Impulsive Control of Lagrangian Systems
and Locomotion in Fluids

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Abstract. Aim of this paper is to provide a survey of the theory of impulsive control of Lagrangian systems. It is assumed here that an external controller can determine the evolution of the system by directly prescribing the values of some of the coordinates. We begin by motivating the theory with a couple of elementary examples. Then we discuss the analytical form taken by the equations of motion, and their impulsive character. The following sections review various results found in the literature concerning the continuity of the control-to-trajectory map, the existence of optimal controls, and the asymptotic controllability to a reference state. In the last section we indicate a further application of the theory, to the control of deformable bodies immersed in a fluid.

1 - Introduction

In the control literature, by “impulsive control system” one usually refers to a system governed by a differential equation

\[
\frac{d}{dt} x(t) = f(t, x(t), u(t)) , \tag{1.1}
\]

but where the state \(x\) is also allowed to jump, at a finite set times:

\[x(t_i^+) = \Phi(t_i^-, x(t_i^-), \alpha_i).\]

Here the controller selects the control function \(u(\cdot)\), as well as the times \(t_i\) and the parameters \(\alpha_i\) determining the jumps [BL].

There is also a quite different framework leading to impulsive control. Namely, consider a Lagrangian system described by coordinates \(q = (q^1, \ldots, q^{n+m})\). Assume that, by imposing suitable mechanical constraints, we can directly assign some of these coordinates as functions of time, say \(q^i(t) = u_i(t)\) for \(i = n+1, \ldots, n+m\). One can then show that the evolution of the remaining free coordinates \(q^1, \ldots, q^n\) is determined by a first order system of \(2n\) differential equations of the form

\[
\begin{aligned}
\dot{q}^i &= \phi_i(t, q(t), p(t), u(t), \dot{u}(t)) , \\
\dot{p}_i &= \psi_i(t, q(t), p(t), u(t), \dot{u}(t)) , \quad i = 1, \ldots, n ,
\end{aligned} \tag{1.2}
\]

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where upper dots denote derivatives w.r.t. time. Here \( p_1, \ldots, p_n \) are the so-called conjugate moments, i.e. \( p_j = \partial T/\partial \dot{q}^j \), with \( T = T(q, \dot{q}) \) denoting the kinetic energy of the system. See (3.6) in Section 3 for details.

Notice that the right hand side of (1.2) also contains the time derivative of the control \( u(\cdot) \). If this control function is discontinuous, the motion will thus have an impulsive character. It is on this second type of impulsive systems that we focus our interest, throughout the present paper.

The theory of control of Lagrangian systems by means of moving constraints was initiated independently by Aldo Bressan and by Charles-Michel Marle, around 1980. The memoir [AB1] was motivated by problems of optimal control for the ski or the swing, later studied in [AB2]. In [Ma] one can find a more general geometric approach, also including some mechanical applications. The connections between the two approaches were clarified in [CF].

The mathematical theory for these problems has been concerned with various issues, which we briefly review.

1. Form of the equations. The equations (1.2) constitute a system of \( n + n \) equations for the first-order time derivatives \( \dot{q}^1, \ldots, \dot{q}^n, \dot{p}_1, \ldots, \dot{p}_n \). In general, one can show that the right hand sides of these O.D.E’s are polynomials of degree \( \leq 2 \) w.r.t. the derivatives \( \dot{u}_i \). Renaming the variables \( x = (x_1, \ldots, x_{2n}) = (q^1, \ldots, q^n, p_1, \ldots, p_n) \), this means we have a system of the form

\[
\dot{x} = \tilde{f}(t, x, u) + \sum_{i=1}^{m} \tilde{g}_i(t, x, u) \dot{u}_i + \sum_{i,j=1}^{m} \tilde{h}_{ij}(t, x, u) \dot{u}_i \dot{u}_j .
\]

(1.3)

The explicit dependence of the vector fields \( \tilde{f}, \tilde{g}_i, \tilde{h}_{ij} \) on the variables \( t \) and \( u \) can be eliminated by introducing the additional state variables \( x_0 = t \) and \( x_{n+1} = u_1, \ldots, x_{n+m} = u_m \), with equations

\[
\dot{x}_0 = 1, \quad \dot{x}_{n+j} = \dot{u}_j \quad j = 1, \ldots, m .
\]

This yields an impulsive system of the form

\[
\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) \dot{u}_i + \sum_{i,j=1}^{m} h_{ij}(x) \dot{u}_i \dot{u}_j .
\]

(1.4)

There are several important cases where all the coefficients \( h_{ij} \) of the quadratic terms vanish identically, and the right hand side of (1.4) is an affine function of the components \( \dot{u}_i \), namely

\[
\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) \dot{u}_i .
\]

(1.5)

Systems of the form (1.5) were called “fit for jumps” in [AB1]. Indeed, as long as the derivative of the control enters linearly in the equations, solutions can be defined also in connection with a control function having jumps at certain points. On the other hand, if in the system (1.4) we insert a control having a jump, a product like \( \dot{u}_i \dot{u}_i \) will formally contain the square of a Dirac delta distribution. Therefore, if the vector field \( h_{ii} \) does not vanish, the state of system will instantly reach infinity. In this case, the model is clearly not well posed.

An analytic characterization of systems “fit for jumps” was first derived in [AB1]. This property also admits an elegant geometric characterization, in terms of orthogonal geodesic curves. This was first obtained in Theorem 5.1 of [AB1], in the case of scalar controls. For general vector-valued controls, see the analysis in [Ra].
Of particular interest is the case where all vector fields \( g_i \) in (1.5) commute, i.e. their Lie brackets \([g_i, g_j] = (Dg_j) g_i - (Dg_i) g_j\) vanish identically. By a suitable change of coordinates one can then remove the presence of the derivatives \( \dot{u}_i \) from the equations [Su]. The evolution is thus described by a standard (non-impulsive) control system, as in (1.1).

2. **Definition of solutions.** Assuming that the function \( u = (u_1, \ldots, u_m) \) is absolutely continuous, we could define \( v_i = \dot{u}_i \) and use \( v = (v_1, \ldots, v_m) \) as our basic control function. In these new variables, (1.4) would become a control system in standard form. This approach, however, is not of much interest. Indeed

(i) In most applications, the dynamics of the system and the constraints on the control functions are naturally formulated using the coordinates themselves as controls, rather then their time derivatives.

(ii) In several optimization problems, the optimal control \( u_{opt}(\cdot) \) is a discontinuous function of time. Restricting the search to absolutely continuous controls would be fruitless.

According to the previous remarks, it is best to study the impulsive system in its original form (1.4). This granted, we now face the issue of how to define solutions for controls which are not Lipschitz continuous.

Let the functions \( f, g_i, \) and \( h_{ij} \) in (1.4) be smooth, and consider the initial data

\[
x(0) = \bar{x}.
\]

In connection with a \( C^1 \) control \( u(\cdot) \), by standard O.D.E. theory, the Cauchy problem (1.4), (1.6) has a unique solution \( t \mapsto x(t, u) \). In order to construct a solution corresponding to a more general (possibly discontinuous) control function \( u(\cdot) \), it is natural to approximate \( u \) by a sequence of \( C^1 \) control functions \( u_k \) and take the limit of the corresponding trajectories. The key problem here is to identify suitable topologies on the space of controls and on the space of trajectories which render continuous the control-to-trajectory map: \( u(\cdot) \mapsto x(\cdot, u) \). Several papers have analyzed this problem, first in the context of stochastic differential equations [Su], then for control systems [LS, B1, BR1, BR2, BR3, BR4, M1]. As soon as we know that the convergence \( u_k(\cdot) \to u(\cdot) \) implies the convergence of the sequence of trajectories \( x(\cdot, u_k) \), we can then define the trajectory \( x(\cdot, u) \) to be the unique limit

\[
x(\cdot, u) \doteq \lim_{k \to \infty} x(\cdot, u_k),
\]

in a suitable topology.

3. **Reduction to a differential inclusion.** A related problem is to characterize the \( L^1 \) closure of the set of all trajectories which correspond to smooth controls. As shown in [BR5], this can be done in terms of a suitable differential inclusion:

\[
\dot{x} \in F(x).
\]

Trajectories of (1.7) can be interpreted as a kind of “generalized solutions” of the impulsive control system (1.4). Quite often, problems of asymptotic stabilization or of optimal control can be more conveniently studied by looking directly at the differential inclusion (1.7).

4. **Stabilization to a constant state.** We say that the impulsive system (1.4) is **locally stabilizable** at a state \( \bar{x} \) if, given any \( \varepsilon > 0 \) one can find \( \delta > 0 \) such that the following holds.
Given any initial state $x^*$ with $|x^* - x| \leq \delta$, one can find a $C^1$ control $u(\cdot)$ such that the corresponding trajectory satisfies

$$|x(t, u) - \bar{x}| \leq \varepsilon \quad \text{for all } t \geq 0.$$  

(1.8)

If a control $u(\cdot)$ can be found such that, in addition to (1.8),

$$\lim_{t \to \infty} x(t, u) = \bar{x},$$

(1.9)

then we say that the system is **locally asymptotically stabilizable** at $\bar{x}$. Notice that here we restrict the attention to $C^1$ controls. This is natural, because the more general trajectories of (1.4) are always defined as limits of solutions corresponding to smooth controls.

Results on the (asymptotic) stabilization of a Lagrangian system, by means of moving constraints, can be found in [BR5]. The key idea here is to reduce the problem to a stabilization problem for a related differential inclusion. In turn, this can be analyzed by well established techniques [Sm].

Quite often, the desired trajectories of the differential inclusion, i.e. those that satisfy (1.8) or (1.9), correspond to limits of highly oscillating control functions. Examples of mechanical systems that can be stabilized by vibrating constraints are well known in the literature, see for example [St, Ka1, Ca2, L1, L2]. The present approach provides a different perspective to this stabilization problem, in the more general framework of control theory.

5. **Optimal control.** Optimization problems can be naturally posed, in connection with the impulsive system (1.4). Indeed, the problem of minimizing the time taken by a skier to reach the end of a trail, studied in [AB2], was one of initial motivations for research on this subject. As shown in [BR2], certain cases can be reduced to an optimization problem for a standard (non-impulsive) control system. Other cases are best studied in terms of a related differential inclusion. Further results on optimal impulsive control can be found in [KP, M2, M3, MR, P].

6. **Non-holonomic constraints.** There are interesting examples of mechanical systems where the time-dependent constraints imposed by the controller are of non-holonomic type. In this case, the equations describing the motion are clear as long as the control is smooth. In connection with discontinuous controls, however, the existence of a unique limit of smooth approximations requires careful analysis. Some results and examples in this direction can be found in [BR6].

7. **Locomotion in fluids.** A further application of the theory of impulsive Lagrangian systems, which we develop in the present paper, relates to the motion of deformable bodies immersed in an incompressible, irrotational fluid. Mathematical models of fish-like swimming have attracted increasing attention in recent years [Ch, Ga, KMR, KM, KO, KR1, KR2, Lg, Sa, ST, Wu]. It is generally assumed that the shape of the body can be assigned as a function of time. To completely determine the swim-like motion, one needs to find the position of the barycenter, and the angular orientation of the body. These are obtained by solving the Newton equations of motion for the immersed body, coupled with the incompressible Euler or Navier-Stokes equations for the surrounding fluid.

