Lecture Notes on the Theory of Incompressible Fluid Motion

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Chapter 1

The equations of fluid motion

1.1 Differential operators

Given a scalar function $u : \mathbb{R}^N \to \mathbb{R}$, its gradient $\nabla u$ is the vector of partial derivatives

$$\nabla u = (u_{x_1}, u_{x_2}, \ldots, u_{x_N}) = (\frac{\partial}{\partial x_1} u, \frac{\partial}{\partial x_2} u, \ldots, \frac{\partial}{\partial x_N} u).$$

For a vector field $u : \mathbb{R}^N \to \mathbb{R}^N$, its divergence is

$$\text{div } u = \sum_{i=1}^{N} u_{x_i}^i.$$

The Laplace operator is defined as

$$\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} = \text{div } \nabla u.$$

Consider a velocity field $u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, so that $u(t, x) = (u^1, \ldots, u^N)(t, x)$ denotes the velocity of the particle located at the point $x = (x^1, \ldots, x^N)$ at time $t$. We then define the material derivative of a function $\phi = \phi(t, x)$ as

$$D_t \phi = \frac{\partial}{\partial t} \phi + \sum_{i=1}^{N} u^i \frac{\partial}{\partial x_i} \phi.$$

This provides the time-derivative of $\phi$ along particle trajectories.

We recall that the wedge product of two vectors $v, w \in \mathbb{R}^3$ can be obtained as follows. Let $e_1, e_2, e_3$ be the vectors in the standard orthonormal basis of $\mathbb{R}^3$. Then

$$v \wedge w = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$
Given a velocity field \( u : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \), its \textit{vorticity} \( \omega \) is defined as
\[
\omega = \text{curl} \ u = \left( \partial_2 u^3 - \partial_3 u^2, \ \partial_3 u^1 - \partial_1 u^3, \ \partial_1 u^2 - \partial_2 u^1 \right).
\]

Note that, formally, one can write
\[
\text{curl} \ u = \nabla \wedge u = \det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ u_1 & u_2 & u_3 \end{pmatrix}.
\]

1.2 The Euler and Navier-Stokes equations

Let \( u = u(t, x) \) denote the velocity of a fluid, at time \( t \) and at a point \( x \) in space. The flow of a homogeneous, incompressible, non-viscous fluid in \( \mathbb{R}^N \) is modelled by the Euler equations
\[
\begin{cases}
  u_t + u \cdot \nabla u = -\nabla p \\
  \text{div} \ u \equiv 0
\end{cases}
\]

balance of momentum, incompressibility condition. \hspace{1cm} (1.1)

Notice that \( u_t + u \cdot \nabla u = D_t u \) is the material derivative of the vector field \( u \). Here
- \( t \) is time,
- \( x = (x_1, \ldots, x_N) \in \mathbb{R}^N \) is the Eulerian space variable,
- \( u = (u^1, \ldots, u^N) \) is the fluid velocity,
- \( p \) is the scalar pressure.

When viscosity is present, an additional diffusion term is present in the momentum equation. This leads to the Navier-Stokes equations
\[
\begin{cases}
  u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u \\
  \text{div} \ u \equiv 0
\end{cases}
\]

balance of momentum, incompressibility condition. \hspace{1cm} (1.2)

The positive constant \( \nu \) is the \textit{coefficient of kinematic viscosity}. Many authors refer to it as the inverse of the Reynolds number: \( \nu = 1/Re \).

The equations (1.1) or (1.2) are naturally supplemented by initial data
\[
u(0, x) = u_0(x) . \hspace{1cm} (1.3)
\]

If the fluid motion takes place in a bounded set \( \Omega \subset \mathbb{R}^N \), one must add suitable boundary conditions. In the case of the inviscid Euler equations, one requires that the velocity \( u \) of the fluid is tangential to the boundary \( \partial \Omega \). Calling \( \mathbf{n} \) the unit outer normal, we thus have
\[
\mathbf{n} \cdot u = 0 \hspace{1cm} x \in \partial \Omega . \hspace{1cm} (1.4)
\]

In the presence of viscosity, instead of (1.4), the Navier-Stokes equations are supplemented by the non-slip boundary conditions
\[
u = 0 \hspace{1cm} x \in \partial \Omega . \hspace{1cm} (1.5)
\]

We now show how the Euler equation can be derived from the balance law of momentum.
Consider any region $W_0 \subset \mathbb{R}^n$. Denote by $W_t$ the region occupied at time $t > 0$ by those fluid particles which are initially in $W_0$. Assume that the only force acting on the fluid through the boundary $\partial W_t$ is the pressure. Then the balance law of momentum takes the form

$$\frac{d}{dt} \int_{W_t} u \, dV = [\text{total forces across the boundary}] = -\int_{\partial W_t} p \, n \, dA.$$  \hfill (1.6)

If $e$ is any fixed vector in $\mathbb{R}^N$ by the divergence theorem we get

$$\int_{\partial W_t} p \, e \cdot n \, dA = \int_{W_t} \text{div}(p \, e) \, dV = \int_{W_t} \nabla p \, e \, dV.$$ \hfill (1.7)

Then:

$$\frac{d}{dt} \int_{W_t} u \, dV = \int_{W_t} D_t u \, dV = -\int_{W_t} \nabla p \, dV,$$ \hfill (1.8)

for any region $W_t$ in the fluid at time $t$. Hence the equality holds also in the differential form:

$$D_t u = -\nabla p \quad \text{Euler equation.}$$

### 1.3 Symmetry groups for the Euler and Navier-Stokes equations

Assume that $u = u(t, x)$ and $p = p(t, x)$ provide a solution to the Euler or the Navier-Stokes equations. Further solutions can then be obtained by various variable transformations.

- **Translation Invariance**: For any constant vector $c$ in $\mathbb{R}^N$

  $$\begin{cases}
  u^c(t, x) = u(t, x - ct) + c, \\
p^c(t, x) = p(t, x - ct),
  \end{cases}$$

  is another solution.

- **Rotation Invariance**: For any orthogonal matrix $Q$

  $$\begin{cases}
  u^Q(t, x) = Q^T u(t, Qx), \\
p^Q(t, x) = p(t, Qx),
  \end{cases}$$

  is another solution.

- **Scale Invariance**: If $u, p$ provide a solution to the Euler equations, then for any $\lambda, \tau > 0$ the functions

  $$\begin{cases}
  u^{\lambda, \tau}(t, x) = \lambda^{\frac{\tau}{\tau}} u(t^{\frac{\tau}{\tau}}, x^{\frac{\tau}{\tau}}), \\
p^{\lambda, \tau}(t, x) = \lambda^{\frac{\tau}{\tau^2}} p(t^{\frac{\tau}{\tau}}, x^{\frac{\tau}{\tau}}),
  \end{cases}$$ \hfill (1.9)

  provide a 2-parameters family of solutions to the Euler equation.
If $u, p$ are a solution to the Navier-Stokes equations, then for all $\tau > 0$ the functions

\[
\begin{align*}
    u^\tau(t, x) &= \tau^{-1/2} u(\tau, x), \\
p^\tau(t, x) &= \tau^{-1} p(\tau, x).
\end{align*}
\]  

(1.10)

provide a 1-parameter family of solutions to the Navier-Stokes equation for $\tau \in \mathbb{R}$. Observe that (1.9) coincides with (1.10) when $\lambda = \tau^{1/2}$.

### 1.4 Particle trajectories

Given a fluid flow with velocity field $u$ (not necessarily incompressible), the particle trajectory mapping

\[
X : [0, \infty) \times \mathbb{R}^N \rightarrow \mathbb{R}^N \quad t, \alpha \mapsto X(t, \alpha)
\]  

(1.11)

describes the trajectory of the particle which is initially located at point $\alpha = (\alpha_1, \ldots, \alpha_N)$ at time $t = 0$. The Lagrangian variable $\alpha$, can be regarded as a particle marker. The function $X(t, \alpha)$ is determined by solving the Cauchy problem

\[
\begin{align*}
    \frac{\partial}{\partial t} X(t, \alpha) &= u(t, X(t, \alpha)), \\
    X(0, \alpha) &= \alpha.
\end{align*}
\]

An initial domain $W_0 \subset \mathbb{R}^N$ in the fluid evolves in time to

\[W_t = X(t, W_0) = \{ X(t, \alpha) \mid \alpha \in W_0 \}.
\]

We shall denote by $\nabla_\alpha X$ the Jacobian matrix of first order partial derivatives of $X$ w.r.t. $\alpha$, so that

\[
\left( \nabla_\alpha X(t, \alpha) \right)_{ij} = \frac{\partial X^i}{\partial \alpha_j}(t, \alpha).
\]

The evolution of $\nabla_\alpha X$ along a particle trajectory is described by the linear evolution equation

\[
\frac{\partial}{\partial t} \left( \nabla_\alpha X(t, \alpha) \right) = \left( \nabla_x u(t, X(t, \alpha)) \right) \cdot \nabla_\alpha X(t, \alpha).
\]

The Jacobian determinant of $X(t, \cdot)$ is denoted as

\[J(t, \alpha) = \det \left( \nabla_\alpha X(t, \alpha) \right).
\]

At the initial time $t = 0$, $\nabla_\alpha X(0, \alpha) = I$ is the identity matrix, hence its determinant is $J(0, \alpha) \equiv 1$. If $u$ is a smooth velocity field then the time evolution of $J$ is given by

\[
\frac{\partial}{\partial t} J(t, \alpha) = \left[ \text{div} u(t, X(t, \alpha)) \right] J(t, \alpha).
\]

In particular, $J(t, X(t, \alpha)) \equiv 1$ if the flow is incompressible.
Theorem 1  The Trasport Formula

Let \( W \) be an open, bounded domain in \( \mathbb{R}^N \) with smooth boundary, and let \( X \) be the particle trajectory mapping of a given smooth velocity field \( u \).

For any smooth function \( f = f(t, x) \) we have

\[
\frac{d}{dt} \int_{W_t} f \, dx = \int_{W_t} \left[ f_t + \text{div} (fu) \right] \, dx.
\]

Proof

\[
\frac{d}{dt} \int_{W_t} f \, dx = \frac{d}{dt} \int_{W} f(t, X(t, \alpha)) J(t, X(t, \alpha)) \, d\alpha
\]

\[
= \int_{W} \left( D_t f J + f J_t \right) \, d\alpha
\]

\[
= \int_{W} \left( f_t + u \cdot \nabla f + f \text{div} u \right) J \, d\alpha
\]

\[
= \int_{W_t} \left[ f_t + \text{div}(fu) \right] \, dx.
\]

\( \square \)

Definition 1  A flow \( X \) is incompressible if for any subregion \( W \) with smooth boundary, and for any time \( t > 0 \), \( X \) is volume preserving:

\[
\text{Vol}(X(t, W)) = \text{Vol}(W).
\]

Applying the transport formula with \( f \equiv 1 \) one obtains

Proposition 1  The following facts are equivalent:

- The fluid flow is incompressible,
- \( \text{div} u = 0 \),
- \( J(t, \alpha) \equiv 1 \)

1.5  Vorticity

In the following, we consider a smooth velocity field \( u : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \). We define the vorticity field \( \omega \) as the curl of the velocity:

\[
\omega = \text{curl} \ u \equiv \left( \partial_2 u^3 - \partial_3 u^2, \ \partial_3 u^1 - \partial_1 u^3, \ \partial_1 u^2 - \partial_2 u^1 \right).
\]

Proposition 2 (Local behavior of an incompressible flow). In a neighborhood of any point \( x_0 \), up to higher order terms w.r.t. \( \|x - x_0\| \), a smooth incompressible velocity field \( u = u(x) \) can be written in a unique way as the sum of an infinitesimal translation, rotation and deformation velocities.
Proof Let \( x_0 \) be a given point in \( \mathbb{R}^N \). A first order Taylor expansion of \( u \) yields
\[
u(x_0 + h) = u(x_0) + \nabla u(x_0) \cdot h + \mathcal{O}(h^2).
\]
The Jacobian matrix \( \nabla u \), can be written as the sum of its symmetric and antisymmetric parts:
\[
\nabla u = \underbrace{\mathcal{D}}_{\text{Symm.}} + \underbrace{\Omega}_{\text{Antisymm.}}
\]
where
\[
\mathcal{D} = \frac{1}{2} \left( \nabla u + (\nabla u)^t \right), \quad \text{hence} \quad \text{tr} \mathcal{D} = 0
\]
\[
\Omega = \frac{1}{2} \left( \nabla u - (\nabla u)^t \right) \quad \text{hence} \quad \Omega h = \frac{1}{2} \omega \wedge h.
\]
Observe that, since every symmetric matrix can be diagonalized and the trace is invariant under orthogonal transformations, \( \mathcal{D} \) can be written as
\[
\mathcal{D} = \begin{pmatrix}
\gamma_1 & 0 & 0 \\
0 & \gamma_2 & 0 \\
0 & 0 & -(\gamma_1 + \gamma_2)
\end{pmatrix}.
\]
The Taylor expansion thus takes the form
\[
u(x_0 + h) = \underbrace{u(x_0)}_{\text{translation}} + \underbrace{\frac{1}{2} \omega \wedge h}_{\text{rotation}} + \underbrace{\mathcal{D} h}_{\text{deformation}} + \mathcal{O}(h^2).
\]

We now study how the vorticity of the fluid evolves in time.

