Dynamic Blocking Problems

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A set $R(t)$ expands as time increases.

To restrain its growth, a barrier $\Gamma$ can be constructed, in real time.

Models: blocking an advancing wild fire, the spatial spreading of a chemical contamination...
Blocking an advancing wildfire
Models of fire propagation


A Differential Inclusion Model for Fire Propagation

\[ R(t) \subset \mathbb{R}^2 = \text{set reached by the fire at time } t \geq 0 \]

determined as the reachable set by a differential inclusion

\[
\dot{x} \in F(x) \quad x(0) \in R_0 \subset \mathbb{R}^2
\]

Fire may spread in different directions with different velocities

\[
R(t) = \left\{ x(t); \ x(\cdot) \text{ absolutely continuous}, \ x(0) \in R_0, \ \dot{x}(\tau) \in F(x(\tau)) \text{ for a.e. } \tau \in [0,t] \right\}
\]
Propagation speed of the fire front

advancing speed of the fire front, in the normal direction:

$$\max_{\nu \in F(x)} \langle \mathbf{n}(x), \nu \rangle$$
The minimum time function

\[ T(x) = \inf \left\{ t \geq 0 ; \ x \in \mathbb{R}(t) \right\} \]

= minimum time taken by the fire to reach the point \( x \)

The minimum time function provides a solution of the Hamilton-Jacobi equation

\[ H(x, \nabla T(x)) = 0, \quad H(x, p) \doteq \max_{v \in F(x)} \langle p, v \rangle - 1 \]

with boundary data \( T(x) = 0 \) for \( x \in R_0 \) (in a viscosity sense)

The level set \( \{ T(x) = t \} \) describes the position of the fire front at time \( t > 0 \)
Confinement Strategies

(A.B., J.Differential Equations, 2007)

Assume: a controller can construct a wall, i.e. a one-dimensional rectifiable curve $\gamma$, which blocks the spreading of the fire.

$$\gamma(t) \subset \mathbb{R}^2 = \text{portion of the wall constructed within time } t$$

$$\sigma = \text{speed at which the wall is constructed}$$

**Definition 1.** A set valued map $t \mapsto \gamma(t) \subset \mathbb{R}^2$ is an **admissible strategy** if:

(H1) For every $t_1 \leq t_2$ one has $\gamma(t_1) \subseteq \gamma(t_2)$

(H2) Each $\gamma(t)$ is a rectifiable set (possibly not connected). Its length satisfies

$$m_1(\gamma(t)) \leq \sigma t$$
Definition 2. The reachable set determined by the blocking strategy $\gamma$ is

$$R^\gamma(t) \doteq \left\{ x(t); \; x(\cdot) \text{ absolutely continuous}, \; x(0) \in R_0 \right\}$$

subject to

$$\dot{x}(\tau) \in F(x(\tau)) \text{ for a.e. } \tau \in [0, t], \quad x(\tau) \notin \gamma(\tau) \text{ for all } \tau \in [0, t]$$

REMARK: Walls must be constructed in real time!

An admissible strategy is described by a set-valued function $t \mapsto \gamma(t) \subset \mathbb{R}^2$

$$\gamma(t) = \text{portion of the wall constructed within time } t$$
A **cost functional** should take into account

- The value of the region destroyed by the fire.
- The cost of building the wall.

\[ \alpha(x) = \text{value of a unit area of land around the point } x \]

\[ \beta(x) = \text{cost of building a unit length of wall near the point } x \]

**Cost Functional**

\[
J(\gamma) \triangleq \lim_{t \to \infty} \left\{ \int_{R^\gamma(t)} \alpha \, dm_2 + \int_{\gamma(t)} \beta \, dm_1 \right\}
\]
1. Blocking Problem.

Given an initial set $R_0$, a multifunction $F$ and a wall construction speed $\sigma$, does there exist an admissible strategy $t \mapsto \gamma(t)$ such that the reachable sets $R^\gamma(t)$ remain uniformly bounded for all $t > 0$?