In most applications, the shape of the body is described by finitely many parameters. In the case of an irrotational, non-viscous fluid, the Euler equations can be reduced to a finite-dimensional system of O.D.E’s. This model, consisting of body + surrounding fluid, fits nicely within our framework of impulsively controlled, finite dimensional Lagrangian systems. In addition, we can now treat a large variety of situations where the overall shape of the body is not entirely prescribed. For example, think of a chain of rigid bodies, where the position of the first one is assigned, and the others trail at fixed distances, “flapping” in the fluid.
A key technical tool in the analysis of impulsive systems is the reparametrization of the graph of the control function \( t \mapsto u(t) \). Given a function \( u(\cdot) \) with bounded variation (BV), one can consider a Lipschitz continuous curve \( \gamma \), parametrized as \( s \mapsto (t(s), u(s)) \), which contains the graph of \( u \). Under suitable conditions, the impulsive equations (1.5) reduce to a standard system of O.D.E's, in terms of this new variable \( s \). This approach relies on the basic concept of \textbf{graph completion} of a BV function, introduced in [BR1]. It is worth mentioning that, beyond the theory of impulsive control, in [DM1, DM2, LF] this same idea was used also for the definition of non-conservative products, and non-conservative solutions to hyperbolic systems in one space dimension.

The outline of the paper is as follows. In Section 2 we review some elementary mechanical applications, which motivate the impulsive control model. Section 3 contains a derivation of the basic equation of motion, as in [AB1], assuming that the controller always implements \textbf{frictionless constraints}. Here we also discuss the analytic form of the equations, linear or quadratic w.r.t. the derivatives of the control functions. In Section 4 we explain the geometric conditions that render a system “fit for jumps”, and the invariant meaning of quadratic terms in the equations of motion. The basic definition of \textbf{graph completion} is introduced in Section 5, where we also examine the continuity of the control-to-trajectory map. In Section 6 we derive a differential inclusion whose set of trajectories describes the closure of all solutions of the control system (1.2). Results on the asymptotic stabilization of impulsive systems are reviewed in Section 7. Finally, in Section 8 we present a new application of the theory, to the motion of deformable bodies immersed in an incompressible, non-viscous fluid.

2 - Some examples

There are two fundamentally different ways to control the dynamics of a mechanical system. On one hand, the controller can apply additional external forces, thus modifying the time evolution of the system. This leads to a standard control problem, where the time derivative of the state depends continuously on the control function.

In other situations, also physically realistic, the controller acts on the system by directly assigning the values of some of the coordinates. The remaining coordinates are then be determined by solving an impulsive control system, where the derivative of the state depends on the time derivative of the control function. We illustrate these two cases with simple examples.

\textbf{Example 1.} Consider a small child riding on a swing, pushed by his mother. His motion is similar to that of a forced pendulum, say of length \( \ell \) and mass \( m \) (see fig. 1, left). In addition to the gravity acceleration \( g \), the child is subject to a force \( F = u(t) \) exerted by the pushing parent. This force represents a control, and its value can be prescribed at will (within certain bounds) as function of time. Calling \( \theta \) the angle with a vertical line, the motion of the swing is described by the equation

\[ m\ell \ddot{\theta} = -m g \ell \sin \theta + u(t). \tag{2.1} \]

This is a control equation in standard (non-impulsive) form.

\textbf{Example 2.} Next, consider an older boy riding on the same swing. By standing up or kneeling down, he can change at will the radius of oscillation (see fig. 1, right). We describe this new system in terms of two variables: the angle \( \theta \) and the radius of oscillation \( r \). The kinetic energy is given
by
\[ T(r, \theta, \dot{r}, \dot{\theta}) = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2), \quad (2.2) \]
while the potential energy is
\[ V(r, \theta) = -mgr \cos \theta. \quad (2.3) \]

The control implemented by the boy amounts to assigning the radius of oscillation as a function of time, i.e. \( r = u(t) \), for some control function \( u \). Calling \( L = T - V \) the associated Lagrangian function, the evolution of the remaining coordinate \( \theta = \theta(t) \) is now determined by the equation
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}, \quad (2.4) \]
which in this case yields
\[ 2mr \dot{\theta} \dot{r} + mr^2 \ddot{\theta} = -mgr \sin \theta. \quad (2.5) \]

Denoting by \( \omega = \dot{\theta} \) the angular velocity, and recalling that \( r = u \), we obtain the impulsive system
\[ \dot{\theta} = \omega, \quad \dot{\omega} = -\frac{g \sin \theta}{u} - \frac{2\omega}{u} \dot{u}. \quad (2.6) \]

Observe that the right hand side of the second equation depends (linearly) on the time derivative of the control function. In this special case, we can remove the dependence on \( \dot{u} \) by a change of variable. Namely, calling \( p = mr^2 \dot{\theta} = mu^2 \dot{\theta} \) the angular momentum, from (2.5) one obtains
\[ \dot{\theta} = \frac{p}{mu^2}, \quad \dot{p} = -mg \sin \theta, \quad (2.7) \]
where the right hand sides do not depend on \( \dot{u} \).

**Example 3.** Consider a skier on a straight, narrow trail with variable slope (fig. 2). It is assumed that he can control his speed only by raising or lowering the position of his barycenter, perpendicularly to the trail.

To describe the motion, let \( s \) be the arc-length parameter along points of the trail, and let \( h \) be the height of the barycenter of the skier, measured on a line perpendicular to the trail. Moreover,
call $R(s)$ the local radius of curvature of the trail, and let $m$ be the mass of the skier. The kinetic and the potential energies of the skier are now given respectively by

$$T(s,h,\dot{s},\dot{h}) = \frac{m}{2} \left( \dot{h}^2 + \frac{(R(s) \pm h)^2 \dot{s}^2}{R^2(s)} \right),$$

$$V(s,h) = m \left( H(s) + h \cos \theta(s) \right),$$

where $H(s)$ the height of the point $s$ of the trail (say, w.r.t. sea level), and $\theta$ is the angle between a vertical line and the perpendicular line to the trail, at $s$. Concerning the $\pm$ sign in the formula for the kinetic energy, one should take the plus sign at points where the trail is concave down, and the minus sign at points where it is concave up.

If the height $h = u(t)$ is assigned as a function of time, the time evolution of the remaining free coordinate $s$ can be derived from the Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{s}} = \frac{\partial L}{\partial s},$$

where $L = T - V$. See [AB2] for details. We remark that, in the case of a trail having constant radius of curvature $R(s) \equiv R_0$, the equations of motion are exactly the same as for the swing.

![Figure 2. Left: a skier on a straight, narrow trail with varying slope. Right: if the radius of curvature is constant, the mathematical formulation is the same as for the swing.](image)

**Example 4.** A bead slides without friction along a bar, while the bar can be rotated around the origin, on a vertical plane (see fig. 3). This system can be described by two lagrangian parameters: the distance $r$ of the bead from the origin, and the angle $\theta$ formed by the bar and a vertical line. Calling $m$ the mass of the bead, its kinetic and potential energy are still given by (2.2)-(2.3). In the present case, however, we assign the angle $\theta = u(t)$ as a function of time, while the radius $r$ is the remaining free coordinate. Instead of (2.4), the equations of motion are derived from the Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r},$$

where $L = T - V$. Using (2.2)-(2.3), and setting $\theta = u$, from (2.8) we obtain the second order equation

$$\ddot{r} = r\dot{u}^2(t) + g \cos u(t)$$

Observe that in this case the right hand side of the equation contains the square of the derivative of the control.

**3 - The equations of motion**
Figure 3. A bead sliding without friction along a rotating bar, in a vertical plane.

Consider a system described by $N$ Lagrangian variables $q^1, \ldots, q^N$. Let the kinetic energy $T = T(q, \dot{q})$ be given by a positive definite quadratic form of the time derivatives $\dot{q}^i$, say

$$T(q, \dot{q}) = \frac{1}{2} \sum_{i,j=1}^{N} A_{ij}(q) \dot{q}^i \dot{q}^j.$$  \hfill (3.1)

In addition, we assume that the system is affected by external forces having components $Q_i = Q_i(t, q, \dot{q})$. The motion of the (uncontrolled) system is thus determined by the equations

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^i} = \frac{\partial T}{\partial q^i} + Q_i(t, q, \dot{q}) \quad i = 1, \ldots, N.$$  \hfill (3.2)

In a common situation, the controller can apply additional forces, whose components $\phi_i(q, u)$ depend continuously on the state $q$ of the system and on the value $u = u(t)$ of the control function. In this case, one obtains the system of equations

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^i} = \frac{\partial T}{\partial q^i} + Q_i(t, q, \dot{q}) + \phi_i(q, u) \quad i = 1, \ldots, N.$$  

This is equivalent to a standard control system, where the right hand side depends continuously on the control $u$.

In a quite different but still realistic situation, a controller can prescribe the values of the last $m$ coordinates $q^{n+1}, \ldots, q^{n+m}$ as functions of time, say

$$q^{n+i}(t) = u_i(t) \quad i = 1, \ldots, m.$$  \hfill (3.3)

We assume that this is achieved by implementing $m$ frictionless constraints. Here frictionless means that the forces produced by the constraints make zero work in connection with any virtual displacement of the remaining free coordinates $q^1, \ldots, q^n$. Calling $\Phi_i(t)$ the components of the additional forces, used to implement the constraints (3.3), the motion is now determined by the equations

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^i} = \frac{\partial T}{\partial q^i} + Q_i(t, q, \dot{q}) + \Phi_i(t) \quad i = 1, \ldots, n + m.$$  \hfill (3.4)

The assumption that the constraints are frictionless is expressed by the identities

$$\Phi_1(t) = \cdots = \Phi_n(t) = 0.$$  \hfill (3.5)
Remarkably, there is no need to compute the remaining components of the forces $\Phi_{n+1}, \ldots, \Phi_{n+m}$, in order to completely determine the evolution of the system. Indeed, the variables $q^{n+1}, \ldots, q^{n+m}$ are already assigned by (3.3). Of course, their time derivatives

$$q^{n+1} = \dot{u}_1(t), \ldots, q^{n+m} = \dot{u}_m(t)$$

are also determined. We now show that the evolution of components $q^1, \ldots, q^n$ can be derived from the first $n$ equations in (3.4), taking (3.5) into account. This is done in two steps.