Proposition 3 \((\text{Evolution equation for the vorticity})\) Let the velocity field \( u = u(t, x) \) provide a solution to the Navier-Stokes equations. Then the vorticity \( \omega = \text{curl} u \) evolves according to the equation
\[
D_t \omega = (\omega \cdot \nabla) u + \nu \Delta \omega.
\]

Proof We use the vector identity:
\[
\frac{1}{2} \nabla |u|^2 = u \wedge \text{curl} u + (u \cdot \nabla) u,
\]
in order to replace the term \((u \cdot \nabla) u\) in the Navier-Stokes equation:
\[
\partial_t u + \frac{1}{2} \nabla |u|^2 - u \wedge \omega = -\nabla p + \nu \Delta u.
\]

We now recall that \( \text{curl}(\nabla \varphi) = 0 \) for every \( \varphi \). Taking the curl of both sides of the above identity one obtains
\[
\partial_t \omega - \text{curl} (u \wedge \omega) = \nu \Delta \omega.
\]
In order to write explicitly the term \( \text{curl} (u \wedge \omega) \) we recall the following general vector identity:
\[
\text{curl} (u \wedge \omega) = (\omega \cdot \nabla) u - \omega \div u - (u \cdot \nabla) \omega - u \div \omega.
\]
Since \( \text{div} \, u \equiv 0 \), this reduces to

\[
\text{curl} \, (u \wedge \omega) = (\omega \cdot \nabla)u - (u \cdot \nabla)\omega.
\]

Therefore, from (1.14) we obtain

\[
\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \nu \Delta \omega. \tag{1.16}
\]

\[\square\]

### 1.6 Vorticity transport formula for the Euler equation

The map \( t \mapsto X(t, \alpha) \) describes the trajectory of the particle that was in \( \alpha \) at \( t = 0 \). In the present section we will show that the inviscid vorticity equation, i.e. the vorticity evolution equation associated to Euler equation,

\[
D_t \omega = (\omega \cdot \nabla)u, \tag{1.17}
\]

can be integrated by means of the particle trajectory equation

\[
\frac{d}{dt}X(t, \alpha) = u(t, X(t, \alpha)).
\]

Throughout the following, we call \( \omega_0(\alpha) = \omega(0, \alpha) \) the initial vorticity at time \( t = 0 \).

**Proposition 4 (Vorticity transport formula).** Let \( X(t, \alpha) \) be the smooth particle trajectory mapping corresponding to the smooth divergence-free vector field \( u(t, x) \). Then

\[
\omega(t, X(t, \alpha)) = \nabla_\alpha X(t, \alpha) \omega_0(\alpha)
\]

solves (1.17).

**Proof** We observe that the vorticity equation has the same form of the evolution equation satisfied by a first order perturbation.

Let \( h(t, x) \) be a smooth vector field in \( \mathbb{R}^N \), and consider the perturbed flow

\[
X(t, \alpha + h) = X(t, \alpha) + \underbrace{\nabla_\alpha X(t, \alpha) \, h}_{(\ast)} + \mathcal{O}(h^2).
\]

Here (\( \ast \)) is the term which tell us how the perturbation evolves in time. Then the evolution equation of a first order perturbation takes the form

\[
D_t h = h \cdot \nabla u.
\]

Notice that the right hand side represents the directional derivative of the velocity \( u \) in the direction of the vorticity \( \omega \). The following lemma applies also to the more general case in which the velocity field \( u \) is not divergence free.
Lemma 1 Let $u = u(t,x)$ be a smooth vector field with associated particle trajectory mapping $X(t,\alpha)$. The smooth vector field $h = h(t,x)$ satisfies

$$D_t h = h \cdot \nabla u$$

(1.18)

if and only if

$$h(t,X(t,\alpha)) = \nabla_\alpha X(t,\alpha) h(0,\alpha).$$

(1.19)

Proof The particle trajectory mapping is the solution of

$$\begin{cases}
\frac{d}{dt} X(t,\alpha) = u(t, X(t,\alpha)), \\
X(0,\alpha) = \alpha.
\end{cases}$$

By differentiation with respect to $\alpha$ we obtain

$$\frac{d}{dt} \nabla_\alpha X(t,\alpha) = (\nabla_x u)_{|x=X(t,\alpha)} \nabla_\alpha X(t,\alpha),$$

and then

$$\frac{d}{dt} \nabla_\alpha X(t,\alpha) h(0,\alpha) = (\nabla_x u)_{|x=X(t,\alpha)} \nabla_\alpha X(t,\alpha) h(0,\alpha).$$

If (1.19) holds then

$$\frac{d}{dt} h(t,X(t,\alpha)) = (\nabla_x u)_{|x=X(t,\alpha)} h(t,X(t,\alpha)).$$

Notice that the right hand side represents the directional derivative of the velocity $u$ in the direction of the vector $h(t,X(t,\alpha))$. This implies (1.18).

Viceversa, if (1.18) holds, then the map $t \mapsto h(t,X(t,\alpha))$ satisfies

$$\frac{d}{dt} h(t,X(t,\alpha)) = (\nabla_x u)_{|x=X(t,\alpha)} h(t,X(t,\alpha)),$$

and, since $h(t,X(t,\alpha))$ and $\nabla_\alpha X(t,\alpha) h(0,\alpha)$ both satisfy the same linear ODE with initial data $h(0,\alpha)$, they coincide. □

Proposition 4 is now proved by applying the lemma to the case $h = \omega$. □

Corollary 1 For a planar flow, the vorticity is constant along particle trajectories:

$$\omega(t,X(t,\alpha)) = \omega_0(\alpha).$$

(1.20)

Indeed, if the flow lies on the plane $\mathbb{R}^2$, then the velocity field is $u = (u^1,u^2,0)$, the vorticity is $\omega = (0,0,\omega_3) = (0,0,u^2_{x_1} - u^1_{x_2})$, and therefore

$$\nabla_\alpha X(t,\alpha) \omega = \begin{pmatrix}
\frac{dX_1}{dx_1} \\
\frac{dX_1}{dx_2} \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
\omega_3
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
\omega_3
\end{pmatrix}.$$
1.7 Vortex lines

At a fixed time \( t \), the integral curves of the vorticity field \( \omega = \text{curl} u \) are called \textit{vortex lines}:

\begin{definition}
A smooth curve \( s \mapsto \gamma(s) \in \mathbb{R}^N \) is a vortex line at time \( t \) if
\[
\frac{d}{ds} \gamma(s) = \lambda(s) \omega(t, \gamma(s)), \quad \text{for some } \lambda(s) \neq 0.
\]
\end{definition}

\begin{corollary}
In a non-viscous fluid flow, the vortex lines move with the fluid.
\end{corollary}

\begin{proof}
Let \( \gamma_0 \) be a vortex line at time \( t = 0 \). At time \( t > 0 \), the fluid particles which are initially along \( \gamma_0 \) will have moved to the curve
\[
\gamma_t = \{ X(t, \gamma_0(s)) \}.
\]
We claim that \( \gamma_t \) is also a vortex line, at time \( t \). Indeed,
\[
\frac{d}{ds} \gamma_t(s) = \nabla_\alpha X(t, \gamma_0(s)) \frac{d}{ds} \gamma_0(s)
= \nabla_\alpha X(t, \gamma_0(s)) \lambda(s) \omega(0, \gamma_0(s))
= \lambda(s) \omega(t, X(t, \gamma_0(s)))
= \lambda(s) \omega(t, \gamma_t(s)).
\]
\( \Box \)
\end{proof}

\begin{definition}
Let \( u \) be a smooth divergence-free vector field and let \( C \) be a smooth oriented closed curve in the fluid. We call \textit{circulation} of \( u \) around \( C \) the line integral
\[
\Gamma_C = \oint_C u \cdot dl.
\]
\end{definition}

\begin{lemma}
\textbf{Transport formula for curves.}
Let \( C_t = X(t,C) \) be a curve transported by the flow. The following identity holds:
\[
\frac{d}{dt} \oint_{C_t} u \cdot dl = \oint_{C_t} D_t u \cdot dl.
\] \hfill (1.22)
\end{lemma}

\begin{proof}
Let \( s \mapsto C(s) \) be a parametrization of \( C \), and let \( s \mapsto X(t,C(s)) = C_t(s) \) be the parametrization of \( C_t \).
\[
\frac{d}{dt} \oint_{C_t} u \cdot dl = \frac{d}{dt} \int_0^1 u(t, X(t,C(s))) \frac{\partial}{\partial s} X(t,C(s)) \, ds
= \int_0^1 D_t u(t, X(t,C(s))) \frac{\partial}{\partial s} X(t,C(s)) \, ds
+ \int_0^1 u(t, X(t,C(s))) \frac{\partial}{\partial t \partial s} X(t,C(s)) \, ds
= \oint_{C_t} D_t u \cdot dl + \int_0^1 \frac{\partial}{\partial s} |u(t, X(t,C(s)))|^2 \frac{2}{2} \, ds.
\] \hfill (1.23)
Since \( C_t \) is a loop the last term of the right hand side is null and we obtain the desired identity (1.22).
\( \Box \)
Theorem 2  **Kelvin’s circulation theorem.** For an inviscid, incompressible fluid the circulation of the velocity \( u \) around any closed curve \( C_t \) moving with the fluid is constant in time.

**Proof** Using the transport formula (1.22) one obtains

\[
\frac{d}{dt} \int_{C_t} u \cdot dl = \int_{C_t} D_t u \cdot dl
\]

\[= - \int_{C_t} \nabla p \cdot dl = 0\] (1.24)

because the line integral of a gradient along a loop is zero.  \( \square \)

Corollary 3  The vorticity flux through a 2-dimensional surface moving with the fluid is constant in time.

**Proof** Indeed, consider any 2-dimensional surface \( \Sigma_t \) whose boundary is a closed curve \( C_t \) moving with the fluid. At any point of the surface, call \( n \) the perpendicular unit vector. Using Stokes’ formula one obtains

\[
\int_{\Sigma_t} \omega \cdot n \, dA = \oint_{C_t} u \cdot dl
\]

The result thus follows from Kelvin’s circulation theorem.  \( \square \)

An example of this phenomena is provided by tornadoes. If the surface of a horizontal section decreases proportionally to the height \( h \) from the ground, then as \( h \to 0 \) the angular velocity around the vertical axis must increase.

### 1.8 Conserved quantities for the Euler equation

A conservation law in \( N \) space dimensions takes the form:

\[
\partial_t u(t, x) + \text{div}_x F(u(t, x)) = 0,
\] (1.25)

where \( t \geq 0 \) is the time variable, \( x = (x_1, \ldots, x_N) \in \mathbb{R}^N \) is the space variable, \( u = (u^1, \ldots, u^N) \) is a vector in \( \mathbb{R}^N \) and \( F \) is a smooth map defined on a convex neighborhood of the origin in \( \mathbb{R}^N \) with values in \( \mathbb{R}^N \). The components of \( u \) are called the **conserved quantities** while the components of \( F \), are the **fluxes**.

Let \( W \subset \mathbb{R}^N \) be a closed region with smooth boundary. Integrating (1.25) on \( W \) we obtain

\[
\frac{d}{dt} \int_W u(t, x) \, dx = \int_W u_t(t, x) \, dx
\]

\[= -\int_W \text{div} F(u(t, x)) \, dx
\]

\[= -\int_{\partial W} F(u(t, x)) \cdot n \, dA
\]

\[= \text{[flow across the boundary]}.
\]
Remark. Notice that
\[
d\frac{d}{dt} \int_{\mathbb{R}^N} u(t,x) \, dx = 0 \quad \text{if} \quad |F| = o(|x|^{1-N}).
\]
Indeed
\[
\frac{d}{dt} \int_{\mathbb{R}^N} u(t,x) \, dx = \lim_{R \to \infty} \frac{d}{dt} \int_{|x| \leq R} u(t,x) \, dx = -\lim_{R \to \infty} \int_{|x| = R} F(u(t,x)) \cdot n \, dA = 0,
\]
provided that \(|F|/|x|^{N-1} \to 0\) as \(|x| \to \infty\).

The following are conserved quantities for the Euler equations.

- **Components of momentum.** The Euler equation is equivalent to the momentum balance law. Then, if \(u\) is a solution to the Euler equation its components satisfy
  \[
u_i^t + \sum_{j=1}^{N} u_j^i u_{x_j} = -p_{x_i}, \quad i = 1, \ldots, N.
  \]
The above equations can be written as a system of conservation laws, namely
  \[
u_i^t + \text{div} (u^i u + P_i) = 0, \quad i = 1, \ldots, N,
  \]
where \(P_i \equiv (0, \ldots, 0, p, 0, \ldots, 0)\).