2. Optimization Problem.

Find an admissible strategy $\gamma(\cdot)$ which minimizes the cost functional $J(\gamma)$. 
- **Existence** of an optimal strategy $\gamma(\cdot)$
- **Necessary conditions** for optimality
- **Sufficient conditions** for optimality
- **Regularity** of the curves $\gamma(t)$ constructed by an optimal strategy
- **Numerical computation** of an optimal strategy
Blocking Problems and Optimization Problems can be reformulated in terms of one single rectifiable set $\Gamma$

$$\Gamma \quad \rightarrow \quad \gamma(t) \doteq \Gamma \cap \overline{R(\Gamma(t))} \quad \text{(walls touched by the fire within time } t\text{)}$$
Blocking the Fire

- Fire propagates in all directions with unit speed: \( F(x) = B_1 \)
- Wall is constructed at speed \( \sigma \)

**Theorem (A.B., J.Differential Equations, 2007)**

On the entire plane, the fire can be blocked if \( \sigma > 2 \), it cannot be blocked if \( \sigma < 1 \).

**Blocking Strategy:** If \( \sigma > 2 \), construct two arcs of logarithmic spirals along the edge of the fire

\[
\gamma(t) = \begin{cases} (r, \theta) ; & r = e^{\lambda|\theta|}, \ 1 \leq r \leq 1 + t \end{cases}, \quad \lambda \doteq \frac{1}{\sqrt{\frac{\sigma^2}{4} - 1}}
\]
No strategy can block the fire if $\sigma \leq 1$

$\bar{x} = \text{position of "last brick of the wall"}$

$$T^\Gamma(\bar{x}) = \sup_{x \in \Gamma} T^\Gamma(x) \geq \frac{|\Gamma|}{\sigma} \quad \text{otherwise the fire escapes}$$

$$|\gamma_2| + |\gamma_3| \leq 2|\Gamma|$$

$$T^\Gamma(\bar{x}) = |\gamma_0| < |\gamma_1| \leq \min \{|\gamma_2|, |\gamma_3|\} \leq |\Gamma|$$
The isotropic case on the half plane

- Fire propagates in all directions with unit speed. \( F(x) = B_1 \)
- Wall is constructed at speed \( \sigma \)


Restricted to a half plane, the fire can be blocked if and only if \( \sigma > 1 \)
When can the fire be blocked?

**Conjecture:** Assume the fire propagates with speed 1 in all directions. On the entire plane the fire can be blocked if and only if $\sigma > 2$.

Single spiral strategy: curve closes on itself if and only if $\sigma > \sigma^\dagger = 2.614430844 \ldots$ (M. Burago, 2006)
Non-isotropic fire propagation

Assume:

\[ F = \left\{ (r \cos \theta, r \sin \theta) ; \quad 0 \leq r \leq \rho(\theta) \right\} \]

\[ \rho(-\theta) = \rho(\theta), \quad 0 \leq \rho(\theta') \leq \rho(\theta) \quad \text{for all } 0 \leq \theta \leq \theta' \leq \pi. \]

**Theorem.** (A.B., M. Burago, A. Friend, J. Jou, Analysis and Applications, 2008)

If the wall construction speed satisfies

\[ \sigma > \text{[vertical width of } F] = 2 \max_{\theta \in [0, \pi]} \rho(\theta) \sin \theta \]

then, for every bounded initial set \( R_0 \), a blocking strategy exists.
Existence of Optimal Strategies

Fire propagation: \( \dot{x} \in F(x) \) \( x(0) \in R_0 \)

Wall constraint: \( \int_{\gamma(t)} \psi \, dm_1 \leq t \) \( (1/\psi(x) = \text{construction speed at } x) \)

Minimize: \( J(\gamma) = \left\{ \int_{R^{\gamma(t)}} \alpha \, dm_2 + \int_{\gamma(t)} \beta \, dm_1 \right\} \)

Assumptions:

(A1) The initial set \( R_0 \) is open and bounded. Its boundary satisfies \( m_2 (\partial R_0) = 0 \).