**STEP 1:** In connection with the quadratic form (3.1), introduce the conjugate moments

$$p_i = p_i(q, \dot{q}) = \frac{\partial T}{\partial \dot{q}^i} = \sum_{i=1}^{n+m} A_{ij}(q) \dot{q}^j.$$  \hspace{1cm} (3.6)

Moreover, define the Hamiltonian function

$$H(q, p) = \frac{1}{2} \sum_{i,j=1}^{n+m} B^{ij}(q) p_i p_j,$$  \hspace{1cm} (3.7)

where $B^{ij}$ are the components of the $(n + m) \times (n + m)$ inverse matrix $B = A^{-1}$. In other words,

$$\sum_{j=1}^{n+m} B^{ij} A_{jk} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases} \hspace{1cm} (3.8)$$

**STEP 2:** Solve the system of Hamiltonian equations for the first $n$ variables

$$\begin{cases} \dot{q}^i = \frac{\partial H}{\partial p_i}(q, p) \\ \dot{p}_i = \frac{\partial H}{\partial q^i}(q, p) + Q_i(t, q, \dot{q}) \end{cases} \hspace{1cm} i = 1, \ldots, n. \hspace{1cm} (3.9)$$

Notice that (3.9) is a system of $2n$ equations for $q^1, \ldots, q^n, p_1, \ldots, p_n$, where the right hand side also depends on the remaining components $q^i, p_i, i = n+1, \ldots, n+m$. We can remove this explicit dependence by inserting the values

$$\begin{cases} q^{n+i} = u_i(t), & \dot{q}^{n+i} = \dot{u}_i(t) \\ p_j = p_j(p_1, \ldots, p_n, q^{n+1}, \ldots, q^{n+m}) \end{cases} \hspace{1cm} i = 1, \ldots, m, \hspace{1cm} j = n+1, \ldots, n+m. \hspace{1cm} (3.10)$$

In (3.10), to express $p_j$ as a linear combinations of $p_1, \ldots, p_n, q^{n+1}, \ldots, q^{n+m}$, we proceed as follows. Let $C = (C_{ij})$ be the inverse of the $m \times m$ submatrix $(B^{ij})_{i,j=n+1,\ldots,n+m}$, so that

$$\sum_{i=n+1}^{n+m} C_{ji} B^{ik} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases} \hspace{1cm} j, k \in \{n+1, \ldots, n+m\}. \hspace{1cm} (3.11)$$

Recalling that $p = A\dot{q}, \dot{q} = Bp$, we multiply by $C_{ji}$, both sides of the identity

$$\dot{q}^i = \sum_{k=1}^{n} B^{ik} p_k + \sum_{k=n+1}^{n+m} B^{ik} p_k,$$
then we sum over \( i = n + 1, \ldots, n + m \). By (3.11), this yields

\[
p_j = \sum_{i=n+1}^{n+m} C_{ji} q_i^\ell - \sum_{i=n+1}^{n+m} \sum_{k=1}^{n} C_{ji} B^{ik} p_k \quad j = n + 1, \ldots, n + m. \tag{3.12}
\]

Inserting in (3.9) the values \( p_{n+1}, \ldots, p_{n+m} \) given at (3.12), we obtain a closed system of 2n equations for the 2n variables \( q^1, \ldots, q^n, p_1, \ldots, p_n \).

We now take a closer look at the equation of motion derived at (3.9)-(3.10). For simplicity, we temporarily assume that there are no external forces, i.e. \( Q_i(t, q, \dot{q}) \equiv 0 \). The extension to the general case is straightforward.

Fix an index \( i \in \{1, \ldots, n\} \). Inserting the values (3.12) for the last \( m \) components in (3.9) and recalling the definition of the Hamiltonian function at (3.7), we obtain

\[
\dot{q}^i = \frac{\partial}{\partial p_i} \left\{ \frac{1}{2} \sum_{j,k=1}^{n+m} B^{jk}(q) p_j p_k \right\} \\
= \sum_{j=1}^{n} B^{ij} p_j + \sum_{j=n+1}^{n+m} B^{ij} p_j \\
= \sum_{j=1}^{n} B^{ij} p_j + \sum_{j=n+1}^{n+m} B^{ij} \left( \sum_{\ell=n+1}^{n+m} C_{j\ell} \dot{q}^\ell - \sum_{\ell=n+1}^{n+m} \sum_{k=1}^{n} C_{j\ell} B^{ik} p_k \right). \tag{3.13}
\]

Next, using again (3.7) and (3.12) we compute

\[
\dot{p}_i = -\frac{\partial}{\partial q^i} \left\{ \frac{1}{2} \sum_{j,k=1}^{n+m} B^{jk}(q) p_j p_k \right\} \\
= - \left( \frac{1}{2} \sum_{j,k=1}^{n} + \sum_{j=1}^{n} \sum_{k=n+1}^{n+m} + \frac{1}{2} \sum_{j,n+1}^{n+m} \right) \frac{\partial B^{jk}}{\partial q^i} p_j p_k \\
= - \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial B^{jk}}{\partial q^i} p_j p_k - \sum_{j=1}^{n} \sum_{k=n+1}^{n+m} \frac{\partial B^{jk}}{\partial q^i} p_j \left( \sum_{h=n+1}^{n+m} C_{kh} q^h - \sum_{h=n+1}^{n+m} C_{kh} B^{hf} p_f \right) \\
- \frac{1}{2} \sum_{j,n+1}^{n+m} \frac{\partial B^{jk}}{\partial q^i} \left( \sum_{h=n+1}^{n+m} C_{jh} q^h - \sum_{h=n+1}^{n+m} \sum_{\ell=1}^{n} C_{j\ell} B^{hf} p_f \right) \left( \sum_{r=n+1}^{n+m} C_{kr} q^r - \sum_{r=n+1}^{n+m} \sum_{\ell=1}^{n} C_{kr} B^{fr} p_f \right). \tag{3.14}
\]

Recalling that \( \dot{q}^{n+i} = \dot{u}_i \), and that the matrices \( C(q) = (C_{ij}(q)) \) are invertible, a direct inspection of the above equations reveals that:

(i) The right hand side of (3.13) is always an affine function of the derivatives \( \dot{u}_1, \ldots, \dot{u}_m \).

(ii) The right hand side of (3.14) is an affine function of the derivatives \( \dot{u}_1, \ldots, \dot{u}_m \) if and only if

\[
\frac{\partial B^{jk}(q)}{\partial q^i} \equiv 0 \quad \text{for all} \quad i \in \{1, \ldots, n\}, \quad j, k \in \{n + 1, \ldots, n + m\}. \tag{3.15}
\]
Following [AB1], systems whose equations of motion are affine w.r.t. the time derivatives of the control will be called \textbf{fit for jumps}. In the special case where the derivatives $\dot{u}_i$ do not appear at all in the equations, we say that the system is \textbf{strongly fit for jumps}. From the above analysis we thus obtain

**Theorem 1.** The system described by (3.3)–(3.5) is “fit for jumps” if and only if the external forces $Q_i$ are affine functions of the derivatives $\dot{q}^j$, $j = n + 1, \ldots, n + m$, and the identities (3.15) hold.

**Theorem 2.** The system at (3.3)–(3.5) is “strongly fit for jumps” provided that the external forces $Q_i$ depend only on the variables $t, q^i$, and moreover the identities (3.15) hold, together with

$$B^{ij}(q) \equiv 0 \quad i \in \{1, \ldots, n\}, \quad j \in \{n + 1, \ldots, n + m\}. \quad (3.16)$$

\section*{4 - Geometric properties of the foliation}

In the present section we investigate the geometric meaning of the properties “fit for jumps” and “strongly fit for jumps”, introduced above. For simplicity, we assume that there are no external forces, so that $Q_i \equiv 0$. Our impulsive system (3.3)–(3.5) is thus defined by the equations

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^i} = \frac{\partial T}{\partial q^i} + \Phi_i(t) \quad i = 1, \ldots, n + m, \quad (4.1)$$

$$q^{n+i}(t) = u_i(t) \quad i = 1, \ldots, m, \quad \Phi_1(t) = \cdots = \Phi_n(t) = 0. \quad (4.2)$$

Consider the manifold $\mathcal{M}$, with coordinates $q^1, \ldots, q^{n+m}$ and with the Riemann metric given by

$$ds^2 = \sum_{i,j=1}^{n+m} A_{ij}(q) \, dq^i \, dq^j. \quad (4.3)$$

Here $A_{ij}$ is the quadratic form describing the kinetic energy, as in (3.1).

![Figure 4](attachment:figure4.png)

Figure 4. The geodesic curve $\gamma$ has perpendicular intersection with all leaves $\Gamma_c$ of the foliation.
By assigning $m$ constant values for the last $m$ coordinates, say $c_1, \ldots, c_m$, we obtain an $n$-dimensional submanifold
\[
\Gamma_c \doteq \{(q^1, \ldots, q^{n+m}); \quad q^{n+1} = c_1, \ldots, q^{n+m} = c_m\}.
\] (4.4)

The union of all these submanifolds constitutes a foliation $\mathcal{F}$ of the original manifold $\mathcal{M}$. The analysis in [AB1] and [Ra] has shown that the geometric properties of this foliation play a crucial role in determining the form of the equations of motion (3.9)-(3.10). We summarize here the main results. Consider the following two properties:

(P1) If a geodesic curve $\gamma$ crosses orthogonally one of the leaves $\Gamma_c$ of the foliation $\mathcal{F}$, then all of the other leaves touched by $\gamma$ will also be crossed orthogonally.

(P2) The $m$-dimensional distribution $T^\perp_{\Gamma}$ orthogonal to the leaves of the foliation is involutive.

The first property is illustrated in fig. 4. We now explain the second property. At each point $q \in \mathcal{M}$, the $(n+m)$-dimensional tangent space $T_{\mathcal{M}}(q)$ can be decomposed as a direct sum (see fig. 5)
\[
T_{\mathcal{M}}(q) = T_{\Gamma}(q) \oplus T^\perp_{\Gamma}(q).
\] (4.5)
Here $T_{\Gamma}(q)$ is the $n$-dimensional space tangent to the leaf of the foliation passing through $q$, while $T^\perp_{\Gamma}(q)$ is its orthogonal complement, w.r.t. the metric (4.3).

![Figure 5](image_url)

Figure 5. The distribution orthogonal to the leaves of the foliation $\mathcal{F}$ can be integrable (right) or not integrable (left).

The requirement that the distribution $T^\perp_{\Gamma}$ is involutive means that, at least locally, it is integrable (see fig. 5, right). One can thus find a system of adapted coordinates, which we still call $(q^1, \ldots, q^{n+m})$, such that
\[
T_{\Gamma}(q) = \text{span}\left\{ \frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n} \right\}, \quad T^\perp_{\Gamma}(q) = \text{span}\left\{ \frac{\partial}{\partial q^{n+1}}, \ldots, \frac{\partial}{\partial q^{n+m}} \right\}.
\] (4.6)
Because of (4.6), for each choice of the constants $b_1, \ldots, b_n$, the submanifold

$$\Gamma^b = \left\{ (q^1, \ldots, q^{n+m}) ; \quad q^1 = b_1, \ldots, q^n = b_n \right\}. \quad (4.7)$$

has a perpendicular crossing with every leaf $\Gamma_c$ of the foliation $\mathcal{F}$, defined at (4.4). Notice that this orthogonality condition implies that, in the adapted coordinates, the symmetric matrix $A = (A_{ij})$ defining the Riemann metric takes the form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where $A_1$ is an $n \times n$ matrix, while $A_2$ is an $m \times m$ matrix. In this case, the inverse matrix $B = A^{-1}$ has the form

$$B = A^{-1} = \begin{pmatrix} A_1^{-1} & 0 \\ 0 & A_2^{-1} \end{pmatrix},$$

hence the identities (3.16) clearly hold.

The main results, relating the form of the impulsive equations (4.1)-(4.2) to the geometric properties of the foliation $\mathcal{F}$ in (4.4), are as follows.

**Theorem 3.** The impulsive system (4.1)-(4.2) is “fit for jumps” if and only if the foliation $\mathcal{F}$ defined at (4.4) satisfies the property (P1).