- **Vorticity.** The equation for the \(i\)-th component of the vorticity takes the form
  \[
  \omega_i^t + (u \nabla) \omega_i = (\omega \nabla) u_i.
  \]
These equations also yield a system of conservation laws
  \[
  \omega_i^t + \text{div} (\omega_i^t u - u_i^t \omega) = 0 \quad i = 1, \ldots, N.
  \]

- **Energy.** Taking the scalar product of both sides of the Euler equation by \(u\), we obtain the energy equation
  \[
  \left(\frac{|u|^2}{2}\right)_t + \sum_{i,j=1}^{N} u_i^j u_{x_j}^i + \sum_{i=1}^{N} u_i^i p_{x_i} = 0.
  \]
It can be written as a conservation law
  \[
  \left(\frac{|u|^2}{2}\right)_t + \text{div} \left(\frac{|u|^2}{2} u + p v\right) = 0.
  \]

- **Helicity.** The helicity measures the component of velocity in the direction of the vorticity:
  \[
  H = u \cdot \omega.
  \]
Differentiating w.r.t. time we obtain

\[
\frac{\partial}{\partial t} H = u_t \cdot \omega + u \cdot \omega_t
\]

\[
= \left[ -(u \cdot \nabla)u - \nabla p \right] \cdot \omega + u \cdot \left[ -(u \cdot \nabla)\omega + (\omega \cdot \nabla)u \right]
\]

\[
= -\sum_{i,j=1}^{N} u^i x_j \omega^j - \sum_{i=1}^{N} p_{x_i} \omega^i - \sum_{i,j=1}^{N} u^i \omega^i \omega^j x_j + \sum_{i,j=1}^{N} u^i \omega^j u^i x_j.
\]

This equation can be written in conservation form, namely

\[
H_t + \text{div} \left[ \sum_{i=1}^{N} \omega^i u^i u - \sum_{i=1}^{N} \frac{|u^i|^2}{2} \omega + p \omega \right] = 0.
\]

### 1.9 Leray formulation of the Navier-Stokes equations

In this section, our goal is to eliminate the pressure term from the Navier-Stokes equations, and obtain an evolution equation for the velocity \( u \) alone. Assume that, at time \( t = 0 \), \( \text{div} u = 0 \). Moreover, let \( u \) satisfy the equation

\[
\frac{du}{dt} = -(u \cdot \nabla)u + \nu \Delta u - \nabla p
\]

(1.26)

How can we determine the pressure function \( p = p(t, x) \), so that the divergence of \( u \) remains zero at all positive times?

To set the ideas, suppose \( u \in L^2 \), so that

\[
\|u\|_{L^2} = \left( \int |u|^2 \, dx \right)^{1/2} < +\infty.
\]

We introduce the subspace \( E \subset L^2 \) containing all vector fields whose divergence (in distribution sense) vanishes. Notice that, if \( u \) is a \( C^1 \) vector field with zero divergence, then for every smooth scalar function \( \varphi \) with compact support we have

\[
\int u \cdot \nabla \varphi \, dx = \int \text{div} (\varphi u) \, dx = 0.
\]

We can thus define the subspace \( E \) by setting

\[
E = \left\{ u \in L^2 \mid \int u \cdot \nabla \varphi \, dx = 0 \text{ for every } \varphi \in \mathcal{C}^\infty \right\}.
\]

(1.27)

If \( u(t) \in E \) for all \( t \geq 0 \), the time derivative \( \partial_t u(t) \) must remain in \( E \) for all times. Looking at the equation (1.26) we observe that

\[
\text{div} u = 0 \Rightarrow \text{div} (\Delta u) = 0,
\]

\[
\text{div} u = 0 \nRightarrow \text{div} (u \cdot \nabla u) = 0.
\]

(1.28)

More precisely, if the \( i \)-th component of \( (u \cdot \nabla)u \) is

\[
[(u \cdot \nabla)u]^i = \sum_j w^j u^i_{x_j},
\]
we can write
\[
\text{div} \ (u \cdot \nabla)u = \sum_{i,j} \left( u_j^j x_i u_i x_j + u_j^j u_i x_j \right) A_{i,j}.
\]
Since \( \text{div} \ u = \sum_i u_i x_i = 0 \), we get \( \sum_{i,j} A_{i,j} = 0 \), so, in general,
\[
\text{div} \ (u \cdot \nabla)u = \sum_{i,j} u_j^j x_i u_i x_j = \text{tr}(\nabla u)^2 \neq 0.
\]
We now seek a scalar function \( p \) such that the sum
\[
\frac{du}{dt} = -(u \cdot \nabla)u + \nu \Delta u - \nabla p
\]
lies in the subspace \( E \). Observing that
\[
\text{div} \ [(u \cdot \nabla)u + \nabla p] = \sum_{i,j} u_j^j x_i u_i x_j + \Delta p = \text{tr}(\nabla u)^2 + \Delta p,
\]
we conclude that the identity
\[
\text{div} \ [(u \cdot \nabla)u + \nabla p] = 0
\]
holds if and only if the function \( p \) solves the elliptic equation
\[
\Delta p = -\text{tr}(\nabla u)^2
\]
on the whole space \( \mathbb{R}^N \). A solution formula for the Poisson equation on \( \mathbb{R}^N \) is given in the following lemma.

**Lemma 3 Solution of the Poisson equation.** Let \( f \) be a smooth function in \( \mathbb{R}^N \), vanishing sufficiently rapidly as \( |x| \rightarrow \infty \). Then the solution to the Poisson equation
\[
\begin{aligned}
\Delta z &= f, \\
\nabla z &\rightarrow 0 \quad \text{as} \ |x| \rightarrow \infty,
\end{aligned}
\]
is given by
\[
z(x) = \mathcal{N}_N * f(x) = \int_{\mathbb{R}^N} \mathcal{N}_N(x-y)f(y) \, dy,
\]
where \( \mathcal{N} \) is the \( N \)-dimensional Newton potential
\[
\mathcal{N}_N(x) = \begin{cases} 
\frac{1}{2\pi} \ln |x|, & \text{if } N = 2, \\
\frac{1}{(2-N)\sigma_N} |x|^{2-N}, & \text{if } N \geq 3 
\end{cases}
\]
By \( \sigma_N \) we denote here the surface area of a unit sphere in \( \mathbb{R}^N \).

We observe that the kernel \( \mathcal{N}_N \) is the distributional solution of the equation
\[
\Delta \mathcal{N} = \delta_0
\]
where $\delta_0$ is the Dirac mass concentrate upon the origin.

By applying Lemma 3, we can solve the equation (1.29) and compute the pressure

$$p(t, x) = \int_{\mathbb{R}^N} N_N(x - y) \text{tr} (\nabla u(t, y))^2 \, dy.$$  

The gradient of $p$ takes the form

$$\nabla p(t, x) = \nabla N_N * \text{tr} (\nabla u)^2 (x) = C_N \int_{\mathbb{R}^N} \frac{x - y}{|x - y|^N} \text{tr} (\nabla u(t, y))^2 \, dy,$$

for a suitable constant $C_N$. Using this formula, we can eliminate the pressure from the standard formulation of the Navier-Stokes equation and obtain a system of closed evolution equations for $u$. This is the Leray formulation of the Navier-Stokes equations:

$$D_t u(t, x) = \nu \Delta u - C_N \int_{\mathbb{R}^N} \frac{x - y}{|x - y|^N} \text{tr} (\nabla u(t, y))^2 \, dy.$$  

(1.31)

**Remark.** According to (1.27), the subspace $E$ of vector fields with zero divergence can be regarded as the space perpendicular to all gradient vector fields (with suitable decay as $|x| \to \infty$).

The Navier-Stokes equation can be equivalently written as

$$u_t = \Pi_E (u \cdot \nabla) u + \nu \Delta u,$$

where $\Pi_E$ is the perpendicular projection operator on the space $E$.

### 1.10 Vorticity-stream formulation of the Navier-Stokes Equation

Taking the curl of both sides of the Navies-Stokes equation we obtain the vorticity equation

$$\omega_t + u \cdot \nabla \omega = \omega \cdot \nabla u + \nu \Delta \omega,$$  

(1.32)

in which the pressure does not appear. However, both sides of the equation (1.32) contain the fluid velocity $u$. In order to obtain an equation involving only the vorticity field $\omega$, we need to express the vector field $u$ in terms of $\omega = \text{curl} u$.

**The 2-dimensional case.**

If the flow lies in a plane, the velocity field is $u = (u^1, u^2, 0)$ and the vorticity field $\omega = (0, 0, u^2_{x_1} - u^1_{x_2})$. We can denote them respectively as

$$u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}, \quad \text{and} \quad \omega = u^2_{x_1} - u^1_{x_2}.$$  

In this case the vorticity equation becomes a scalar equation

$$\partial_t \omega + (u \cdot \nabla) \omega = \nu \Delta \omega.$$  

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We now show how to recover the vector field $u$, knowing its vorticity $\omega$. Recall that the system of PDEs
\[
\begin{align*}
\psi_{x_1} &= a(x_1, x_2), \\
\psi_{x_2} &= b(x_1, x_2),
\end{align*}
\]
has a solution if and only if $a_{x_2} = b_{x_1}$. Since $\text{div} \, u = u^1_{x_1} + u^2_{x_2} = 0$, there exists a so-called stream function $\psi$, such that
\[
\begin{align*}
\psi_{x_1} &= u^2(x_1, x_2), \\
\psi_{x_2} &= -u^1(x_1, x_2).
\end{align*}
\tag{1.33}
\]
By (1.33) it follows
\[
\begin{align*}
u &= \nabla^\perp \psi = \left( -\frac{\psi_{x_2}}{\psi_{x_1}} \right), \\
\omega &= \text{curl} \, u = \text{curl} (\nabla^\perp \psi) = \Delta \psi. 
\end{align*}
\tag{1.34}
\tag{1.35}
\]
Let $\omega : \mathbb{R}^2 \mapsto \mathbb{R}$ be a given vorticity function. We can find the corresponding stream function $\psi$ by solving the Poisson equation (1.35) as in Lemma 3, namely
\[
\psi(x) = N_2 * \omega(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln |x-y| : \omega(t,y) \, dy.
\]
In turn, the velocity field $u : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is now computed by (1.34). In the end, we see that $u$ can be recovered from $\omega$ by the Biot-Savart formula
\[
u(x) = K_2 * \omega(x) = \int_{\mathbb{R}^2} K_2(x-y) \omega(y) \, dy. 
\tag{1.36}
\]
Here the kernel $K_2$ is the fundamental solution of the equation
\[
\text{curl} \, z = \delta_0,
\]
with $\delta_0$ a Dirac mass at the origin. More precisely,
\[
K_2(x) = \frac{1}{2\pi |x|^2} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, 
\tag{1.37}
\]
Notice that $K_2$ has a singularity at the origin, namely
\[
|K_2(x)| = \frac{1}{2\pi |x|}.
\]
Observe that, in this second formulation of the Navier-Stokes equations, the pressure can be obtained from the the Poisson equation
\[
\Delta p = -\text{tr}(\nabla u)^2.
\]

The 3-dimensional case.

In the 3-dimensional case, the term $\omega \cdot \nabla u$ in the vorticity equation does not vanish. Therefore we have to recover both $u$ and $\nabla u$ from the vorticity $\omega$. 

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Lemma 4 Let $\omega : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a smooth vector field with a sufficiently fast decay as $|x| \rightarrow \infty$. The over-determined elliptic system

$$
\begin{align*}
\text{curl } u &= \omega, \\
\text{div } u &= 0,
\end{align*}
$$

(1.38)

admits a solution if and only if $\text{div } \omega = 0$.

In the positive case, the solution is constructed by means of the equation

$$
u = -\text{curl } \psi.$$

(1.39)

where the stream function $\psi$ solves the Poisson equation

$$
\Delta \psi = \omega,
$$

(1.40)

Remark. The equation (1.40) is solved using Lemma 3

$$
\psi(x) = N_3 * \omega(x) = \frac{1}{2\pi} \int_{\mathbb{R}^3} \frac{1}{|x - y|} \omega(t, y) \, dy.
$$

The explicit solution to (1.39) is then provided by the Biot-Savart formula

$$
u(x) = K_3 * \omega = \int_{\mathbb{R}^3} K_3(x - y) \omega(y) \, dy,
$$

where the kernel $K_3$ is defined by the relation

$$
K_3(x) \cdot h = \frac{1}{4\pi} \frac{x \wedge h}{|x|^3} \quad \text{for } h \in \mathbb{R}^3.
$$

(1.41)

Proof

Necessity: If $\omega = \text{curl } u$ then $\text{div } \omega = \text{div } (\text{curl } u) = 0$.

 Sufficiency: Start with the vector identity

$$
\Delta \psi = \nabla (\text{div } \psi) - \text{curl } (\text{curl } \psi)
$$

(1.42)

and consider the vector fields

$$
\omega \doteq \Delta \psi, \\
k \doteq \nabla (\text{div } \psi), \\
h \doteq -\text{curl } (\text{curl } \psi).
$$

Assuming that $\text{div } \omega = 0$, we will show that there exists a vector field $u$ that solves the system (1.38). As a preliminary, we prove that gradient vector fields are perpendicular to divergence-free vector fields, in the $L^2(\mathbb{R}^3)$ inner product.

