(f, \hat{P}) - L| < \varepsilon.
\]

Theorem (existence and properties of Riemann integrals).

(i) If \( f \) is continuous on \([a, b]\), then \( f \) is Riemann integrable.

(ii) If \( f \) is increasing (or decreasing) on \([a, b]\), then \( f \) is Riemann integrable.

(iii) If \( f \) is not bounded on \([a, b]\), then \( f \) is NOT Riemann integrable.

(iv) If \( f \) and \( g \) are Riemann integrable on \([a, b]\), the same is true for the function \( f + g \), and \( cf \) for any constant \( c \). One has:

\[
\int_a^b (f + g)(x) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \quad \int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx.
\]
18 - The Fundamental Theorem of Calculus

• Riemann sums provide a way to define the integral \( \int_a^b f(x) \, dx \), and to approximately compute its value.

• The fundamental theorem of Calculus allows us to exactly compute the Riemann integral
  \( \int_a^b f(x) \, dx \), whenever an antiderivative of \( f \) is known.

**Theorem I.** Let \( f, F : [a, b] \rightarrow \mathbb{R} \) be functions such that
  (i) \( F \) is continuous on \([a, b]\),
  (ii) \( f \) is Riemann integrable on \([a, b]\),
  (iii) \( F'(x) = f(x) \) for all except at most finitely many points \( x \in [a, b] \).

Then we have
  \[ \int_a^b f(x) \, dx = F(b) - F(a). \]

**Theorem II.** Let \( f \) be Riemann integrable on the interval \([a, b]\), and define the integral function
  \[ F(x) = \int_a^x f(t) \, dt. \]

Then \( F \) is continuous. Moreover, \( F'(x) = f(x) \) at every point \( x \) where \( f \) is continuous.

**Substitution rule:** Assume that the function \( \varphi : [a, b] \rightarrow \mathbb{R} \) has a continuous derivative. Let \( f \) be a continuous function, defined on the image \( \varphi([a, b]) \). Then
  \[ \int_a^b f(\varphi(t)) \varphi'(t) \, dt = \int_{\varphi(a)}^{\varphi(b)} f(x) \, dx. \]

**Integration by parts.** Let the functions \( F, G \) be differentiable on the interval \([a, b]\). Assume that their derivatives \( F'(x) \) and \( G'(x) \) are Riemann integrable. Then
  \[ \int_a^b F'G \, dx = FG \bigg|_a^b - \int_a^b FG' \, dx. \]

19 - Sequences of functions

For each \( n \geq 1 \) let \( f_n \) be a function defined on the interval \([a, b]\).

• The sequence \( (f_n)_{n \geq 1} \) converges pointwise to the function \( f \) if
  \[ \lim_{n \to \infty} f_n(x) = f(x) \quad \text{for each point} \ x \in [a, b]. \]
• The sequence \((f_n)_{n \geq 1}\) converges uniformly to the function \(f\) if for every \(\varepsilon > 0\) there exists an integer \(N_\varepsilon\) such that
\[
|f_n(x) - f(x)| < \varepsilon \quad \text{for every } n > N_\varepsilon \text{ and every } x \in [a, b].
\]

**Theorem.** If all functions \(f_n\) are continuous and converge to \(f\) uniformly on \([a, b]\), then \(f\) is continuous as well.

**Basic Problems**

• Compute the limit of a quotient \(f(x)/g(x)\) using L'Hôpital’s rule.

• Write the Taylor approximation \(P_n(x)\) to a function \(f(x)\) at a point \(x_0\). Estimate how big is the error \(|P_n(x) - f(x)|\).

• Decide if a function \(f\) is Riemann integrable on an interval \([a, b]\).

• Determine the mesh of a partition \(\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}\) of an interval \([a, b]\). Compute a Riemann sum which approximates the integral \(\int_a^b f(x) \, dx\).

• Compute the exact value of an integral \(\int_a^b f(x) \, dx\) using the fundamental theorem of calculus.

• Decide if a sequence of functions \((f_n)_{n \geq 1}\) converges to a function \(f\), pointwise or uniformly -