**Theorem 4.** Assume that the foliation $\mathcal{F}$ defined at (4.2) satisfies both properties (P1) and (P2). Then, in the adapted coordinates satisfying (4.6), the system (4.1)-(4.2) is “strongly fit for jumps”.

For a proof we refer to [Ra]. We remark that, in the case where the control is scalar (i.e. $m = 1$), the orthogonal distribution $T_\Gamma^\perp$ is always involutive. Therefore, if property (P1) is satisfied, one can construct a local system of coordinates satisfying (4.6), and the conclusion of Theorem 4 holds.

![Figure 6](image-url)

Figure 6. Left: a foliation satisfying the property (P1). Right: a foliation for which (P1) fails.

Theorem 3 is best illustrated by our earlier examples. In Example 2 (boy on a swing), we take the radius of oscillation as controlled coordinate. The foliation whose leaves are the circumferences

$$\Gamma_c = \left\{ (r, \theta) ; \quad r = c \right\}$$
satisfies the property (P1). Indeed, in this case the geodesics are straight lines. If a straight line $\gamma$ crosses any of the circumferences $\Gamma_c$ perpendicularly, then it still crosses perpendicularly all the others (fig. 6, left). Hence the system is fit for jumps.

In Example 4 (bead sliding along a rotating axis), we consider a system with the same kinetic energy. However, we take the angle as a controlled coordinate. The foliation whose leaves are the rays through the origin

$$\Gamma_c = \{(r, \theta); \quad \theta = c\}$$

do not satisfy the property (P1). A geodesic $\gamma$ can cross one of the leaves perpendicularly at a point $q$, without crossing perpendicularly the other leaves $\Gamma_c$ of the foliation (fig. 6, right). In accordance with Theorem 3, this system is not fit for jumps. Indeed, the equation of motion (2.9) contains the square of the time derivative of the control function.

The invariant character of the equations of motion was already pointed out in [Ma]. In other words, the motion depends only on the Riemannian metric tensor $A_{ij}$ and on the foliation $\mathcal{F}$ itself, not on the particular choice of coordinates which define the foliation.

In particular, recalling (3.14), consider the quadratic mapping $Q = (Q_1, \ldots, Q_n)$, with

$$Q_i(\xi_1, \ldots, \xi_m) = -\frac{1}{2} \sum_{j,k=n+1}^{n+m} \frac{\partial B_{j,k}}{\partial q^i} \left( \sum_{h=1}^{m} C_{j,n+h} \xi_h \right) \left( \sum_{r=1}^{m} C_{k,n+r} \xi_r \right),$$

which describes the contribution of the quadratic terms $\dot{q}^{n+i} \dot{q}^{n+j} = \dot{u}_i \dot{u}_j$ to the dynamics of the system. The intrinsic meaning of this quadratic mapping $Q$ can be illustrated by the following construction (fig. 7, left).

Let $\mathcal{M}$ be a Riemannian manifold of dimension $n + m$, where the metric is defined by the quadratic tensor $A_{ij}$ in (4.3). Consider a foliation $\mathcal{F}$, with leaves $\Gamma_c$, which in a suitable coordinate system are described by (4.4). Notice that each leaf has dimension $n$, while the quotient manifold $\mathcal{M}/\mathcal{F}$ has dimension $m$.

Fix a point $\bar{q}$, say on the leaf $\Gamma_c$. Observe that there is a canonical bijection between the tangent space $T_{\mathcal{M}/\mathcal{F}}(\Gamma_c)$ and the space $T^\perp(\bar{q})$ of tangent vectors $v \in T(\bar{q})$ at the point $\bar{q}$ which are perpendicular to the leaf $\Gamma_c$.

For a given vector $v \in T^\perp(\bar{q})$, construct the geodesic curve $\gamma$, which is tangent to $v$ at $\bar{q}$. For a given $\varepsilon > 0$, let $q_\varepsilon$ be the point on the curve $\gamma$ which has distance $\varepsilon |v|$ from $\bar{q}$. This point will be on some other leaf of the foliation, say $q_\varepsilon \in \Gamma_{c(\varepsilon)}$.

Next, construct a second geodesic curve $\gamma'$, starting from $q_\varepsilon$, which is perpendicular to the leaf $\Gamma_{c(\varepsilon)}$ and crosses the original leaf $\Gamma_c$ at some point $q'_\varepsilon$. For $\varepsilon$ sufficiently small, using the implicit function theorem one can show that this second curve is unique. The limit

$$Q(v) = \lim_{\varepsilon \to 0} \frac{q'_\varepsilon - \bar{q}}{\varepsilon^2}$$

now defines a tangent vector at $\bar{q}$. More precisely, the assignment $v \mapsto Q(v)$ is a homogeneous quadratic mapping from the space $T^\perp(\bar{q})$ of vectors perpendicular to the leaf $\Gamma_c$ to the space $T\Gamma(\bar{q})$ of vectors parallel to the leaf. We can extend this mapping to a symmetric bilinear form

$$T^\perp(\bar{q}) \otimes T^\perp(\bar{q}) \mapsto T\Gamma(\bar{q})$$

by setting

$$\langle v, w \rangle = \frac{1}{4} [Q(v + w) - Q(v) - Q(w)] .$$
The expression (4.8) corresponds to this quadratic mapping, in a suitable set of coordinates. For details of this construction, we refer to the forthcoming paper [BR5].

Two special cases of this construction are worth mentioning.

(I) If the system is “fit for jumps”, then by the property (P1) the geodesic \( \gamma \) is perpendicular to the leaf \( \Gamma_{c(\epsilon)} \) at the point \( q_\epsilon \). This implies \( \gamma' = \gamma \) and hence \( q_\epsilon' = \bar{q} \) for every \( \epsilon \). Hence \( Q \equiv 0 \).

(II) Let \( \mathcal{M} = \mathbb{R}^{n+1} \) with the Euclidean metric. Assume that \( m = 1 \), so that the leaves of the foliation are \( n \)-dimensional hypersurfaces. Then the orthogonal space \( T_{\Gamma}(\bar{q}) \) is 1-dimensional. The mapping \( Q \) is now defined in terms of the principal curvature of the curves \( \gamma \) orthogonal to the leaves \( \Gamma_c \) of the foliation (fig. 7, right). This case was studied in detail in [LR].

5 - Continuity of the control-to-trajectory map

Consider the Cauchy problem for the impulsive control system

\[
\dot{x} = \tilde{f}(t, x, u) + \sum_{i=1}^{m} \tilde{g}_i(t, x, u) \dot{u}_i + \sum_{i,j=1}^{m} \tilde{h}_{ij}(t, x, u) \dot{u}_i \dot{u}_j ,
\]  

(5.1)

with initial data

\[
x(0) = \bar{x} , \quad u(0) = \bar{u} .
\]  

(5.2)

Assume that all functions \( f, g_i, h_{ij} \) are smooth. If the control function \( t \mapsto u(t) \) is \( C^1 \), then the local existence and uniqueness of a solution follows from classical O.D.E. theory. A natural question now is:

What is the most general class of control functions \( u(\cdot) \) for which the corresponding trajectory of (5.1)-(5.2) is well defined ?

The answer strongly depends on the form of the equations (5.1). For example, if the coefficients \( \tilde{h}_{ij} \) do not vanish, then the control components \( u_i \) must be absolutely continuous and have a square integrable derivative. On the other hand, if \( \tilde{g}_i \equiv 0, \tilde{h}_{ij} \equiv 0 \), then the derivatives of the control
function do not enter at all in the equation, and a unique solution can be constructed for any bounded, measurable control \( u(\cdot) \).

An interesting case, on which we shall focus the attention, is when the system (3.3)-(3.5) is fit for jumps, but not strongly fit. Renaming coordinates, this leads to a system of the form

\[
\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) \dot{u}_i ,
\]

where the vector fields \( g_i \) do not commute, i.e. their Lie brackets \([g_i, g_j]\) do not vanish identically. In this setting, an appropriate class of control functions, for which a solution of (5.3)-(5.2) can be defined, appears to be the space \( BV \) of functions with bounded variation. However, some care must be taken. Given a discontinuous control \( u \in BV \), if the vector fields \( g_i \) do not commute, one can find two sequences of uniformly Lipschitz continuous controls, say \( u^{(\nu)}(\cdot) \), \( v^{(\nu)}(\cdot) \), \( \nu \geq 1 \), such that

\[
\|u^{(\nu)} - u\|_{L^1} \to 0 , \quad \|v^{(\nu)} - u\|_{L^1} \to 0 ,
\]

but the corresponding trajectories of (5.3)-(5.2) converge to different limits. This will be shown in Example 5. To uniquely determine a trajectory \( x(\cdot, u) \) corresponding to a control \( u \in BV \), one also needs to specify the curve along which \( u \) is varied, at times of jumps. This leads to the concept of “graph completion”, discussed later in this section.

**Example 5.** Consider the impulsive system on \( IR^2 \)

\[
(\dot{x}_1, \dot{x}_2) = (1, 0) \dot{u}_1 + (0, x_1) \dot{u}_2 = g_1(x) \dot{u}_1 + g_2(x) \dot{u}_2 ,
\]

with initial conditions

\[
(x(0), x_2(0)) = (0, 0) , \quad (u_1(0), u_2(0)) = (0, 0) .
\]

Observe that in this case the vector fields \( g_1, g_2 \) do not commute. Indeed, their Lie bracket is

\[
[g_1, g_2] = (Dg_2) g_1 - (Dg_1) g_2 = (0, 1) .
\]

Consider the discontinuous control function

\[
(u_1(0), u_2(0)) = \begin{cases} (0, 0) & \text{if } t < 1 , \\ (1, 1) & \text{if } t > 1 . \end{cases}
\]

In the \( L^1 \) norm, we can approximate \( u \) by a sequence of Lipschitz continuous control functions \( u^{(\nu)} \), defined as

\[
(u^{(\nu)}_1, u^{(\nu)}_2)(t) = \begin{cases} (0, 0) & \text{if } t \in [0, 1 - 1/\nu] , \\ (0, 1 + \nu(t - 1)) & \text{if } t \in [1 - 1/\nu, 1] , \\ (\nu(t - 1), 1) & \text{if } t \in [1, 1 + 1/\nu] , \\ (1, 1) & \text{if } t \in [1 + 1/\nu, 2] . \end{cases}
\]

The corresponding Carathéodory solutions of the Cauchy problem (5.4)-(5.5) are computed as

\[
(x_1, x_2)(t, u^{(\nu)}) = \begin{cases} (0, 0) & \text{if } t \in [0, 1] , \\ (\nu(t - 1), 0) & \text{if } t \in [1, 1 + 1/\nu] , \\ (1, 0) & \text{if } t \in [1 + 1/\nu, 2] . \end{cases}
\]
As $\nu \to \infty$, the above sequence of trajectories converges (pointwise and in the $L^1$ norm) to the limit trajectory
\[
(x_1, x_2)(t) = \begin{cases} 
(0, 0) & \text{if } t \in [0, 1], \\
(1, 0) & \text{if } t \in ]1, 2].
\end{cases} 
\tag{5.9}
\]

Next, consider a second approximating sequence
\[
(v^{(\nu)}_1, v^{(\nu)}_2)(t) = \begin{cases} 
(0, 0) & \text{if } t \in [0, 1], \\
(\nu(t - 1), \nu(t - 1)) & \text{if } t \in [1, 1 + 1/\nu], \\
(1, 1) & \text{if } t \in [1 + 1/\nu, 2].
\end{cases} 
\tag{5.10}
\]

The corresponding solutions are now
\[
(x_1, x_2)(t, v^{(\nu)}) = \begin{cases} 
(0, 0) & \text{if } t \in [0, 1], \\
(\nu(t - 1), \nu^2(t - 1)^2/2) & \text{if } t \in [1, 1 + 1/\nu], \\
(1, 1/2) & \text{if } t \in [1 + 1/\nu, 2].
\end{cases} 
\tag{5.11}
\]

As $\nu \to \infty$, in this second case the limit trajectory is
\[
(x_1, x_2)(t) = \begin{cases} 
(0, 0) & \text{if } t \in [0, 1], \\
(1, 1/2) & \text{if } t \in ]1, 2].
\end{cases} 
\tag{5.12}
\]

This limit is still well defined, but different from (5.9).