Lemma 5 Let $w$ be a smooth, divergence-free vector field in $\mathbb{R}^N$ and let $q$ be a smooth scalar function such that

$$
|w(x)||q(x)| = o(|x|^{1-N}) \quad \text{as } |x| \rightarrow \infty.
$$

Then $w$ and $\nabla q$ are orthogonal:

$$
\int_{\mathbb{R}^N} w \cdot \nabla q \, dx = 0.
$$
Indeed,
\[
\int_{\mathbb{R}^N} w \cdot \nabla q \, dx = \lim_{R \to \infty} \int_{|x| \leq R} \text{div} \, (w q) \, dx = \lim_{R \to \infty} \int_{|x|=R} (w q) \cdot n \, dA = 0.
\]

We now return to the proof of the theorem. Taking the inner product of the vector identity (1.42) with \( k \), by Lemma 5 it follows
\[
\langle \omega, k \rangle = \langle k, k \rangle + \langle h, k \rangle = \langle k, k \rangle.
\]
Therefore,
\[
\langle k, k \rangle = \int_{\mathbb{R}^N} |\nabla (\text{div} \, \psi)|^2 \, dx = 0.
\]
This shows that \( \psi \) is divergence-free and
\[
\omega = \Delta \psi = -\text{curl} (\text{curl} \, \psi).
\]
Defining
\[
u = -\text{curl} \, \psi
\]
we obtain \( \omega = \text{curl} \, u \), and of course \( \text{div} \, u = \text{div} (-\text{curl} \, \psi) = 0 \). \( \square \)

1.11 Lagrangian formulation of the Euler equation

In this section we show that the Euler equation can be reformulated as an integro-differential equation for the particle trajectories. Given a smooth velocity field \( u \), the particle trajectory mapping \( X \) satisfies the following ODE, for each value of the Lagrangian variable \( \alpha \in \mathbb{R}^N \):
\[
\begin{cases}
\frac{d}{dt} X(t, \alpha) = u(t, X(t, \alpha)), \\
X(0, \alpha) = \alpha.
\end{cases}
\]
Our goal here is to recover \( u \) from \( X \) and \( \nabla X \). We recall the vorticity transport formula
\[
\omega(t, X(t, \alpha)) = \nabla_\alpha X(t, \alpha) \omega_0(\alpha),
\]
and the Biot-Savart formula
\[
u(t, x) = \int_{\mathbb{R}^N} K_N(x - y) \omega(t, y) \, dy.
\]
Putting together these equations we obtain
\[
u(t, X(t, \alpha)) = \frac{d}{dt} X(t, \alpha) = \int_{\mathbb{R}^N} K_N(X(t, \alpha) - y) \omega(t, y) \, dy.
\]
Consider the change of integration variable given by
\[
y = X(t, \alpha'),
\]
for a suitable value of the Lagrangian variable $\alpha'$. Then

$$u(t, X(t, \alpha)) = \int_{\mathbb{R}^N} \mathcal{K}_N(X(t, \alpha) - X(t, \alpha')) \omega(t, X(t, \alpha')) d\alpha'$$

$$= \int_{\mathbb{R}^N} \mathcal{K}_N(X(t, \alpha) - X(t, \alpha')) \nabla_\alpha X(t, \alpha') \omega_0(\alpha') d\alpha'.$$

(1.43)

In the 3-dimensional case the kernel $\mathcal{K}_3$ is defined by (1.41). We remark that, because of the growth of $\mathcal{K}_3$ as $x \to 0$, the integrand is not absolutely integrable. The right hand side of (1.43) must therefore be interpreted as a principal value of the integral. This is well defined almost everywhere.

In the 2-dimensional case the situation is much simpler. Indeed, according to (1.20), the vorticity is constant along particle trajectories. The integral formula (1.43) thus reduces to

$$u(t, X(t, \alpha)) = \int_{\mathbb{R}^2} \mathcal{K}_2(X(t, \alpha) - X(t, \alpha')) \omega_0(\alpha') d\alpha',$$

with $\mathcal{K}_2$ defined at (1.37).
Chapter 2

Construction of solutions

In this chapter we study the existence, uniqueness, and regularity properties of solutions to the Navier-Stokes equation

\[ u_t + (u \cdot \nabla)u = -\nabla p + \nu \Delta u, \quad (2.1) \]

\[ \text{div } u \equiv 0, \quad (2.2) \]

assuming that the initial data is smooth enough.

2.1 Uniqueness of solutions

Let \( v_1 \) and \( v_2 \) be two smooth solutions of the same Navier-Stokes equation, vanishing sufficiently fast as \(|x| \to \infty\). In particular, we assume \( v_1, v_2 \in L^2(\mathbb{R}^N) \). If \( p_1 \) and \( p_2 \) are the corresponding pressures, we define

\[ \tilde{v} = v_1 - v_2, \quad \tilde{p} = p_1 - p_2. \]

These two functions \( \tilde{v}, \tilde{p} \) satisfy the evolution equation

\[ \tilde{v}_t + (v_1 \cdot \nabla) \tilde{v} + (\tilde{v} \cdot \nabla) v_2 = -\nabla \tilde{p} + \nu \Delta \tilde{v}. \quad (2.3) \]

Observe that this equation is linear w.r.t. \( \tilde{v} \) and \( \tilde{p} \). Taking the inner product of both sides of (2.3) with \( \tilde{v} \) we obtain

\[
\int_{\mathbb{R}^N} \tilde{v} \cdot \tilde{v}_t \, dx + \int_{\mathbb{R}^N} \tilde{v} \cdot (v_1 \cdot \nabla) \tilde{v} \, dx + \int_{\mathbb{R}^N} \tilde{v} \cdot (\tilde{v} \cdot \nabla) v_2 \, dx =
\]

\[
- \int_{\mathbb{R}^N} \tilde{v} \cdot \nabla \tilde{p} \, dx - \nu \int_{\mathbb{R}^N} \tilde{v} \cdot \Delta \tilde{v} \, dx \quad (2.4)
\]

We observe that the two terms (⋆) and (⋆⋆) both vanish. Indeed, (⋆⋆) is the scalar product in \( L^2 \) of a divergence free vector field with the gradient of a scalar function. Moreover, integrating (⋆) by parts one obtains

\[
(⋆) = \sum_{i=1}^{N} \int_{\mathbb{R}^N} v_1 \cdot \nabla \left( \frac{|	ilde{v}_i|^2}{2} \right) \, dx = \sum_{i=1}^{N} \int_{\mathbb{R}^N} \text{div} \left( v_1 \cdot \frac{|	ilde{v}_i|^2}{2} \right) \, dx = 0
\]
because of the fast decay of $v_1$ as $|x| \to \infty$.

Integrating by parts the other terms the equation (2.4), we now obtain

$$
\frac{d}{dt} \int_{\mathbb{R}^N} \frac{|\tilde{v}|^2}{2} \, dx = - \int_{\mathbb{R}^N} \tilde{v} \cdot (\tilde{v} \cdot \nabla)v_2 \, dx - \nu \int_{\mathbb{R}^N} |\nabla \tilde{v}|^2 \, dx.
$$

Therefore

$$
\frac{d}{dt} \|\tilde{v}\|_{L^2}^2 \leq \|\tilde{v}\|_{L^2}^2 \|\nabla v_2\|_{L^\infty}.
$$

(2.5)

Thanks to (2.5), we can now prove that $\tilde{v} \equiv 0$ by a direct comparison with the corresponding ODE. Indeed, the function

$$
z(t) = \exp \left( \int_0^t 2\|\nabla v_2(\tau)\|_{L^2} \, d\tau \right) z_0
$$

solves the Cauchy problem

$$
\begin{aligned}
\dot{z}(t) &= 2\|\nabla v_2(t)\|_{L^\infty} z(t), \\
z(0) &= z_0 = \|\tilde{v}(0)\|_{L^2}^2.
\end{aligned}
$$

By comparison, for any time $t \geq 0$ we have

$$
\|\tilde{v}(t)\|_{L^2} \leq z(t).
$$

Therefore, if the solutions $v_1$ and $v_2$ coincide at $t = 0$, then they coincide for all $t \geq 0$. We have thus proved

**Theorem 3 (Uniqueness of solutions).** Let $v_1, v_2$ be solutions of the Navier-Stokes equations, having bounded energy. If $v_1(0) = v_2(0)$ and if the gradient of $v_2$ satisfies the bound

$$
\int_0^T \|\nabla v_2(t)\|_{L^\infty} \, dt < \infty,
$$

then $v_1(t) = v_2(t)$ for all $t \in [0, T]$.

### 2.2 The Sobolev spaces $H^m(\mathbb{R}^N)$

Toward a proof of existence of solutions, we recall here some basic facts about Sobolev spaces. The Hilbert-Sobolev space $H^m(\mathbb{R}^N)$ consists of all functions $v$ in $L^2(\mathbb{R}^N)$ such that $D^\alpha v$ is in $L^2(\mathbb{R}^N)$ for every multindex $\alpha$, $0 \leq |\alpha| \leq m$, where $D^\alpha$ denotes the distributional derivative $\partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N}$. The space $H^m(\mathbb{R}^N)$ is equipped with the norm

$$
\|v\|_{H^m} = \left( \sum_{0 \leq |\alpha| \leq m} \|D^\alpha v\|_{L^2}^2 \right)^{1/2}.
$$

**Theorem 4 (Sobolev embedding theorem).** If $m > k + \frac{N}{2}$ then $H^m(\mathbb{R}^N)$ is continuously embedded in $C^k$, i.e. there exist a constant $c > 0$ such that

$$
\|v\|_{C^k} \leq c \|v\|_{H^m}.
$$
Lemma 6 (Gagliardo-Nirenberg). Let \( \alpha \) be a multi-index, with \(|\alpha| = k \leq m\). Then
\[
\|D^\alpha v\|_{L^{2m/k}} \leq C \|v\|_{L^\infty}^{1-k/m} \|v\|_{H^m}^{k/m}
\]

Lemma 7 (An inequality in Sobolev spaces). For every \( m \geq 1 \) there exists a constant \( C_m \) such that, for every \( u, v \in L^\infty \cap H^m(\mathbb{R}^N) \), there holds
\[
\|u v\|_{H^m} \leq C_m \left\{ \|u\|_{L^\infty} \|v\|_{H^m} + \|v\|_{L^\infty} \|u\|_{H^m} \right\}.
\]

Proof For any multi-index \( \alpha \) with \(|\alpha| = m\), using the Leibnitz differentiation formula and then Hölder’s inequality, we obtain
\[
\|D^\alpha (u v)\|_{L^2} \leq c_\alpha \sum_{\beta \leq \alpha} \|D^\beta u D^{\alpha-\beta} v\|_{L^2} \leq c_\alpha \sum_{\beta \leq \alpha} \|D^\beta u\|_{L^{2m/|\beta|}} \|D^{\alpha-\beta} v\|_{L^{2m/|\alpha-\beta|}}.
\]
Using the Gagliardo-Nirenberg inequality with \( k = |\beta| \) and \( k = |\alpha - \beta| \), we obtain
\[
\|D^\alpha (u v)\|_{L^2} \leq C \cdot \sum_{\beta \leq \alpha} \|u\|_{L^\infty}^{1-|\beta|/m} \|u\|_{H^m}^{|\beta|/m} \|v\|_{L^\infty}^{1-|\alpha-\beta|/m} \|v\|_{H^m}^{|\alpha-\beta|/m}
\leq C \cdot \sum_{\beta \leq \alpha} \left( \|u\|_{L^\infty} \|v\|_{H^m} \right)^{|\alpha-\beta|/m} \left( \|v\|_{L^\infty} \|u\|_{H^m} \right)^{|\beta|/m}.
\]

For convenience, here and in the sequel the numerical value of the constant \( C \) is allowed to change from one line to the next. We now apply the following Geometric-Arithmetic Inequality. Let \( \alpha, \beta \) be positive numbers, and let \( \theta \in [0, 1] \). Then
\[
\alpha^\theta \beta^{1-\theta} \leq \theta \alpha + (1-\theta) \beta \leq \alpha + \beta.
\]
This yields
\[
\|D^\alpha (u v)\|_{L^2} \leq C \cdot \left\{ \|u\|_{L^\infty} \|v\|_{H^m} + \|v\|_{L^\infty} \|u\|_{H^m} \right\}.
\]
Of course, an estimate of the same type holds for lower order derivatives \( D^\alpha (uv) \), for \( 0 \leq \alpha < m \). This proves (2.6). \( \square \)

From (2.6) and the Sobolev embedding \( H^m(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N) \) for \( m > N/2 \), we deduce

Corollary 4 If \( m > N/2 \), then the space \( H^m(\mathbb{R}^N) \) is closed under multiplication. Namely, there exists a constant \( c \) such that
\[
\|uv\|_{H^m} \leq \|u\|_{H^m} \|v\|_{H^m}.
\]

Corollary 5 (A further inequality). For \(|\alpha| \leq m\), one has
\[
\|D^\alpha (u v) - u D^\alpha v\|_{L^2} \leq c_\alpha \left\{ \|\nabla u\|_{L^\infty} \|v\|_{H^{m-1}} + \|v\|_{L^\infty} \|\nabla u\|_{H^{m-1}} \right\}.
\]
Indeed,
\[
\|D^\alpha (u v) - u D^\alpha v\|_{L^2} \leq c_\alpha \sum_{\beta \leq \alpha, |\beta| > 0} \|D^\beta u D^{\alpha-\beta} v\|_{L^2} \leq c_\alpha \sum_{|\beta| + |\alpha-\beta| \leq m-1} \|D^{\beta'} (\nabla u) D^{\alpha-\beta'} v\|_{L^2}.
\]
Applying (2.6) with \( m \) replaced by \( m - 1 \) to each term on the right hand side of (2.9), we obtain (2.8). \( \square \)
2.3 Local-in-time existence of solutions

In this section, our goal is to construct a solution to the Cauchy problem for the Navier-Stokes equation

\[
\begin{aligned}
    u_t + (u \cdot \nabla)u &= -\nabla p + \nu \Delta u, \\
    \text{div } u &= 0, \\
    u(0, x) &= u_0(x),
\end{aligned}
\]  

(2.10)

assuming that the initial data is smooth enough.

The key step in the proof is to derive an a-priori estimate showing that the \( \mathbf{H}^m \) norm of the solution remains bounded, at least during a short interval of time \([0, T]\).

**Lemma 8** Let \( u \) be a solution to the Navier-Stokes equations. There exists a constant \( c_m \) such that, as long as the norms \( \|u(t)\|_{\mathbf{H}^m} \) and \( \|\nabla u(t)\|_{L^\infty} \) remain bounded, one has

\[
\frac{d}{dt} \|u(t)\|_{\mathbf{H}^m}^2 \leq c_m \|\nabla u(t)\|_{L^\infty} \|u(t)\|_{\mathbf{H}^m}^2.
\]  

(2.11)

**Proof** Applying the differential operator \( \mathcal{D}^\alpha \) to both sides of the Navier-Stokes equation (2.1) and then multiplying by \( \mathcal{D}^\alpha u \), we obtain

\[
\mathcal{D}^\alpha u \mathcal{D}^\alpha u_t + \mathcal{D}^\alpha u \mathcal{D}^\alpha ((u \cdot \nabla)u) = -\mathcal{D}^\alpha u \mathcal{D}^\alpha \nabla p + \nu \mathcal{D}^\alpha u \Delta \mathcal{D}^\alpha u.
\]

After we integrate this equation, it will be convenient to subtract the following integral term, which vanishes because of the decay of \( u \) as \( |x| \to \infty \).

\[
\int_{\mathbb{R}^N} \mathcal{D}^\alpha u (u \cdot \nabla) \mathcal{D}^\alpha u \, dx = \lim_{R \to \infty} \int_{|x|=R} \text{div} \left( \frac{\mathcal{D}^\alpha u^2}{2} \right) \, dx = 0.
\]

The above yields

\[
\frac{d}{dt} \int_{\mathbb{R}^N} \left| \frac{\mathcal{D}^\alpha u}{2} \right|^2 \, dx + \int_{\mathbb{R}^N} \mathcal{D}^\alpha u \left[ \mathcal{D}^\alpha ((u \cdot \nabla)u) - (u \cdot \nabla) \mathcal{D}^\alpha u \right] \, dx =
\]

\[
= - \int_{\mathbb{R}^N} \mathcal{D}^\alpha u \nabla \mathcal{D}^\alpha p \, dx + \nu \int_{\mathbb{R}^N} \mathcal{D}^\alpha u \Delta \mathcal{D}^\alpha u \, dx
\]

\[
= \frac{d}{dt} \int_{\mathbb{R}^N} \left| \frac{\mathcal{D}^\alpha u}{2} \right|^2 \, dx + \nu \int_{\mathbb{R}^N} |\nabla \mathcal{D}^\alpha u|^2 \, dx - \int_{\mathbb{R}^N} \mathcal{D}^\alpha u \left[ \mathcal{D}^\alpha ((u \cdot \nabla)u) - (u \cdot \nabla) \mathcal{D}^\alpha u \right] \, dx
\]  

(2.12)

Using Cauchy’s inequality, and then (2.8), we finally obtain

\[
\int_{\mathbb{R}^N} \left[ \mathcal{D}^\alpha ((u \cdot \nabla)u) - (u \cdot \nabla) \mathcal{D}^\alpha u \right] \, dx \leq \|\mathcal{D}^\alpha u\|_{L^2} \|\mathcal{D}^\alpha (u \cdot \nabla u) - u \mathcal{D}^\alpha \nabla u\|_{L^2}
\]

\[
\leq c_m \|\mathcal{D}^\alpha u\|_{L^2} \|\nabla u\|_{L^\infty} \|u\|_{\mathbf{H}^m}.
\]  

(2.13)

Since the above holds for all multi-indices \( \alpha \) with \( |\alpha| \leq m \), from (2.12) and (2.13) we obtain (2.11).

The inequality (2.11) has two important consequences.
Corollary 6 Consider a solution of (2.10), with initial data \( u_0 \in H^m \). If the map \( t \mapsto \| \nabla u(t) \|_{L^\infty} \) is integrable on \([0, T] \), then the norm \( \| u(t) \|_{H^m} \) of the solution remains uniformly bounded for \( t \in [0, T] \).

Indeed, let \( t \mapsto z(t) \) be the solution to the linear O.D.E.
\[
\dot{z}(t) = c_m \| \nabla u(t) \|_{L^\infty} z(t), \quad z(0) = \| u(0) \|_{H^m}.
\]
By (2.11), a comparison argument yields
\[
\| u(t) \|_{H^m}^2 \leq z(t) = \exp \left( \int_0^t c_m \| \nabla u(t) \|_{L^\infty} \, dt \right) \| u(0) \|_{H^m}^2.
\]
\[\square\]

Corollary 7 Let \( t \mapsto u(t) \) be a solution of (2.10), with initial data \( u_0 \in H^m(\mathbb{R}^N) \), with \( m > 1 + N/2 \). Then the norm \( \| u(t) \|_{H^m} \) remains bounded, on some initial interval \([0, T]\).

Indeed, by the Sobolev embedding theorem, there exists a constant \( C \) such that
\[
\| \nabla u \|_{L^\infty} \leq C \| u \|_{H^m}.
\]
Using this in (2.11) we obtain
\[
\frac{d}{dt} \| u(t) \|_{H^m}^2 \leq C_m \| u(t) \|_{H^m}^3,
\]
for a suitable constant \( C_m \). Call \( t \mapsto z(t) \) the solution to the O.D.E.
\[
\dot{z} = C_m z^{3/2}, \quad z(0) = \| u_0 \|_{H^m}^2.
\]
Observe that \( z \) is well defined and remains bounded on a suitably small interval \([0, T]\). By a comparison argument, from (2.14) we now obtain
\[
\| u(t) \|_{H^m}^2 \leq z(t) \quad t \in [0, T].
\]
\[\square\]

Before we state a local existence result, we wish to clarify what type of solutions will be constructed. As in (1.27), let \( E \subset L^2(\mathbb{R}^N) \) be the subspace of vector fields \( u \) having zero divergence, in distribution sense, and call \( P_E : L^2 \to E \) the perpendicular projection. By a solution of the initial value problem (2.10), we mean a Lipschitz continuous map \( u : [0, T] \to E \) such that \( (u \cdot \nabla)u \in L^2 \) and \( \Delta u \in L^2 \) for a.e. time \( \tau \in [0, T] \), and moreover
\[
u \Delta u(\tau) + \nu \Delta u(\tau) \}
d\tau \end{equation}
for every \( t \in [0, T] \).

Theorem 5 (Existence of solutions to the Navier-Stokes equations, locally in time). Let the initial data satisfy \( u(0, \cdot) = u_0 \in H^m \), with \( m > 2 + N/2 \). Then the equations (2.10) admit a solution defined for \( t \in [0, T] \), with \( T > 0 \) sufficiently small.
We sketch the proof, consisting of various steps:

1. First, one constructs a family of approximate solutions \((u^\varepsilon)_{\varepsilon > 0}\) by a mollification procedure. Consider a smooth, radially symmetric function \(\rho = \rho(|x|)\) in \(C_0^\infty(\mathbb{R}^N)\) satisfying

\[ \rho \geq 0, \quad \text{and} \quad \int_{\mathbb{R}^N} \rho \, dx = 1. \]

Define the family of mollifiers

\[ \rho_\varepsilon(x) = \varepsilon^{-N} \rho\left( \frac{x}{\varepsilon} \right) \]

and the associate family of mollification operators \(J_\varepsilon\)

\[ J_\varepsilon u(x) = \rho_\varepsilon \ast u(x) = \varepsilon^{-N} \int_{\mathbb{R}^N} \rho\left( \frac{x-y}{\varepsilon} \right) u(y) \, dy. \]

It is useful to recall here some basic properties of mollifiers:

(i) **Uniform convergence.** If \(v \in C^0\), then

\[ \|J_\varepsilon v\|_{L^\infty} \leq \|v\|_{L^\infty} \quad \text{and} \quad \lim_{\varepsilon \to 0} J_\varepsilon v(x) = v(x) \quad \text{(2.17)} \]

uniformly on bounded sets.

(ii) **Commuting with derivatives.** If \(v \in H^m\), then for every multi-index \(\alpha\) with \(|\alpha| \leq m\) there holds

\[ D^\alpha (J_\varepsilon v) = J_\varepsilon (D^\alpha v) \quad \text{(2.18)} \]

(iii) **Commuting with integrals.** For every \(u \in L^p\), \(v \in L^q\) with \(p^{-1} + q^{-1} = 1\), there holds

\[ \int_{\mathbb{R}^N} (J_\varepsilon u) \, v \, dx = \int_{\mathbb{R}^N} u \, (J_\varepsilon v) \, dx. \quad \text{(2.19)} \]

(iv) **Rate of convergence.** If \(v \in H^m\), then for some constant \(C\) one has

\[ \lim_{\varepsilon \to 0} \|J_\varepsilon v - v\|_{H^m} = 0; \quad \|J_\varepsilon v - v\|_{H^{m-1}} \leq C\varepsilon \|v\|_{H^m}. \quad \text{(2.20)} \]

(v) **Regularization.** For every \(v \in H^m\) and \(k \geq 0\),

\[ \|J_\varepsilon v\|_{H^{m+k}} \leq C_{mk} \varepsilon^{-k} \|v\|_{H^m}, \quad \|J_\varepsilon D^k v\|_{L^\infty} \leq C_k \varepsilon^{-(N/2+k)} \|v\|_{L^2}. \quad \text{(2.21)} \]

2. Consider the equation obtained by applying the \(\varepsilon\)-mollifier operator to Navier-Stokes equation.

\[ u^\varepsilon_t + J_\varepsilon \left[ (J_\varepsilon u^\varepsilon) \cdot \nabla (J_\varepsilon u^\varepsilon) \right] = -\nabla p^\varepsilon + \nu J_\varepsilon (J_\varepsilon \Delta u^\varepsilon). \]

This mollification procedure transforms the unbounded differential operators into bounded operators on the Sobolev space \(H^m\). Observing that the perpendicular projection operator \(P_E\)
commutes with the derivation and with the mollification operators, we can write the previous equation as an O.D.E. in the Banach space $E \subset L^2$, namely

$$
\frac{d}{dt} u^\varepsilon = - P_E J_\varepsilon [(J_\varepsilon u^\varepsilon) \cdot \nabla (J_\varepsilon u^\varepsilon)] + \nu J_\varepsilon (J_\varepsilon \Delta u^\varepsilon). \tag{2.22}
$$

The right hand side of this equation is Lipschitz continuous, because of the presence of the mollifiers. Hence, given the initial data $u^\varepsilon(0) = u_0$, for each $\varepsilon > 0$ this O.D.E. has a unique solution, which we denote as $t \mapsto u^\varepsilon(t)$.

3. Following the proof of Lemma 8, one can show that *the inequality

$$
\frac{d}{dt} \|u^\varepsilon(t)\|_{H^m}^2 \leq C \cdot \|u^\varepsilon(t)\|_{H^m}^{3/2}
$$

remains valid for approximate solutions of (2.22), with some constant $C$ independent of $\varepsilon$. Hence, on a suitably small interval $[0, T]$, the norms $\|u^\varepsilon(t)\|_{H^m}$ remain uniformly bounded. In turn, since we are assuming $m > 2 + N/2$, this yields an uniform a priori bounds on the norms $\|u^\varepsilon(t)\|_{C^1}$, independent of $\varepsilon > 0$.