The above example shows that, in the non-commutative case, the limit of the approximating trajectories depends not only on the pointwise values of $u$, but also on the way we approximate $u$ by more regular controls. Observe that in the first case the values of $u^{(\nu)}$ change from $(0,0)$ to $(0,1)$, and then to $(1,1)$. In the second case, the values of $v^{(\nu)}$ vary from $(0,0)$ directly to $(1,1)$ along a straight line. This suggests that, in the noncommutative case, a unique trajectory can be determined only if, at every time $\tau$ where $u$ has a jump, we specify along which path the transition from $u(\tau^-)$ to $u(\tau^+)$ takes place. The next definition makes this more precise.

**Definition 1.** A **graph-completion** of a $BV$ function $u : [0, T] \mapsto \mathbb{R}^m$ is a Lipschitz continuous path $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_m) : [0, S] \mapsto [0, T] \times \mathbb{R}^m$ such that

(i) $\gamma(0) = (0, u(0)), \quad \gamma(S) = (T, u(T))$,

(ii) $\gamma_0(s_1) \leq \gamma_0(s_2)$ for all $0 \leq s_1 < s_2 \leq S$,

(iii) for each $t \in [0, T]$ there exists some $s$ such that $\gamma(s) = (t, u(t))$.

Notice that the path $\gamma$ provides a continuous parametrization of the graph of $u$ in the $(t, u)$ space. At a time $\tau$ where $u$ has a jump, the curve $\gamma$ must include an arc joining the left and right points $(\tau, u(\tau^-)), (\tau, u(\tau^+))$.

**Remarks.** If the function $u$ itself is Lipschitz continuous, one can construct a graph completion of $u$ simply by taking $\gamma(s) = (s, u(s))$.

If $u$ is a BV function taking values in an Euclidean space, there is a canonical way to construct a graph completion. Namely:

**STEP 1:** Bridge each jump of $u$ with a straight segment,
STEP 2: Reparametrize the entire curve by arc-length.

In the case where \( u \) takes values in a Riemann manifold, one could move from \( u(\tau^-) \) to \( u(\tau^+) \) along a geodesic (assuming that the shortest path connecting the two values is unique). In general, however, the choice of the specific path can only be justified case by case.

Using graph-completions, we can now construct generalized trajectories for impulsive control systems with non-commutative vector fields.

Consider again the impulsive Cauchy problem (5.3), (5.2). Let \( u : [0, T] \rightarrow \mathbb{R}^m \) be a control function with bounded variation, and let \( \gamma = (\gamma_0, \gamma_1, \ldots, \gamma_m) \) be a graph-completion of \( u \). Consider the related Cauchy problem

\[
\frac{d}{ds} y(s) = f(y(s)) \dot{\gamma}_0(s) + \sum_{i=1}^{m} g_i(y(s)) \dot{\gamma}_i(s), \quad y(0) = \bar{x}.
\] (5.13)

**Definition 2.** Let \( y(\cdot, \gamma) \) be the unique Carathéodory solution of (5.13). Then the (possibly multivalued) function

\[
t \mapsto x(t, \gamma) = \{ y(s, \gamma) ; \ \gamma_0(s) = t \}
\]

is the **generalized trajectory** of (5.2)-(5.3) determined by the graph-completion \( \gamma \) of \( u \).

Observe that, by definition, the path \( \gamma \) is absolutely continuous, hence the Carathéodory solution of (5.13) is well defined.

It can be shown that the trajectory \( x(\cdot, \gamma) \) depends on the path \( \gamma \) itself, but not on the way it is parametrized. In particular, let \( \tilde{\gamma} : [0, \tilde{S}] \mapsto [0, T] \times \mathbb{R}^m \) be another graph-completion of \( u \) such that

\[
\tilde{\gamma}(s) = \gamma(\phi(s)) \quad s \in [0, \tilde{S}]
\]

for some absolutely continuous, strictly increasing \( \phi : [0, \tilde{S}] \mapsto [0, S] \). Then the generalized trajectories \( x(\cdot, \tilde{\gamma}) \) and \( x(\cdot, \gamma) \) coincide.

**Example 5 (continued).** For the discontinuous function \( u \) in (5.6), consider the graph-completion...
Figure 9. The curves $\gamma$ and $\tilde{\gamma}$ provide two different graph-completions of the same function $u(\cdot)$.

$\gamma : [0, 4] \rightarrow [0, 2] \times \mathbb{R}^2$ defined as (see fig. 9)

$$
\gamma(s) = \begin{cases} 
(s, 0, 0) & \text{if } s \in [0, 1], \\
(1, 0, s - 1) & \text{if } s \in [1, 2], \\
(1, s - 2, 1) & \text{if } s \in [2, 3], \\
(s - 3, 1, 1) & \text{if } s \in [3, 4]. 
\end{cases}
$$

The generalized trajectory $t \mapsto x(t, \gamma)$ is

$$
x(t, \gamma) = \begin{cases} 
(0, 0) & \text{if } t \in [0, 1], \\
(1, 0) & \text{if } t \in [1, 2], 
\end{cases}
$$

$$
x(1, \gamma) = \{(\xi, 0); \quad 0 \leq \xi \leq 1\}.
$$

Observe that the curve $\gamma$ in this case is precisely the limit of the graphs of the approximating functions $u^{(\nu)}$ at (5.7). For all $t \in [0, 2]$, $t \neq 1$, the generalized trajectory coincides with the limit in (5.9). At the time $\tau = 1$ where $u$ has a jump, the generalized trajectory $x(1, \gamma)$ is multivalued.

A different graph-completion is achieved by bridging the jump at time $\tau = 1$ with one single straight segment. This yields the path $\tilde{\gamma} : [0, 3] \rightarrow [0, 2] \times \mathbb{R}^2$ defined as (see fig. 9)

$$
\tilde{\gamma}(s) = \begin{cases} 
(s, 0, 0) & \text{if } s \in [0, 1], \\
(1, s - 1, s - 1) & \text{if } s \in [1, 2], \\
(s - 1, 1, 1) & \text{if } s \in [2, 3]. 
\end{cases}
$$

The corresponding (multivalued) trajectory of (5.13) is given by

$$
x(t, \tilde{\gamma}) = \begin{cases} 
(0, 0) & \text{if } t \in [0, 1], \\
(1, 1/2) & \text{if } t \in [1, 2], 
\end{cases}
$$

$$
x(1, \tilde{\gamma}) = \{(\xi, \xi^2/2); \quad 0 \leq \xi \leq 1\}.
$$

In this case, the curve $\tilde{\gamma}$ is the limit of the graphs of the approximating functions $v^{(\nu)}$ at (5.10). For all $t \in [0, 2]$, $t \neq 1$, the generalized trajectory coincides with the limit in (5.12), while $x(1, \tilde{\gamma})$ is multivalued.

It is important to understand how the trajectory of the system (5.13) depends on the choice of graph completion. The main results in this direction, proved in [BR1], are as follows.
Let \( \gamma : [0, S] \rightarrow \mathbb{R}^{1+m} \) and \( \tilde{\gamma} : [0, \tilde{S}] \rightarrow \mathbb{R}^{1+m} \) be any two graph completions of the same control function \( u \in BV \). Define their distance as

\[
\Delta(\gamma, \tilde{\gamma}) = \inf_{\phi} \max_{s \in [0, S]} |\gamma(s) - \tilde{\gamma}(\phi(s))| ,
\]

where the infimum is taken over all continuous, strictly increasing, surjective maps \( \phi : [0, S] \rightarrow [0, \tilde{S}] \). In addition, we recall that the Hausdorff distance between two compact sets \( A, B \subset \mathbb{R}^N \) is

\[
d_H(A, B) = \max \left\{ \max_{a \in A} d(a, B), \max_{b \in B} d(b, A) \right\} .
\]

The distances of a point from a set are here defined as

\[
d(a, B) = \inf_{x \in B} |x - a| , \quad d(b, A) = \inf_{x \in A} |x - b| .
\]

We then have

**Theorem 5.** Let the vector fields \( f, g \) in (5.3) be Lipschitz continuous. Let \( u_n : [0, T] \rightarrow \mathbb{R}^m \) be a sequence of control functions. For each \( n \geq 0 \), let \( \gamma_n \) be a graph-completion of \( u_n \). Assume that the total variation of the maps \( \gamma_n \) remains uniformly bounded, and that

\[
\Delta(\gamma_n, \gamma_0) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty .
\]

Then the graphs of the corresponding (possibly multivalued) trajectories \( x(\cdot, \gamma_n) \) converge to the graph of \( x(\cdot, \gamma_0) \) in the Hausdorff metric.

For a proof, see [BR1]. The following result, also proved in [BR1], relates the generalized solution obtained from a graph completion to the limit of more regular solutions, corresponding to smooth control functions.

**Theorem 6.** Let \( \gamma : [0, S] \rightarrow \mathbb{R}^{1+m} \) be a graph completion of a control \( u : [0, T] \rightarrow \mathbb{R}^m \). Let \( (u_n)_{n \geq 1} \) be a sequence of Lipschitz continuous controls with uniformly bounded total variation, which approximates \( \gamma \) in the following sense: setting \( \gamma_n(s) = (s, u_n(s)) \), one has

\[
\Delta(\gamma, \gamma_n) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty .
\]

Then the generalized solution \( x(\cdot, \gamma) \) of (5.13) corresponding to the graph completion \( \gamma \) satisfies

\[
x(t, \gamma) = \lim_{n \rightarrow \infty} x(t, u_n)
\]

at every time \( t \) where \( x(\gamma, t) \) is single valued, hence almost everywhere.

**Example 5 (continued).** The sequence of controls \( u^{(\nu)} \) at (5.7) approximates the graph completion \( \gamma \) at (5.14). The corresponding trajectories \( x(\cdot, u^{(\nu)}) \) converge pointwise to the generalized trajectory \( t \mapsto x(t, \gamma) \) in (5.15), for every \( t \neq 1 \).