4. Next, we claim that the family of mappings $u^\varepsilon : [0, T] \mapsto E \subset L^2$ is a Cauchy, as $\varepsilon \to 0$. Indeed, let $\varepsilon > \varepsilon' > 0$. We need to estimate

$$
\frac{d}{dt} \frac{1}{2} \|u^\varepsilon - u^{\varepsilon'}\|_{L^2}^2 = A_1 + A_2, \tag{2.23}
$$

where

$$
A_1 \doteq \left\langle P_E J_\varepsilon [(J_\varepsilon u^\varepsilon) \cdot \nabla (J_\varepsilon u^\varepsilon)] - P_E J_\varepsilon [(J_\varepsilon u^{\varepsilon'}) \cdot \nabla (J_\varepsilon u^{\varepsilon'})], u^\varepsilon - u^{\varepsilon'} \right\rangle,
$$

and

$$
A_2 \doteq \nu \left\langle J_\varepsilon^2 \Delta u^\varepsilon - J_\varepsilon^2 \Delta u^{\varepsilon'}, u^\varepsilon - u^{\varepsilon'} \right\rangle.
$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\mathbb{R}^N)$. Observing that $u^\varepsilon, u^{\varepsilon'} \in E$ and $P_E$ is a perpendicular projection, we can estimate

$$
A_1 = \left\langle (J_\varepsilon' - J_\varepsilon) [(J_\varepsilon u^\varepsilon) \cdot \nabla (J_\varepsilon u^\varepsilon)], u^\varepsilon - u^{\varepsilon'} \right\rangle \\
+ \left\langle J_\varepsilon' [(J_\varepsilon' - J_\varepsilon) u^\varepsilon \cdot \nabla (J_\varepsilon u^\varepsilon)], u^\varepsilon - u^{\varepsilon'} \right\rangle \\
+ \left\langle J_\varepsilon' [(J_\varepsilon' u^\varepsilon' - u^\varepsilon) \cdot \nabla (J_\varepsilon u^\varepsilon)], u^\varepsilon - u^{\varepsilon'} \right\rangle \\
+ \left\langle J_\varepsilon' [(J_\varepsilon' u^\varepsilon') \cdot \nabla ((J_\varepsilon' - J_\varepsilon) u^\varepsilon)], u^\varepsilon - u^{\varepsilon'} \right\rangle \\
+ \left\langle J_\varepsilon' [(J_\varepsilon' u^\varepsilon') \cdot \nabla (J_\varepsilon' (u^\varepsilon' - u^\varepsilon))], u^\varepsilon - u^{\varepsilon'} \right\rangle
$$

$$
= A_{11} + A_{12} + A_{13} + A_{14} + A_{15}.
$$

We observe that $A_{11}, A_{12}$ and $A_{14}$ contain the term $J_\varepsilon' - J_\varepsilon$. Using the second inequality in (2.20), we obtain the bound

$$
|A_{11}| + |A_{12}| + |A_{14}| \leq C(\varepsilon + \varepsilon') \left( \|u^\varepsilon\|_{H^m}^2 + \|u^{\varepsilon'}\|_{H^m}^2 \right) \|u^\varepsilon - u^{\varepsilon'}\|_{L^2}.
$$

Moreover, we have

$$
|A_{13}| \leq C \|u^\varepsilon\|_{H^m} \|u^\varepsilon - u^{\varepsilon'}\|_{L^2}^2.
$$

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Finally, an integration by parts yields
\[ A_{15} = \left\langle (J_\varepsilon u^{\varepsilon}) \cdot \nabla J_\varepsilon (u^{\varepsilon} - u^\nu), J_\varepsilon (u^\nu - u^{\varepsilon}) \right\rangle = -\frac{1}{2} \int_{\mathbb{R}^N} J_\varepsilon u^{\varepsilon} \cdot \nabla \left( |J_\varepsilon (u^\nu - u^{\varepsilon})|^2 \right) \, dx = 0. \]

The last term in (2.23) is estimated by
\[ A_2 = \left\langle (J_\varepsilon^2 - J_\nu^2) \Delta u^\nu, u^{\varepsilon} - u^\nu \right\rangle + \left\langle J_\nu^2 \Delta (u^{\varepsilon} - u^\nu), u^{\varepsilon} - u^\nu \right\rangle \leq C(\varepsilon + \varepsilon')\|u^\nu\|_{H^3} \|u^{\varepsilon} - u^\nu\|_{L^2} - \|J_\nu \nabla (u^{\varepsilon} - u^\nu)\|_{L^2}. \]

Combining all the previous estimates, from (2.23) we obtain
\[ \frac{d}{dt} \|u^{\varepsilon} - u^\nu\|_{L^2} \leq C(1 + \|u^\nu\|_{H^m} + \|u^\nu\|_{H^{m+1}}) \cdot [(\varepsilon + \varepsilon') + \|u^{\varepsilon} - u^\nu\|_{L^2}]. \]

By construction, at time \( t = 0 \) we have \( u^{\varepsilon}(0) = u^\nu(0) \). Moreover, by step 3 the \( H^m \) norms of the approximate solutions remain uniformly bounded, for \( t \in [0, T] \). This implies an estimate of the form
\[ \|u^{\varepsilon}(t) - u^\nu(t)\|_{L^2} \leq (\varepsilon + \varepsilon') e^{Kt}, \quad t \in [0, T]. \]

showing that as \( \varepsilon \to 0 \), the family of approximations \( u^{\varepsilon} \) is Cauchy.

5. By the previous step, there exists a continuous map \( u : [0, T] \to L^2(\mathbb{R}^N) \) such that
\[ \lim_{\varepsilon \to 0} \|u^{\varepsilon}(t) - u(t)\|_{L^2} = 0 \]
uniformly on the time interval \( [0, T] \). We claim that all the derivatives \( D^\alpha u^{\varepsilon} \), with \( |\alpha| < m \), converge to the corresponding derivatives of \( u \). Indeed, for any \( 0 < k < m \) the following interpolation inequality holds:
\[ \|v\|_{H^k} \leq C_k \|v\|_{L^2}^{1-k/m} \|v\|_{H^m}^{k/m}. \quad (2.24) \]

An application of this inequality yields
\[ \lim_{\varepsilon \to 0} \|u^{\varepsilon} - u\|_{H^k} \leq \lim_{\varepsilon \to 0} C_k \|u^{\varepsilon} - u\|_{L^2}^{1-k/m} \|u^{\varepsilon} - u\|_{H^m}^{k/m} = 0. \]

In particular, if \( m > 2 + N/2 \), we have the convergence
\[ \|(u^\nu \cdot \nabla)u^\nu - (u \cdot \nabla)u\|_{L^2} \to 0, \quad \|\Delta u^{\varepsilon} - \Delta u\|_{L^2} \to 0, \]
uniformly for \( t \in [0, T] \). From the relations
\[ u^{\varepsilon}(t) = u_0 - \int_0^t P_E J_\nu [(J_\varepsilon u^{\varepsilon}(\tau)) \cdot \nabla (J_\nu u^{\varepsilon}(\tau))] \, d\tau + \nu \int_0^t J_\varepsilon^2 \Delta u^{\varepsilon}(\tau) \, d\tau, \]
letting \( \varepsilon \to 0 \) and using the convergence \( \|u^{\varepsilon} - u\|_{H^m} \to 0 \), one checks that the limit function \( u \) satisfies (2.16). \( \square \)

Remarks. By choosing even larger exponent \( m \), one can prove the local existence of classical solutions with \( C^k \) regularity, for any given \( k \geq 1 \). One should also note that the above proof did not use in any way the regularization effect produced by the diffusion term \( \nu \Delta u \). Indeed, the previous estimates remain valid if \( \nu = 0 \).

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2.4 Global weak solutions of the Navier-Stokes equations

The regular solutions obtained in the previous section are only locally defined. Indeed, their $H^m$ norm may in principle blow up in finite time. In the present section we study the existence of globally defined distributional solutions, on a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary.

Consider the equations

\[
\begin{align*}
  &u_t + (u \cdot \nabla)u = -\nabla p + \nu \Delta u, \\
  &\text{div } u \equiv 0, \\
  &u(t, x) = 0, \quad \text{for all } x \in \partial \Omega, \\
  &u(0, x) = u_0(x).
\end{align*}
\]

(2.25)

Notice that the boundary condition is appropriate because $\nu > 0$.

**Definition 4** A distributional solution $u$ of the Navier-Stokes equation (2.25) is characterized by the following properties:

- **The energy of** $u$ **has to be finite**

  \[
  \int_{\mathbb{R}^3} |u(x,t)|^2 \, dx \leq E_0, \quad \text{for a.e. } t \in [0, T].
  \]

  \[\text{(2.26)}\]

- **The loss of energy due to the viscosity has to be also finite**

  \[
  \int_0^T \int_{\mathbb{R}^3} |\nabla u(x,t)|^2 \, dx \, dt \leq E_0.
  \]

  \[\text{(2.27)}\]

- **The function** $u = u(t, x)$ **satisfies the evolution equation and the incompressibility condition in distributional sense.** Namely, for any $\psi \in C_0^\infty(\mathbb{R}^3)$ and for any $\phi \in C_0^\infty((\mathbb{R},T] \times \Omega; \mathbb{R}^3)$ we have

  \[
  \int_{\mathbb{R}^3} u \nabla \psi \, dx = 0 \quad \text{for a.e. } t \in [0,T],
  \]

  \[
  \int_0^T \int_{\mathbb{R}^3} \left\{u[\phi_t + \nu \Delta \phi] - (u \cdot \nabla u)\phi + p \, \text{div } \phi \right\} \, dx \, dt + \int_{\mathbb{R}^3} u_0(x)\phi(0, x) \, dx = 0
  \]

  \[\text{(2.28)}\]

The proof of existence of a solution will be given in three steps. First, we derive some uniform a priori estimates on exact solutions to the Navier-Stokes equations. In a second step, by a Galerkin method, we construct a sequence of approximate solutions satisfying the same a priori estimates. In turn, these estimates allow us to use a compactness argument, and obtain a subsequence converging to a weak solution, in suitable function spaces.

**Proposition 5** Assume $u$ to be a solution of (2.25), we claim that the following estimates hold:

- $u \in L^\infty([0, T], L^2(\Omega))$, i.e. there exists a constant $E_0$ such that

  \[
  \int_{\Omega} |u(t, x)|^2 \, dx \leq 2E_0 \quad \text{for a.e. } t \in [0, T].
  \]

  \[\text{(2.29)}\]
\[ u \in L^2([0, T], H^1_0(\Omega)), \text{ i.e.} \]
\[ \int_0^T \int_\Omega |\nabla u(t, x)|^2 \, dx \, dt \leq \frac{2}{\nu} E_0 \]  
(2.30)

\[ \frac{du}{dt} \in L^{4/3}([0, T], H^{-1}), \text{ i.e. there exists a constant } C_2 \text{ such that} \]
\[ \int_0^T \|u_t(t)\|_{H^{-1}}^{4/3} \, dt \leq C_2. \]  
(2.31)

**Proof** If \( u \) is a solution of the Navier-Stokes equation, then
\[ (u_t, u) + (u, u \cdot \nabla u) - \nu (u, \Delta u) + (u, \nabla p) = 0. \]

By integrating over \( \Omega \) we obtain
\[
\frac{d}{dt} \int_\Omega \frac{|u|^2}{2} \, dx + \int_\Omega \sum_{i,j=1}^3 u^i u^j u^j_i \, dx + \int_\Omega u \cdot \nabla p \, dx + \nu \int_\Omega |\nabla u|^2 \, dx = 0
\]
\[
\frac{d}{dt} \int_\Omega \frac{|u|^2}{2} \, dx + \int_\Omega \text{div} \left( \frac{|u|^2}{2} \cdot u \right) \, dx + \int_\Omega \text{div}(u \cdot p) \, dx + \nu \int_\Omega |\nabla u|^2 \, dx = 0
\]
\[
\frac{d}{dt} \int_\Omega \frac{|u|^2}{2} \, dx + \nu \int_\Omega |\nabla u|^2 \, dx = 0
\]
(2.32)

The last identity shows that the \( L^2 \) norm of the solution \( u \) decreases in time. Indeed,
\[ \int_\Omega |u(t,x)|^2 \, dx \leq \int_\Omega |u_0(x)|^2 \, dx, \quad \text{for all } t \geq 0. \]

Moreover
\[ \nu \int_0^T \int_\Omega |\nabla u(t,x)|^2 \, dx \, dt \leq \int_\Omega \frac{|u_0(x)|^2}{2} \, dx - \int_\Omega \frac{|u(T,x)|^2}{2} \, dx. \]

This proves the first two estimates, taking \( E_0 \) to be the total energy at the initial time \( t = 0 \).

In order to prove (2.31) we recall that
\[ u_t = P_E (\nu \Delta u - u \cdot \nabla u) \]  
(2.33)

where \( P_E \) denotes the perpendicular projection on on the subspace \( E \subset L^2(\Omega; \mathbb{R}^3) \) consisting of divergence free vector fields.

The \( H^{-1} \) norm of (2.33) can be estimated as follows:
\[ \|\nu \Delta u\|_{H^{-1}} = \sup_{\|w\|_{H^1_0} \leq 1, \text{div} w \equiv 0} \int_\Omega -\nu \nabla u \cdot \nabla w \, dx \leq \nu \|u\|_{H^1_0}, \]
\[ \|u \cdot \nabla u\|_{H^{-1}} = \sup_{\|w\|_{H^1_0} \leq 1, \text{div} w \equiv 0} \int_\Omega (u \cdot \nabla u) \cdot w \, dx. \]
Observing that \( \frac{1}{3} + \frac{1}{2} + \frac{1}{6} = 1 \), using a generalized Hölder inequality and a Sobolev embedding theorem we obtain:

\[
\left| \int_{\Omega} (u \cdot \nabla u) w \, dx \right| \leq \int_{\Omega} \left| \sum_{i,j=1}^{3} u_i u_j w^j \right| \, dx \\
\leq K_1 \|u\|_{L^3} \|\nabla u\|_{L^2} \|w\|_{L^6} \leq K_2 \|u\|_{L^3} \|\nabla u\|_{L^2}.
\]

By Sobolev embedding and interpolation there holds

\[
\|u\|_{L^3} \leq C \|u\|_{H^{1/2}} \leq C \|u\|_{H^0}^{1/2} \|u\|_{H^1}.
\]

Hence

\[
\|u \cdot \nabla u\|_{H^{-1}} \leq C \|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2}^{3/2} \\
\int_0^T \|u\|_{H^{-1}}^{4/3} \, dt \leq C \int_0^T \left( \|u\|_{H^0}^{4/3} + \|u\|_{L^2}^{1/2} + \|\nabla u\|_{L^2}^{3/2} \right)^{4/3} \, dt \\
\leq C \left( 1 + \int_0^T \|\nabla u\|_{L^2}^2 \, dt \right) \leq C_1
\]

2.5 Construction of finite dimensional approximate solutions

In this section we establish the following global existence theorem

**Theorem 6** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with smooth boundary. Let \( u_0 \in L^2(\Omega) \) be a vector field with zero divergence. Then the equations (2.25) admit a weak solution, defined for all times \( t \geq 0 \).

**Proof** Fix any time interval \([0, T]\). Let \( E \subset L^2(\Omega; \mathbb{R}^3) \) be the closed subspace of vector fields whose distributional divergence vanishes. On \( E \), consider the linear differential operator

\[
Au = -\Delta u,
\]

and the eigenvalue problem

\[
\left\{ \begin{array}{lcl}
A u &=& \lambda u \quad x \in \Omega, \\
 u &=& 0 \quad x \in \partial \Omega.
\end{array} \right.
\]

Since the operator \( A \) is self-adjoint, there exists an orthonormal set \( \{w_1, w_2, w_3, \ldots\} \subset L^2(\Omega) \) consisting of eigenfunctions of \( A \). We will denote by \( E_m \subset E \) the subspace generated by the first \( m \) eigenfunctions:

\[
E_m : = \text{span} \{w_1, w_2, \ldots, w_m\},
\]

and write \( P_m : E \mapsto E_m \) for the orthogonal projection onto \( E_m \).

Projecting the equation (2.33) into the finite dimensional subspace \( E_m \) we obtain the Galerkin system

\[
\left\{ \begin{array}{lcl}
\frac{d}{dt} u_m + \nu Au_m + P_m (u \cdot \nabla) u &=& 0, \\
u_m(0, x) &=& P_m u_0(x).
\end{array} \right. \]

(2.36)
Notice that (2.36) is a Cauchy problem in the finite dimensional space $E_m$.

For each fixed time $t$, we can write $u_m(t, \cdot)$ as linear combination of the functions $w_1, w_2, \ldots, w_m$:

$$u_m(t, x) = \sum_{i=1}^{m} \xi_i(t) w_i(x),$$

where $\xi_i(t) \doteq \langle u_m(t), w_i \rangle_{L^2}$. By construction, one has

$$\Delta u_m(t, x) = -\sum_{i=1}^{m} \lambda_j \xi_j(t) w_j(x),$$

and the problem (2.36) can be written as a system of $m$ ODEs for the coefficients $\xi_1, \ldots, \xi_m$.

$$\frac{d}{dt} \xi_i = -\nu \lambda_i \xi_i + \sum_{j,k=1}^{m} b_{ijk} \xi_j \xi_k, \quad (2.37)$$

The constant coefficients $b_{ijk}$ are here computed as

$$b_{ijk} = -\int_{\Omega} [(w_j \cdot \nabla) w_k] w_i \, dx = -\langle (w_j \cdot \nabla) w_k, w_i \rangle_{L^2}.$$

Since the right hand sides of (2.37) are quadratic polynomials, for any given initial data the Cauchy problem has a unique solution defined locally in time. To prove that this solution exists globally, we need to show that its norm remains bounded.

Toward this goal, observe that the coefficients $b_{ijk}$ satisfy

$$b_{ijk} + b_{ikj} = \int_{\Omega} \nabla (w_j \cdot w_k) \cdot w_i \, dx = \int_{\Omega} \text{div} \left( (w_j \cdot w_k) w_i \right) \, dx = 0.$$

Therefore

$$\frac{d}{dt} \frac{|\xi_i(t)|^2}{2} = \sum_{i,j,k=1}^{m} b_{ijk} \xi_j \xi_k \xi_k - \nu \sum_{i=1}^{m} \lambda_i \xi_i^2(t) = -\nu \sum_{i=1}^{m} \lambda_i \xi_i^2(t) < 0. \quad (2.38)$$

Indeed, all the eigenvalues $\lambda_i$ of the operator $A = -\Delta$ are strictly positive. Hence for all $m \geq 1$ we have a well defined approximate solution $u_m : [0, T] \to E_m$.

Repeating the arguments in the proof of Proposition 5, one can show that

- $u_m \in L^\infty([0, T]; L^2(\Omega))$,
- $u_m \in L^2([0, T]; H^1_0(\Omega))$,
- $(u_m)_t \in L^{4/3}([0, T], H^{-1}(\Omega)).$

Moreover, the norms of the approximate solutions $u_m$ in these various function spaces remain uniformly bounded as $m \to \infty$. Using a compactness lemma, one can extract a subsequence, still called $(u_m)_{m \geq 1}$, and a limit function $u$ such that

- $u_m \rightharpoonup u$ weakly in $L^2([0, T]; H^1_0(\Omega))$,
\( (u_m)_t \rightharpoonup u_t \) weakly in \( L^{4/3}([0, T], H^{-1}(\Omega)) \),

\( u_m \rightharpoonup u \) weakly* in \( L^\infty([0, T]; L^2(\Omega)) \).

In turn, by extracting a further subsequence, we can achieve the further convergence

\( u_m \rightharpoonup u \) strongly in \( L^2([0, T]; L^2(\Omega)) \),

\( u_m \rightarrow u \) in \( C^0([0, T], H^{-1}(\Omega)) \).

Observe that the uniform bound on the norm of \( \|u_m\|_{L^{4/3}([0, T]; H^{-1})} \) implies that the maps \( t \mapsto u_m(t) \in H^{-1} \) are uniformly Holder continuous of exponent \( 1/4 \). Indeed,

\[
\|u(t_2) - u(t_1)\|_{H^{-1}} \leq \int_{t_1}^{t_2} 1 \cdot \|u_t\|_{H^{-1}} dt \leq \|u_t\|_{L^{4/3}([t_1, t_2]; H^{-1})} \|1\|_{L^{1/4}([t_1, t_2])} \leq \|u_t\|_{L^{4/3}} |t_2 - t_1|^{1/4}.
\]

Since \( u_m(0) = P_m u_0 \rightharpoonup u_0 \) as \( m \to \infty \), this uniform continuity implies

\[
\lim_{t \to 0^+} \|u(t) - u_0\|_{H^{-1}} = 0.
\]

Hence the limit function \( u \) takes the correct initial data.

Since every function \( u_m \) maps \([0, T]\) into the closed subspace \( E \subset H^1_0 \) consisting of divergence-free vector fields, the same holds for \( u \). It now remains to show that the limit function \( u \) satisfies the Navier-Stokes equation in a weak sense.

Fix an integer \( k \geq 1 \) and consider a smooth test function \( \phi : [0, T] \mapsto E_k \) with compact support, so that \( \phi(0) = \phi(T) = 0 \in E_k \). Then for all \( m \geq k \) we have

\[
\int_0^T \int_{\Omega} (-\phi_t - \nu \Delta \phi) u_m + [(u_m \cdot \nabla) u_m] \cdot \phi \, dx \, dt = -\int_{\Omega} (\phi_t + \nu \Delta \phi) u_m + [(u_m \cdot \nabla) \phi] \cdot u_m \, dx \, dt = 0.
\]

Taking the limit as \( m \to \infty \), this yields

\[
\int_{\Omega} (\phi_t + \nu \Delta \phi) u + [(u \cdot \nabla) \phi] \cdot u \, dx \, dt = 0.
\]

Observing that, as \( k \to \infty \), the set of test functions of the form \( \phi : [0, T] \mapsto E_k \) becomes dense in the set of all test functions \( \phi : [0, T] \mapsto E \), we conclude that \( u \) provides a weak solution. \( \square \)
Chapter 3

Vortex sheets

3.1 Velocity behavior near a vortex sheet

We have a vortex sheet when vorticity concentrates along a curve \( \gamma : \mathbb{R} \rightarrow \mathbb{R}^2 \) with linear density \( \phi \).

Let \( s \mapsto \gamma(s) \) be a parametrization of the curve \( \gamma \). Given \( s_1 < s_2 \), the total vorticity on the arc between \( \gamma(s_1) \) and \( \gamma(s_2) \) is thus given by the integral

\[
\int_{s_1}^{s_2} \phi(s) \, ds.
\]

If the vorticity is located only along the curve \( \gamma \), and vanishes outside \( \gamma \), then the velocity of the fluid at a point \( x \in \mathbb{R}^2 \) is computed by

\[
u(x) = \int K(x - \gamma(s)) \, \phi(s) \, ds.
\]

Here \( K \) denotes the two-dimensional Biot-Savart kernel, i.e.

\[
K(x) = \frac{1}{2\pi |x|^2} \left( \begin{array}{c} -x_2 \\ x_1 \end{array} \right).
\] (3.1)

To determine how the vorticity changes in time, we should now determine how points on the curve \( \gamma \) move. If \( x \) is a point on the curve, i.e. \( x = \gamma(s_0) \) for some \( s_0 \), then the above formula yields

\[
u(\gamma(s_0)) = \int K(\gamma(s_0) - \gamma(s)) \, \phi(s) \, ds.
\] (3.2)

However, since \( K(\gamma(s_0) - \gamma(s)) = O(1) \cdot |s_0 - s|^{-1} \), the above integral is not absolutely convergent.

We now show that, if \( \gamma \) and \( \phi \) are smooth, then the integral (3.2) can be uniquely defined as a principal value. Indeed, assume that \( \gamma'(s_0) \neq 0 \). Then \( \gamma \) can be written in the following way

\[
\gamma(s) = \gamma(s_0) + \tilde{\gamma}'(s_0)(s - s_0),
\]

where

\[
\tilde{\gamma}'(s) = \frac{1}{s - s_0} \int_{s_0}^{s} \gamma'() \, d\sigma.
\]
Similarly, we can write
\[ \phi(s) = \phi(s_0) + \phi'(s)(s - s_0). \]

We now compute

\[
\int_{s_0}^{s_0 + \delta} K(\gamma(s_0) - \gamma(s)) \phi(s) \, ds
= \int_{s_0}^{s_0 + \delta} K(\gamma'(s)(s - s_0)) \cdot [\phi(s_0) + \phi'(s)(s - s_0)] \, ds
= \int_{s_0}^{s_0 + \delta} K(\gamma'(s)(s - s_0)) \phi(s_0) \, ds
+ \int_{s_0}^{s_0 + \delta} [K(\gamma'(s)(s - s_0)) - K(\gamma'(s_0)(s - s_0))] \phi(s) \, ds
+ \int_{s_0}^{s_0 + \delta} K(\gamma'(s)(s - s_0)) \phi'(s)(s - s_0) \, ds.
\]

Of the three integrals on the right hand side of (3.3), the first one vanishes, while the remaining two are absolutely integrable. Indeed,

\[ K(\gamma'(s)(s - s_0)) = O(1) \cdot |s - s_0|^{-1}, \]

\[ |\gamma'(s) - \gamma'(s_0)||s - s_0| = O(1)(s - s_0)^2, \]

hence

\[ K(\gamma'(s)(s - s_0)) - K(\gamma'(s_0)(s - s_0)) = O(1). \]

Next, let \( \gamma \) be a vortex sheet. We wish to compute the jump in the velocity of the fluid across \( \gamma \). To simplify computations, we assume here \( ||\gamma'(s)|| \equiv 1 \), so that the curve is parametrized by arc length.

Call \( n \) be the unit normal vector, perpendicular to the curve. The variation of velocity between a point on the curve \( \gamma(s_0) \) and a point near the curve \( \gamma(s_0) + \varepsilon n \) is given by the integral

\[
u(\gamma(s_0) + \varepsilon n) - \nu(\gamma(s_0)) = \int \left[ K(\gamma(s_0) + \varepsilon n - \gamma(s)) - K(\gamma(s_0) - \gamma(s)) \right] \phi(s) \, ds. \tag{3.4}
\]

To compute the limit as \( \varepsilon \to 0^+ \), it suffices to restrict the above integral to any interval of the form \([s_0 - \delta, s_0 + \delta]\). Indeed, outside this interval the integrand in (3.4) is a continuous function of \( \varepsilon \) and goes to 0 as \( \varepsilon \to 0 \). Notice that, as \( s \to s_0 \), we have

\[ \gamma'(s) \to \gamma'(s_0), \quad \phi(s) \to \phi(s_0). \]

From (3.4) we obtain

\[
\lim_{\varepsilon \to 0^+} \left[ \nu(\gamma(s_0) + \varepsilon n) - \nu(\gamma(s_0)) \right] = \lim_{\varepsilon \to 0} \int_{s_0 - \delta}^{s_0 + \delta} \left[ K(\gamma'(s)(s - s_0) + \varepsilon n) - K(\gamma'(s)(s - s_0)) \right] \phi(s) \, ds
= \lim_{\varepsilon \to 0} \int_{s_0 - \delta}^{s_0 + \delta} \left[ K(\gamma'(s_0)(s - s_0) + \varepsilon n) - K(\gamma'(s_0)(s - s_0)) \right] \phi(s_0) \, ds.
\]

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Performing the change of variable \( \xi = (s - s_0)/\varepsilon \), the above yields
\[
\lim_{\varepsilon \to 0^+} [u(\gamma(s_0) + \varepsilon n) - u(\gamma(s_0))] = \lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} [K(\gamma'(s_0)\varepsilon \xi + \varepsilon n) - K(\gamma'(s_0)\varepsilon \xi)] \phi(s_0) \varepsilon \, d\xi.
\]

To compute this integral, we fix the orthonormal basis
\[
e_1 = \gamma'(s_0), \quad e_2 = n
\]
and write the Biot-Savart kernel \( K \) coordinate-wise w.r.t. this basis. This yields
\[
\lim_{\varepsilon \to 0^+} [u(\gamma(s_0) + \varepsilon n) - u(\gamma(s_0))] = \lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} \left( -\frac{1}{1 + \xi^2}, 0 \right) d\xi = \left( -\frac{\phi(s_0)}{2}, 0 \right).
\]

This shows that the jump of the tangential component of the velocity across a vortex sheet is equal to the vorticity density \( \phi(s_0) \) (up to the sign), while the normal component is continuous.