On the other hand the sequence of controls \( v^{(\nu)} \) at (5.10) approximates the graph completion \( \tilde{\gamma} \) at (5.16). The corresponding trajectories \( x(\cdot, v^{(\nu)}) \) converge pointwise to the generalized trajectory \( t \mapsto x(t, \tilde{\gamma}) \) in (5.17), for every \( t \neq 1 \).
6 - A related differential inclusion

We now consider a general control system where the right hand side is a linear or quadratic function of the derivative of the control:

\[ \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) \dot{u}_i + \sum_{i,j=1}^{m} h_{ij}(x) \dot{u}_i \dot{u}_j. \]  

(6.1)

We always assume that all functions \( f, g_i, \) and \( h_{ij} = h_{ji} \) are Lipschitz continuous. Given the initial conditions

\[ x(0) = \bar{x}, \quad u(0) = \bar{u}, \]  

(6.2)

for every smooth control function \( u : [0,T] \mapsto \mathbb{R}^m \) one obtains a unique solution \( t \mapsto x(t; u) \) of the Cauchy problem (6.1)-(6.2). More generally, since the equation (6.1) is quadratic w.r.t. the derivative \( \dot{u} \), it is natural to consider as set of admissible controls all the absolutely continuous functions \( u(\cdot) \) with derivative in \( L^2 \). For example

\[ \mathcal{U} = \left\{ u : [0,T] \mapsto \mathbb{R}^m ; \quad \int_0^T \| \dot{u}(t) \|^2 dt < \infty \right\}. \]  

(6.3)

In this connection, a natural problem is to describe the set of all admissible trajectories. The main goal of the following analysis is to provide a characterization of the closure of this set of trajectories, in terms of an auxiliary differential inclusion.

By \( \mathbb{co}(A) \) we denote here the closed convex hull of a set \( A \subset \mathbb{R}^{1+n} \). Notice that the multifunction \( F \) is Lipschitz continuous w.r.t. the Hausdorff metric \( \mathcal{AC} \), and has convex, compact values. For a given interval \([0,S]\), the set of trajectories of the differential inclusion

\[ \dot{y}(s) \in F(y), \quad y(0) = \begin{pmatrix} 0 \\ \bar{x} \end{pmatrix} \]  

(6.5)

is a non-empty, closed, bounded subset of \( C([0,S] ; \mathbb{R}^{1+n}) \). Consider one particular solution, say \( s \mapsto y(s) = \begin{pmatrix} x_0(s) \\ x(s) \end{pmatrix} \), defined for \( s \in [0,S] \). Assume that \( T = x_0(S) > 0 \). Since the map \( s \mapsto x_0(s) \) is non-decreasing, it admits a generalized inverse

\[ s = s(t) \quad \text{if} \quad x_0(s) = t. \]  

(6.6)

Indeed, for all but countably many times \( t \in [0,T] \) there exists a unique value of the parameter \( s \) such that the identity on the right of (6.6) holds. We can thus define a corresponding trajectory

\[ t \mapsto x(t) = x(s(t)) \in \mathbb{R}^n. \]  

(6.7)
This map is well defined for almost all times $t \in [0,T]$.

To establish a connection between the original control system (6.1) and the differential inclusion (6.5), consider first a smooth control function $u : [0,T] \mapsto \mathbb{R}^m$. Define a reparametrized time variable by setting

$$s(t) = \int_0^t \left( 1 + \sum_i \dot{u}_i^2(\tau) \right)^{1/2} d\tau. \quad (6.8)$$

Notice that the map $t \mapsto s(t)$ is strictly increasing. The inverse map $s \mapsto t(s)$ is uniformly Lipschitz continuous and satisfies

$$\frac{dt}{ds} = \left( 1 + \sum_i \dot{u}_i^2(t) \right)^{-1/2}.$$

Let now $x : [0,T] \mapsto \mathbb{R}^m$ be a solution of (6.1) corresponding to the smooth control $u(\cdot)$. We claim that the map $s \mapsto y(s) = \left( t(s), x(t(s)) \right)$ is a solution to the differential inclusion (6.5). Indeed, setting

$$v_0(s) = \frac{1}{\left( 1 + \sum_j \dot{u}_j^2(t(s)) \right)^{1/2}}, \quad v_i(s) = \frac{\dot{u}_i(s(t))}{\left( 1 + \sum_j \dot{u}_j^2(t(s)) \right)^{1/2}} \quad i = 1,\ldots,m, \quad (6.9)$$

one has

$$\frac{d}{ds} \begin{pmatrix} t(s) \\ x(s) \end{pmatrix} = \begin{pmatrix} v_0^2(s) \\ f(x(s)) v_0^2(s) + \sum_{i=1}^m g_i(x(s)) v_0(s) v_i(s) + \sum_{i,j=1}^m h_{ij}(x(s)) v_i(s) v_j(s) \end{pmatrix}.$$

Hence (6.5) holds, because by (6.9)

$$\sum_{i=0}^m v_i^2(s) \equiv 1.$$

The following theorem shows that every solution of the differential inclusion (6.5) can be approximated by smooth solutions of the original control system (6.1). More precisely, one can achieve: (i) convergence of trajectories in the space $L^1([0,T])$, and (ii) pointwise convergence at the terminal time.

**Theorem 7.** Let $y : [0,S] \mapsto \mathbb{R}^{1+n}$ be a solution to the multivalued Cauchy problem (6.5). Let the first component satisfy $x_0(S) = T > 0$. Then there exists a sequence of smooth control functions $u^{(\nu)} : [0,T] \mapsto \mathbb{R}^m$ such that the following properties hold.

(i) Define the rescaled time $t \mapsto s_\nu(t)$ as in (6.8), call $s \mapsto t_\nu(s)$ the inverse map and set $x_\nu(s) = x(t_\nu(s), u^{(\nu)})$. Here $t \mapsto x(t, u^{(\nu)})$ is the solution of (6.1)-(6.2) using the control $u^{(\nu)}$. Then one has

$$u^{(\nu)}(0) = \bar{u}, \quad s_\nu(T) = S \quad \text{for all } \nu \geq 1. \quad (6.10)$$

Moreover, the corresponding solutions $s \mapsto \begin{pmatrix} t_\nu(s) \\ x_\nu(s) \end{pmatrix}$ converge to the map $s \mapsto y(s)$ uniformly on $[0,S]$. 

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(ii) Defining the rescaled time \( t \mapsto s(t) \) as in (6.6) and setting \( \left( \frac{t}{x(t)} \right) = y(s(t)) \), we have
\[
\lim_{\nu \to \infty} x(T, u^{(\nu)}) = x(T), \quad \lim_{\nu \to \infty} \int_0^T |x(t) - x_\nu(t)| \, dt = 0. \tag{6.11}
\]

**Proof.** By assumptions, the extended vector fields
\[
\hat{f} = \begin{pmatrix} 1 \\ f \end{pmatrix}, \quad \hat{g}_i = \begin{pmatrix} 0 \\ g_i \end{pmatrix}, \quad \hat{h} = \begin{pmatrix} 0 \\ h_{ij} \end{pmatrix}
\]
are Lipschitz continuous. Consider the set of trajectories of the control system on \( \mathbb{R}^{1+n} \)
\[
\frac{d}{ds} y(s) = \hat{f} v_0^2 + \sum_{i=1}^m \hat{g}_i v_0 v_i + \sum_{i,j=1}^m \hat{h}_{ji} v_i v_j, \quad y(0) = \begin{pmatrix} 0 \\ x \end{pmatrix}, \tag{6.12}
\]
where the controls \( v_i \) satisfy the constraints
\[
v_0(s) \in [0,1], \quad \sum_{i=0}^m v_i^2(s) = 1 \quad s \in [0,S].
\]
In the above setting, it is well known that the set of trajectories of (6.12) is dense on the set of solutions of the differential inclusion (6.5). In particular, there exists a sequence of control functions \( s \mapsto v^{(\nu)}(s) = (v_0^{(\nu)}, \ldots, v_m^{(\nu)}) \), \( \nu \geq 1 \), such that the corresponding solutions \( s \mapsto y^{(\nu)}(s) = y(s, v^{(\nu)}) \) of (6.12) converge to \( y(\cdot) \) uniformly for \( s \in [0,S] \). Notice that this implies
\[
y_0^{(\nu)}(S) = \int_0^S (v_0^{(\nu)})^2(s) \, ds \to y_0(S) = T. \tag{6.13}
\]
We now observe that the “input-output map” \( v(\cdot) \mapsto y(\cdot, v) \) from controls to trajectories is uniformly continuous as a map from \( L^1([0,S]); \mathbb{R}^{1+m} \) into \( C([0,S]; \mathbb{R}^{1+n}) \). Thanks to the assumption \( x_0(S) = T > 0 \), we can slightly modify the controls \( v^{(\nu)} \) in \( L^1 \) and replace the sequence \( v^{(\nu)} \) by a new sequence of smooth control functions \( w^{(\nu)} : [0,S] \to \mathbb{R}^{1+m} \) with the following properties:
\[
w_0^{(\nu)}(s) > 0 \quad \text{for all} \ s \in [0,S], \ \nu \geq 1. \tag{6.14}
\]
\[
\int_0^S (w_0^{(\nu)}(s))^2 \, ds = T \quad \text{for all} \ \nu \geq 1, \tag{6.15}
\]
\[
\lim_{\nu \to \infty} \int_0^S |w^{(\nu)}(s) - v^{(\nu)}(s)| \, ds = 0. \tag{6.16}
\]
This implies the uniform convergence
\[
y(\cdot, w^{(\nu)}) = \left( \frac{x_0(\cdot, w^{(\nu)})}{x(\cdot, w^{(\nu)})} \right) \to y(\cdot)
\]
in $\mathcal{C}([0, S] ; \mathbb{R}^{1+n})$. In particular, looking at the last $n$ coordinates, we have
\[
\lim_{\nu \to \infty} \left\| x(\cdot, w^{(\nu)}) - x(\cdot) \right\|_{\mathcal{C}([0, S])} = 0 \quad (6.17)
\]

By (6.14), for each $\nu \geq 1$ the map
\[
s \mapsto x_0(s, w^{(\nu)}) \doteq \int_0^s \left[ w_0^{(\nu)}(\xi) \right]^2 d\xi
\]
is strictly increasing. Therefore it has a smooth inverse $s = s_\nu(t)$. We now define the sequence of smooth control functions $u^{(\nu)} = (u_1^{(\nu)}, \ldots, u_m^{(\nu)})$ from $[0, T]$ into $\mathbb{R}^m$ by setting
\[
u = \min \{ \nu \geq 1 : \nu \in [1, m] \}
\]
and denote by $s_\nu(t)$ its inverse, as in (6.6). For each $\nu \geq 1$, consider the map $s \mapsto s_\nu(s)$ together with its inverse $t \mapsto s_\nu(t)$. Notice that each $s_\nu$ is smooth. Moreover we have
\[
\left| \frac{d}{ds} t(s) \right| \leq 1, \quad \left| \frac{d}{ds} s_\nu(s) \right| \leq 1, \quad (6.21)
\]
Next, by (6.15) we have $t_\nu(S) = T$ for every $\nu$. Therefore, the first limit in (6.11) is an immediate consequence of (6.20).

To establish the second limit in (6.11), let $t(s) = x_0(s)$ be the first coordinate of the map $s \mapsto y(s)$, and denote by $t \mapsto s(t)$ its inverse, as in (6.6). For each $\nu \geq 1$, consider the map $s \mapsto t_\nu(s)$ together with its inverse $t \mapsto s_\nu(t)$. Notice that each $s_\nu$ is smooth. Moreover we have
\[
\lim_{\nu \to \infty} \int_0^T |s(t) - s_\nu(t)| dt = \lim_{\nu \to \infty} \int_0^S |t(s) - t_\nu(s)| ds = 0. \quad (6.22)
\]
Using (6.19), and then (6.21) to ensure that $|dt| \leq |ds|$, we obtain the estimate
\[
\int_0^T \left| s(t) - s_\nu(t) \right| dt = \lim_{\nu \to \infty} \int_0^T \left| x(s(t)) - x(s(t), u^{(\nu)}) \right| dt + \int_0^T \left| x(s(t), u^{(\nu)}) - x(s_\nu(t), w^{(\nu)}) \right| dt 
\leq \int_0^S \left| x(s) - x(s, w^{(\nu)}) \right| ds + C \cdot \int_0^T \left| s(t) - s_\nu(t) \right| dt. \quad (6.23)
\]
Here the constant $C$ denotes an upper bound for the derivative w.r.t. $s$, say,
\[
C \doteq \sup \left\{ \left| f(x) \right| + \sum_i \left| g_i(x) \right| + \sum_{ij} \left| h_{ij}(x) \right| \right\}.
\]
By (6.20) and (6.22), the right hand side of (6.23) vanishes as \( \nu \to \infty \). This completes the proof of the theorem.

For further results on the closure of the set of trajectories of (6.1), we refer to [BR3], [BR4], [BR5]. In particular, for a system which is “fit for jumps”, as in (5.3), one can consider iterated Lie brackets of the vector fields \( g_1, \ldots, g_m \), for example \([g_i, g_j], [g_j, g_k], \ldots\). By a classical result in geometric control theory [AS], [J], the set of solutions of (5.3) is dense on the set of solutions for the more general control system

\[
\dot{x} = f(x) + \sum_{\alpha} G_\alpha(x) v_\alpha.
\]

As vector fields \( G_\alpha \) one can take here any collection of iterated Lie brackets of the fields \( g_i \).

### 7 - Stabilization

Aim of this section is to review various concepts of stability for the quadratic impulsive system (6.1), and relate them to the weak stability of a differential inclusion.

**Definition 3.** We say that the impulsive system (6.1) is **stabilizable** at the point \( x^{\dagger} \in \mathbb{R}^n \) if, for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that the following holds. For every initial state \( \bar{x} \) with \( |\bar{x} - x^{\dagger}| \leq \delta \) there exists a \( C^1 \) control function \( t \mapsto u(t) = (u_1, \ldots, u_m)(t) \) such that the corresponding trajectory of (6.1)-(6.2) satisfies

\[
|x(t, u) - x^{\dagger}| \leq \varepsilon \quad \text{for all } t \geq 0. \tag{7.1}
\]

We say that the system (6.1) is **asymptotically stabilizable** at the point \( \bar{x} \) if a control \( u(\cdot) \) can be found such that, in addition to (7.1), the trajectory satisfies

\[
\lim_{t \to \infty} x(t, u) = x^{\dagger}. \tag{7.2}
\]

**Remark.** In the above definition we are not putting any constraint on the control function \( u : [0, \infty[ \to \mathbb{R}^m \). In principle, one may well have \( |u(t)| \to \infty \) as \( t \to \infty \). If one wishes to stabilize the system (6.1) and at the same time keep the control values within a small neighborhood of a given value \( u^{\dagger} \), it suffices to consider the stabilization problem for an augmented system, adding the variables \( x_{n+1}, \ldots, x_{n+m} \) together with the equations

\[
\dot{x}_{n+j} = \dot{u}_j, \quad j = 1, \ldots, m.
\]

Similar stability concepts (see for example [Sm]) can be also defined for a differential inclusion

\[
\dot{x} \in F(x). \tag{7.3}
\]

**Definition 4.** The point \( x^{\dagger} \) is **weakly stable** for the differential inclusion (7.3) if, for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that the following holds. For every initial state \( \bar{x} \) with \( |\bar{x} - x^{\dagger}| \leq \delta \) there exists a trajectory of (7.3) such that

\[
x(0) = \bar{x}, \quad |x(t) - x^{\dagger}| \leq \varepsilon \quad \text{for all } t \geq 0. \tag{7.4}
\]
Moreover, $x^\dagger$ is **weakly asymptotically stable** if, there exist a trajectory such that, in addition to (7.4), satisfy
\[
\lim_{t \to \infty} x(t) = x^\dagger. \tag{7.5}
\]

There is an extensive literature, in the context of O.D.E’s, and of control systems or differential inclusions, relating the stability of an equilibrium state to the existence of a Lyapunov function. We recall below the basic definition, in a form suitable for our applications. For simplicity, we shall consider the case $\bar{x} = 0 \in \mathbb{R}^n$, which of course is not restrictive.

**Definition 5.** A scalar function $V$ defined on a neighborhood $\mathcal{N}$ of the origin is a **weak Lyapunov function** for the differential inclusion (7.3) if the following holds.

(i) $V$ is continuous on $\mathcal{N}$, and continuously differentiable on $\mathcal{N} \setminus \{0\}$.

(ii) $V(0) = 0$ while $V(x) > 0$ for all $x \neq 0$.

(iii) For every $\delta > 0$ sufficiently small, the sublevel set $\{x; V(x) \leq \delta\}$ is compact.

(iv) At each $x \neq 0$ one has
\[
\inf_{y \in F(x)} \nabla V(x) \cdot y \leq 0. \tag{7.6}
\]

In connection with the multifunction $F$ defined at (6.4) we consider a second multifunction $F^\circ$ obtained by projecting the sets $F(y) \subset \mathbb{R}^{1+n}$ into the subspace $\mathbb{R}^n$. More precisely, we set
\[
F^\circ(x) = \overline{co} \left\{ f(x) v_0^2 + \sum_{i=1}^m g_i(x) v_0 v_i + \sum_{i,j=1}^m h_{ij}(x) v_i v_j \mid v_0 \geq 0, \sum_{i=0}^m v_i^2 = 1 \right\}. \tag{7.7}
\]

Observe that, if the vector fields $f, g_i$, and $h_{ij}$ are Lipschitz continuous, then the multifunction $F^\circ$ is Lipschitz continuous with compact, convex values. The next result, proved in [BR5], relates the asymptotic stabilization of the impulsive system (6.1) to the stability of a related differential inclusion.

**Theorem 8.** The impulsive system (6.1) is asymptotically stabilizable at the origin if and only if the origin is weakly asymptotically stable for the differential inclusion
\[
\frac{d}{ds} x(s) \in F^\circ(x(s)). \tag{7.8}
\]

The following result, also proved in [BR5], deals with the stabilization of the impulsive system (6.1), relying on the existence of a Lyapunov function.

**Theorem 9.** Assume that the differential inclusion (7.8) admits a Lyapunov function $V = V(x)$ defined on a neighborhood $\mathcal{N}$ of the origin. More precisely, referring to the original multifunction $F$ at (6.4), assume that for every $x \in \mathcal{N} \setminus \{0\}$ there exists $\dot{v} = (v_0, v) \in F(x)$ such that
\[
\nabla V(x) \cdot v \leq 0 \quad v_0 > 0. \tag{7.9}
\]

Then the quadratic impulsive system (6.1) can be stabilized at the origin.

**Example 6.** On $\mathbb{R}^2$, consider the constant vector fields $f = (1, 0)$, $h_{11} = (0, 1)$, $h_{22} = (0, -1)$, $g_1 = g_2 = h_{12} = h_{21} = (0, 0)$. Then, choosing $v_0 = 0$, $v_1 = v_2 = 1/\sqrt{2}$ in (7.7) we see that
$(0,0) \in F^\Diamond(x)$ for every $x \in \mathbb{R}^2$. Hence the condition (7.6) is trivially satisfied by any function $V$. However, it is clear that in this case the system (6.1) is not stabilizable at the origin. This motivates the stronger assumption (7.9) needed in the above theorem.

**Example 7.** Consider a Lagrangian system (see fig. 10) consisting of two equal point masses, moving in a vertical plane with coordinates $x$-$z$, connected by a bar with length $\rho$. The first mass is constrained to lie on the vertical $z$-axis. The system has two degrees of freedom. Let $A, B$ be the positions of the two masses. Its configuration can be described in terms of two variables $(h, \theta)$, where $h$ is the $z$-coordinate of $A$, while $\theta$ is the angle between the $z$-axis and the segment $AB$. Notice that

\[
A = (0, h) , \quad B = (\rho \sin \theta , \ h + \rho \cos \theta)
\]

\[
\dot{A} = (0, \dot{h}) , \quad \dot{B} = (\rho \theta \cos \theta , \ \dot{h} - \rho \theta \sin \theta).
\]

The kinetic energy and the potential energy of the system are given by

\[
T(h, \theta, \dot{h}, \dot{\theta}) = \frac{m}{2} (2\dot{h}^2 + \rho^2 \dot{\theta}^2 - 2\rho \dot{h} \sin \theta) , \quad V(h, \theta) = m(2h + \cos \theta).
\]

Assigning the coordinate $h = u(t)$ as a function of time, the motion of the remaining free coordinate $\theta$ is determined by the equation (2.4). In the present case, this yields

\[
\frac{d}{dt} \left[ \frac{m}{2} (2\rho^2 \dot{\theta} - 2\rho \dot{h} \sin \theta) \right] = -\frac{m}{2} 2\rho \dot{h} \cos \theta + m \sin \theta , \quad (7.10)
\]

\[
\rho^2 \ddot{\theta} - \rho \ddot{h} \sin \theta - \rho \dot{h} \dot{\theta} \cos \theta = -\frac{m}{2} 2\rho \dot{h} \cos \theta + m \sin \theta . \quad (7.11)
\]

Notice that the equation (7.11) is not in the desired form, because it contains the second derivative of the control function: $\ddot{h} = \ddot{u}$. Following the procedure described in Section 3, we introduce the generalized angular momentum

\[
p = \frac{\partial T}{\partial \dot{\theta}} = \left[ \frac{m}{2} (2\rho^2 \dot{\theta} - 2\rho \dot{h} \sin \theta) \right] .
\]

Observing that

\[
\dot{\theta} = \frac{p + m \dot{h} \sin \theta}{m \rho^2} , \quad (7.12)
\]

from (7.10) we now obtain

\[
\dot{p} = -mp \dot{h} \cos \theta \left( \frac{p + m \dot{h} \sin \theta}{m \rho^2} \right) + m \sin \theta . \quad (7.13)
\]

Since $h = u(t)$, the equations (7.12)-(7.13) yield

\[
\begin{pmatrix}
\dot{\theta} \\
\dot{p}
\end{pmatrix} = \begin{pmatrix}
\frac{p}{m \rho^2} \\
\frac{m \sin \theta}{m \rho^2}
\end{pmatrix} + \begin{pmatrix}
\frac{\sin \theta}{\rho^2} \\
-\frac{p \cos \theta}{\rho}
\end{pmatrix} \dot{u} + \begin{pmatrix}
0 \\
-\frac{m \sin \theta}{2 \rho}
\end{pmatrix} \dot{u}^2. \quad (7.14)
\]

Observe that the coefficient of $\dot{u}^2$ does not vanish, hence this system is not “fit for jumps”.