### 3.2 The Birkhoff-Rott equation

Assume that the vorticity density is \( \phi(s) \) is smooth and uniformly positive. Then we can reparametrize \( \gamma \) by means of the vorticity. More precisely, fix a value \( s_0 \). We then define the new parametrization
\[
\sigma \mapsto z(\sigma) = \gamma(s(\sigma)) \in \mathbb{R}^2,
\]
where the change of variable \( \sigma \mapsto s(\sigma) \) is implicitly defined by the relation
\[
\sigma = \int_{s_0}^{s(\sigma)} \phi(s) \, ds.
\]

Observe that the new parametrization has the property that the amount of vorticity contained in the arc between \( z(\sigma_1) \) and \( z(\sigma_2) \) is precisely \( \sigma_2 - \sigma_1 \).

The Birkhoff-Rott equation is a differential equation defining the motion of the vortex sheet. We now regard the 2-dimensional vector \((x_1, x_2)\) as a point \( z = x_1 + ix_2 \) of the complex plane. Using (3.2) with \( \phi \equiv 1 \), we can write
\[
\frac{d}{dt} z(t, \sigma) = \int_{-\infty}^{+\infty} K(z(\sigma) - z(\sigma')) \, d\sigma' = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{d\sigma'}{\bar{z}(\sigma) - z(\sigma')}.
\]

Here the upper bar denotes conjugation. Equivalently, this can be written as
\[
\frac{d}{dt} \bar{z}(t, \sigma) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{d\sigma'}{z(\sigma) - z(\sigma')}, \tag{3.5}
\]
where the left hand side is the time derivative of the conjugate of \( z \). We remark that all these integrals are not absolutely convergent, and should be interpreted as principal values.
3.3 Kelvin-Helmholtz instability

One very simple solution of the Birkhoff-Rott equation is the constant function \( z(t, \sigma) = \sigma \). In this case the curve \( \gamma \) is precisely the real axis. We wish to understand the stability of this solution w.r.t. small perturbations. Consider the perturbed function

\[
\tilde{z}(t, \sigma) = \sigma + \varepsilon g(t, \sigma).
\]

This should satisfy

\[
\frac{d}{dt} \tilde{z}(t, \sigma) = \frac{d}{dt} \varepsilon \tilde{g}(t, \sigma) = \frac{1}{2\pi i} P.V. \int_{-\infty}^{+\infty} \frac{d\sigma'}{(\sigma - \sigma')^2 + \varepsilon(g(\sigma) - g(\sigma'))}.
\]

Performing the expansion

\[
\frac{1}{a + \varepsilon b} = \frac{1}{a} \left( \frac{1}{1 + \varepsilon b/a} \right) = \frac{1}{a} \left( 1 - \varepsilon \frac{b}{a} + o(\varepsilon^2) \right).
\]

inside the integral, and letting \( \varepsilon \to 0 \), we obtain

\[
\frac{d}{dt} \tilde{g}(t, \sigma) = -\frac{1}{2\pi i} P.V. \int_{-\infty}^{+\infty} \frac{g(\sigma) - g(\sigma')}{(\sigma - \sigma')^2} d\sigma' = \frac{1}{2\pi i} \lim_{\delta \to 0^+} \left( \int_{-\infty}^{-\delta} + \int_{\delta}^{+\infty} \right) \frac{g(\sigma) - g(\sigma')}{(\sigma - \sigma')^2} d\sigma'.
\]

Integrating by parts and observing that the boundary terms cancel each other as \( \delta \to 0^+ \), we obtain

\[
\frac{d}{dt} \tilde{g}(t, \sigma) = -\frac{1}{2\pi i} P.V. \int_{-\infty}^{+\infty} \frac{g'(\sigma')}{\sigma - \sigma'} d\sigma' = \frac{1}{2} \mathcal{H}g'.
\]

(3.6)

Here \( \mathcal{H}g' \) denotes the Hilbert transform of \( g' \). We recall that the Hilbert transform of a function \( f \) is defined as the principal value

\[
\mathcal{H}f(x) \overset{\Delta}{=} \frac{1}{\pi i} P.V. \int_{-\infty}^{+\infty} \frac{f(y)}{y - x} dy.
\]

Using the Fourier transform, \( f \) can be written as

\[
f(x) = \int e^{i\xi x} \hat{f}(\xi) d\xi.
\]

Setting \( w = y - x \) we compute

\[
\mathcal{H}e^{i\xi x} = \frac{1}{\pi i} P.V. \int_{-\infty}^{+\infty} \frac{e^{i\xi(x-y)} \cdot \frac{1}{\pi i} \lim_{\delta \to 0^+} \left( \int_{-\infty}^{-\delta} + \int_{\delta}^{+\infty} \right) \frac{e^{i\xi w} - e^{-i\xi w}}{w} dw}{y} dy = e^{i\xi x} \left( \frac{1}{\pi i} \lim_{\delta \to 0^+} \int_{-\infty}^{+\infty} \frac{2i \sin \xi w}{w} dw = e^{i\xi x} \cdot \frac{2 \pi}{\pi} \int_{0}^{\infty} \frac{\sin w}{w} dw \right) = e^{i\xi x} \cdot \text{sign } \xi.
\]

Therefore

\[
\mathcal{H}f(x) \overset{\Delta}{=} \int_{\xi > 0} e^{i\xi x} \hat{f}(\xi) d\xi - \int_{\xi < 0} e^{i\xi x} \hat{f}(\xi) d\xi.
\]
Using a Fourier transform, we can now write
\[ g(t, x) = \int_{0}^{+\infty} A(t, \xi) e^{i\xi x} d\xi + \int_{0}^{+\infty} B(t, \xi) e^{-i\xi x} d\xi, \]
where \( A(t, \xi) = \hat{g}(t, \xi), B(t, \xi) = \hat{g}(t, -\xi), \xi > 0 \). With the above notations, we have
\[ \mathcal{H}g'(t, x) = \int_{0}^{+\infty} A(t, \xi) i\xi e^{i\xi x} d\xi + \int_{0}^{+\infty} B(t, \xi) i\xi e^{-i\xi x} d\xi. \]
Recalling (3.6), the time evolution of \( A(t, \xi) \) and \( B(t, \xi) \) is described by the equation
\[
\begin{aligned}
\frac{d}{dt} A(t, \xi) &= -\frac{i\xi}{2} B(t, \xi), \\
\frac{d}{dt} B(t, \xi) &= -\frac{i\xi}{2} A(t, \xi).
\end{aligned}
\]
Notice that (3.7) implies
\[ \frac{d^2}{dt^2} A(t, \xi) = \frac{\xi^2}{4} A(t, \xi), \quad \frac{d^2}{dt^2} B(t, \xi) = \frac{\xi^2}{4} B(t, \xi). \]
The general solution of (3.7) has the form
\[
\begin{aligned}
A(t, \xi) &= A_1(\xi) e^{\xi t/2} + A_2(\xi) e^{-\xi t/2}, \\
B(t, \xi) &= B_1(\xi) e^{\xi t/2} + B_2(\xi) e^{-\xi t/2}.
\end{aligned}
\]
It is now clear that the coefficient of the Fourier mode \( e^{i\xi \sigma} \) can grow like \( e^{\frac{\xi}{2} t} \). Therefore the solution is very unstable. In order to achieve at least the local existence of solutions, one has to start with initial data whose Fourier transform decreases very fast as \( |\xi| \to \infty \), namely analytic data.

### 3.4 Spiralling solutions

Since a vortex sheet is highly unstable, it very likely that it loses regularity within finite time. One may conjecture that singularities have the form of an infinite double spiral (vortex roll-up). To get some understanding of these singularities, we can reduce the dimension of the problem using a rescaling symmetry. We recall that, if \( u \) solves the equation
\[ u_t + P_E(u \cdot \nabla u) = 0, \]
then
\[ u^{\lambda, \tau}(t, x) = \frac{\lambda}{\tau} u \left( \frac{t}{\tau}, \frac{x}{\lambda} \right), \]
is also a solution, for all \( \lambda, \tau > 0 \). Here \( P_E \) the perpendicular projection on the space \( E \subset L^2 \) of divergence free vector fields.

Given a solution \( u \) of (3.9), and a constant \( k \), we will call \((u^\tau)_{\tau>0}\) the 1-parameter family of solutions defined by setting \( \lambda = \tau^k \), namely
\[ u^\tau(t, x) = \tau^{k-1} u \left( \frac{t}{\tau^k}, \frac{x}{\tau^k} \right). \]
Suppose that $u$ is invariant with respect to the transformation $\tau \mapsto u^\tau$ for all $\tau > 0$. Then

$$u(t, x) = u^t(t, x) = t^{k-1}u(1, \frac{x}{tk}).$$

In other words, knowing the velocity $u$ at time $t = 1$, the entire solution is determined for all times $t > 0$.

We conclude this section by deriving an equation for the self-similar solution of the Birkhoff-Rott equation, in the form of a spiral.

Assume that the vorticity is initially concentrated on the real axis, which is parametrized by $\gamma(s) = s$. Let the vorticity density be

$$\phi(s) = a|s|^\beta,$$

for some constant $a$. Then, re-parametrizing the curve in term of the vorticity, we obtain

$$\sigma(s) = \int_0^s a\sigma^\beta d\sigma = \frac{a}{\beta + 1}s^{\beta+1}.$$

In terms of the rescaled parameter $\sigma$, we have

$$z(\sigma) = \left[\frac{\beta + 1}{a}\right]^{\frac{1}{\beta+1}}.$$

(3.10)

Next, consider the Birkhoff-Rott equation

$$\frac{d}{dt}z(t, \sigma) = \frac{1}{2\pi i} \int \frac{d\sigma'}{z(\sigma) - z(\sigma')}$$

(3.11)

with initial data where the vorticity is concentrated along the real axis, with density as in (3.10), with $\beta = -1/2$. Namely

$$z_0(\sigma) = \frac{1}{4\xi^2} \cdot \sigma^2 \text{sign}(\sigma).$$

Here $\xi > 0$ is an arbitrary constant. Introduce the variable

$$\tau = \frac{\sigma}{\xi^{4/3} \cdot t^{1/3}}.$$

We seek a special, self-similar solution of (3.11), having the form

$$z(t, \sigma) = (\xi t)^{2/3} \rho(\tau),$$

(3.12)

where $\tau \mapsto \rho(\tau)$ is a parametrization of the curve, at time $t = \xi^{-1}$. Notice that, as time increases, the spiral keeps the same shape, while its size is amplified by $t^{2/3}$.

$$\frac{d}{dt}z(t, x) = \frac{d}{dt} \left[\left((\xi t)^{2/3} \rho\left(\frac{\sigma}{\xi^{4/3} t^{1/3}}\right)\right)\right]$$

$$= \frac{2}{3}\xi^{2/3}t^{-1/3}\rho + (\xi t)^{2/3} \cdot \frac{d\rho}{d\sigma} \cdot \left(-\frac{1}{3}\right) \cdot \frac{1}{\xi^{4/3} t^{4/3}}$$

$$= \frac{2}{3}\xi^{2/3}t^{-1/3}\rho - \frac{1}{3}\xi^{2/3}t^{-1/3} \cdot \frac{d\rho}{d\sigma}.$$
Next, the Birkhoff-Rott equation becomes

$$\frac{d}{dt} \bar{z}(t, x) = \frac{1}{2\pi i} \int \frac{1}{(\xi t)^{2/3}} \frac{d\sigma'}{\rho(\xi t)^{2/3} \bar{\sigma} - \rho(\bar{\xi t})^{2/3}} \cdot \frac{d\sigma'}{\rho(\bar{\xi t})^{2/3} \bar{\sigma} - \rho(\xi t)^{2/3}} \cdot \xi^{2/3} t^{-1/3}.$$  

Comparing (3.13) with (3.12) we obtain

$$\frac{2}{3} \dot{\rho} - \frac{\tau}{3} \frac{d}{d\tau} \rho = \frac{1}{2\pi i} \int \frac{d\tau'}{\rho(\tau) - \rho(\tau')}.$$ 

hence

$$\frac{d}{d\tau} \rho(\tau) = \frac{1}{\tau} \left( 2\rho(\tau) + \frac{1}{2\pi i} \int \frac{d\tau'}{\rho(\tau) - \rho(\tau')} \right).$$ 

A solution of this equation is the Kaden spiral.