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Fix any angle $\bar{\theta}$, with $|\bar{\theta}| < \pi/2$. We claim that the above system can be asymptotically stabilized at the point $(\bar{\theta}, 0)$. Indeed, according to Theorem 8, it suffices to show that the corresponding differential inclusion (7.8) is weakly asymptotically stable at $(\bar{\theta}, 0)$.

Toward this goal, we observe that, by (7.14),

$$\left(\begin{array}{c}
\frac{p}{mp^2} \\
m \sin \theta
\end{array}\right) \in F^{\diamond}(\theta, p), \quad \left(\begin{array}{c}
0 \\
-\frac{m \sin 2\theta}{2p}
\end{array}\right) \in F^{\diamond}(\theta, p). \quad (7.15)$$

Therefore, since $F^{\diamond}$ is convex, the set of trajectories of the differential inclusion (7.8) contains the set of all trajectories of the control system

$$\frac{d}{ds} \left(\begin{array}{c}
\theta \\
p
\end{array}\right) = \left(\begin{array}{c}
\frac{p}{mp^2} \\
m \sin \theta
\end{array}\right) w(s) + \left(\begin{array}{c}
0 \\
-\frac{m \sin 2\theta}{2p}
\end{array}\right) (1 - w(s)), \quad (7.16)$$

where the scalar control function $s \mapsto w(s)$ enters linearly. It now remains to check that $(\bar{\theta}, 0)$ is an asymptotically stable equilibrium point for the system (7.16), provided that $|\bar{\theta}| < \pi/2$. We work out details, assuming $\bar{\theta} \neq 0$.

$$w_0 = \frac{\cos \bar{\theta}}{\rho + \cos \bar{\theta}},$$

so that

$$w_0 - \frac{\cos \bar{\theta}}{\rho} (1 - w_0) = 0.$$

Notice that $0 < w_0 < 1$. In terms of the new control variable $\omega \equiv w - w_0$, the system (7.16) can be rewritten as

$$\frac{d}{ds} \left(\begin{array}{c}
\theta \\
p
\end{array}\right) = \left(\begin{array}{c}
\frac{p}{mp^2} \\
m \left(w_0 - \frac{\cos \bar{\theta}}{\rho} (1 - w_0)\right) \sin \theta
\end{array}\right) w(s) + \left(\begin{array}{c}
\frac{p}{mp^2} \\
m \left(1 + \frac{\cos \bar{\theta}}{\rho}\right) \sin \theta
\end{array}\right) \omega(s). \quad (7.17)$$

where the control $\omega(s)$ now takes values in the interval $[-w_0, 1 - w_0]$. Linearizing (7.17) at the point $(\bar{\theta}, 0)$ we obtain the linear control system with constant coefficients

$$\frac{d}{ds} \left(\begin{array}{c}
\theta - \bar{\theta} \\
p
\end{array}\right) = \left(\begin{array}{c}
0 \\
m (1 - w_0) \sin^2 \bar{\theta} \\
0
\end{array}\right) \left(\begin{array}{c}
\theta - \bar{\theta} \\
p
\end{array}\right) + \left(\begin{array}{c}
0 \\
m \left(1 + \frac{\cos \bar{\theta}}{\rho}\right) \sin \theta
\end{array}\right) \omega(s). \quad (7.18)$$

For $|\bar{\theta}| < \pi/2, \bar{\theta} \neq 0$, the linearized system (7.18) is completely controllable, hence (7.17) is locally asymptotically stabilizable at the origin, as claimed. In the special case $\bar{\theta} = 0$, the proof that the two-dimensional system (7.16) is locally asymptotically controllable at the origin is somewhat more lengthy. For a detailed analysis of planar control systems we refer to [BP].
Figure 10. This system can be stabilized at any angle $\theta < \pi/2$, vibrating the point $A$ up and down.

8 - Swim-like motion of bodies immersed in a fluid

In this section, we describe a further application of the theory of impulsive Lagrangian systems. Consider a body whose shape depends on a finite number of parameters $q^1, \ldots, q^N$ and is immersed in a homogeneous incompressible, non-viscous fluid.

To fix the ideas, let $\Omega \subset \mathbb{R}^d$ be a reference configuration of the body. We assume that $\Omega$ is a bounded, open set with smooth boundary. The case where $\Omega$ consists of several connected components (modelling several bodies swimming in the same environment) is also of interest. For each given value $q = (q^1, \ldots, q^N)$ of the Lagrangian parameters, let $\xi \mapsto \varphi^q(\xi)$ be a volume preserving diffeomorphism. Assigning the coordinates $q = q(t)$ as functions of time, the image $\varphi^q(t)(\Omega)$ thus describes the region of space occupied by the body at time $t$. Let

$$T(q, \dot{q}) = \sum_{i,j=1}^{N} A_{ij}(q)\dot{q}^i\dot{q}^j$$

describe the kinetic energy of the body. For simplicity, we assume that the surrounding fluid has unit density. Calling $v = v(x)$ its velocity at the point $x$, the kinetic energy of the surrounding fluid is given by

$$K = \int_{\mathbb{R}^d \setminus \varphi^q(\Omega)} \frac{|v(x)|^2}{2} \, dx.$$
For \( n + m = N \), we assume that the last \( m \) Lagrangian coordinates \( q^{n+1}, \ldots, q^{n+m} \) can be prescribed by a controller, as functions of time. As in (3.3), these assignments will be implemented by means of frictionless constraints. Assuming that no other forces are present, we wish to derive a system of equations describing the motion of the remaining \( n \) coordinates and of the surrounding fluid. Calling \( v = v(t, x) \) the fluid velocity, if the only active forces are due to the scalar pressure \( p \), the motion is governed by the Euler equation for non-viscous, incompressible fluids:

\[
v_t + v \cdot \nabla v = -\nabla p, \tag{8.3}\]

supplemented by the incompressibility condition

\[
\text{div} \ v \equiv 0. \tag{8.4}\]

In addition, we need a boundary condition along \( \partial(\varphi^q(\Omega)) = \varphi^q(\partial\Omega) \), namely

\[
\left< v(\varphi^q(\xi)) - \sum_{i=1}^{n} \frac{\partial}{\partial q^i} \varphi^q(\xi) \cdot \dot{q}^i, \ n^q(\xi) \right> = 0. \tag{8.5}\]

Here \( n^q(\xi) \) denotes the unit outer normal to the set \( \varphi^q(\Omega) \) at the point \( \varphi^q(\xi) \).

To find the evolution of the coordinates \( q^1, \ldots, q^n \), we observe that

\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^i} = \frac{\partial T}{\partial q^i} + F_i, \tag{8.6}\]

where \( T \) is the kinetic energy of the body and \( F_i \) are the components of the external pressure forces, acting on the boundary of \( \varphi^q(\Omega) \). To determine these forces, we observe that, in connection with a small displacement of the \( q^i \) coordinate, the work done by the pressure forces is

\[
\delta W = -\delta q^i \cdot \int_{\partial\Omega} \left< n^q(\xi), \frac{\partial \varphi^q}{\partial q^i}(\xi) \right> p(\varphi^q(\xi)) J^q(\xi) d\sigma. \tag{8.7}\]

Notice that the above integral is computed along the surface of the reference configuration. The factor \( J^q(\xi) \) gives the ratio between the area of an infinitesimal portion of the surface \( \varphi^q(\partial\Omega) \) and the corresponding portion in the reference configuration \( \partial\Omega \). According to (8.6) and (8.7), the equation of motion for the first \( n \) coordinates is derived from

\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^i} = \frac{\partial T}{\partial q^i} - \int_{\partial\Omega} \left< n^q(\xi), \frac{\partial \varphi^q}{\partial q^i}(\xi) \right> p(\varphi^q(\xi)) J^q(\xi) d\sigma. \tag{8.8}\]

We now show that, in the case of irrotational flow, the coupled system of equations (8.3)–(8.5) and (8.8) can be reduced to a finite dimensional impulsive Lagrangian system. In particular, the techniques discussed earlier in this paper can be applied to this situation as well.

To fix the ideas, we consider the two-dimensional case, assuming that the body has one connected component, and that the initial velocity of the fluid is irrotational with zero circulation around the body. Since the flow is inviscid, this implies that vorticity and circulation will vanish identically at all times.

For any given values of \( q = (q^1, \ldots, q^N) \), and \( \dot{q} = (\dot{q}^1, \ldots, \dot{q}^N) \), consider the unique irrotational velocity field \( v = v(q, \dot{q}) : \mathbb{R}^2 \setminus \varphi^q(\Omega) \mapsto \mathbb{R}^2 \) that satisfies the boundary conditions (8.5) and has
zero circulation around the body. It is well known \cite{MP} that this velocity field can be computed by setting \( v = \nabla \psi \) and solving the Neumann problem in the exterior domain

\[
\Delta \psi = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \varphi^q(\Omega),
\]

(8.9)

\[
\mathbf{n} \cdot \nabla \psi = \sum_{i=1}^N \mathbf{n} \cdot \frac{\partial \varphi^q}{\partial q^i} \dot{q}^i \quad \text{on} \quad \partial \varphi^q(\Omega),
\]

(8.10)

\[
|\psi(x)| \to 0 \quad \text{as} \quad |x| \to \infty.
\]

(8.11)

Call

\[
T^v(q, \dot{q}) = \int_{\mathbb{R}^2 \setminus \varphi^q(\Omega)} \frac{|v^2(x)|}{2} \, dx
\]

(8.12)

the kinetic energy of the fluid. Notice that this is entirely determined by \( q, \dot{q} \). For given values of \( q^1, \ldots, q^N \), the solution \( v(\cdot) \) of the Neumann problem (8.9)–(8.11) depends linearly on the boundary conditions (8.10), hence it is a linear functional of the time derivatives \( \dot{q}^i, \ldots, \dot{q}^N \). The kinetic energy in (8.12) is thus a quadratic function of these variables, say

\[
T^v(q, \dot{q}) = \sum_{i,j=1}^N A'_{ij}(q) \dot{q}^i \dot{q}^j.
\]

(8.13)

Summing (8.1) with (8.13), we can now consider the total kinetic energy of the body and of the fluid:

\[
\tilde{T}(q, \dot{q}) \doteq T^{(\text{body})} + T^{(\text{fluid})} = \sum_{i,j=1}^N A_{ij}(q) \dot{q}^i \dot{q}^j + \sum_{i,j=1}^N A'_{ij}(q) \dot{q}^i \dot{q}^j.
\]

(8.14)

This achieves the desired reduction to a finite dimensional system.

If the last \( m \) coordinates are directly assigned as functions of time, say \( q^{n+1} = u_1(t), \ldots, q^{n+m} = u_m(t) \), the motion of the remaining \( n \) free coordinates \( q^1, \ldots, q^n \) can be determined as in Section 3. The only difference is that now we use \( \tilde{T}(q, \dot{q}) \) as the total kinetic energy of the system.

\section*{References}


\cite{BL} A. Bensoussan and J. L. Lions, \textit{Impulse Control and Quasivariational Inequalities}. Heyden & Son, Philadelphia, 1984.


[KO] V. V. Kozlov and D. A. Onishchenko, Motion of a body with undeformable shell and variable mass geometry in an unbounded perfect fluid, *J. Dynam. Diff. Equat.* **15** (2003), 553-570.