(A2) The multifunction \( F \) is Lipschitz continuous w.r.t. the Hausdorff distance. For each \( x \in \mathbb{R}^2 \) the set \( F(x) \) is compact, convex, and contains a ball of radius \( \rho_0 > 0 \) centered at the origin.

(A3) For every \( x \in \mathbb{R}^2 \) one has \( \alpha(x) \geq 0, \beta(x) \geq 0, \) and \( \psi(x) \geq \psi_0 > 0 \). \( \alpha \) is locally integrable, while \( \beta \) and \( \psi \) are both lower semicontinuous.

Assume (A1)-(A3), and $\inf_{\gamma \in \mathcal{S}} J(\gamma) < \infty$.
Then the minimization problem admits an optimal solution $\gamma^*$.

Direct method: Consider a minimizing sequence of strategies $\gamma_n(\cdot)$
Define the optimal strategy $\gamma^*$ as a suitable limit.
**Key step in the proof:** For each rational time $\tau$, order the connected components of $\gamma_n(\tau)$ according to decreasing length:

$$\ell_{n,1} \geq \ell_{n,2} \geq \ell_{n,3} \geq \cdots \quad \gamma_n(t) = \gamma_{n,1} \cup \gamma_{n,2} \cup \gamma_{n,3} \cup \cdots$$

Taking a subsequence, as $n \to \infty$ we can assume

$$\ell_{n,i}(\tau) \to \ell_i(\tau) \quad \gamma_{n,i}(\tau) \to \gamma_i(\tau) \quad \tau \in \mathbb{Q}$$

We then define $\gamma(\tau) \doteq \bigcup_{i \geq 1, \ell_i(\tau) > 0} \gamma_i(\tau)$
New approach: the minimum time function with obstacle


\[ \dot{x} \in F(x), \quad x(0) \in R_0 \subset \mathbb{R}^2 \]

Minimum time function with obstacle: \[ T^\Gamma(x) = \inf \{ t \geq 0 ; \ x \in R^\Gamma(t) \} \]

\( R^\Gamma(t) = \) set of points reached by \( F \)-trajectories which start in \( R_0 \)
and do not cross \( \Gamma \)

**Goal:** characterize \( T^\Gamma \) in terms of a H-J equation
Definition. A function $u : \mathbb{R}^2 \mapsto [0, \infty]$ is in the set $\mathcal{S}_\Gamma$ if

- $u \in SBV$ \hspace{1cm} \left( Du = \nabla u + D^{jump} u + D^{Cantor} u, \text{ with } D^{Cantor} u \equiv 0 \right)$
- $m_1(J_u \setminus \Gamma) = 0$ \hspace{1cm} \left( J_u \equiv \text{set of jump points of } u \right)$
- $u = 0$ \hspace{1cm} \text{on } R_0$
- $H(x, \nabla u(x)) \leq 0$ \hspace{1cm} \text{for a.e. } x$

$\nabla u \equiv$ absolutely continuous part of the distributional derivative $Du$. 
Theorem. (C. De Lellis & R. Robyr)

The minimum time function $T^\Gamma$ is the unique maximal element of $S^\Gamma$.

$\implies$ alternative proof of existence of optimal blocking strategies

- Take a minimizing sequence of admissible barriers $\Gamma_k$
- Consider the corresponding minimum time functions $T_k \uparrow T^\Gamma_k$
- Using the Ambrosio-De Giorgi compactness theorem for SBV functions, one obtains a convergent subsequence $T_k \rightarrow U$, with $U \in SBV$
- The jump set $J_U$ yields the optimal barrier
Necessary conditions for optimality

**Problem:** find an admissible barrier $\Gamma$ which minimizes

$$J(\Gamma) = \alpha \cdot [\text{total burned area}] + \beta \cdot [\text{length of the curve}]$$

**GOAL:** derive a set of ODE’s describing the walls built by an optimal strategy

Classification of arcs in an optimal strategy

Minimum time function \( T^\Gamma(x) \doteq \inf \left\{ t \geq 0 ; \ x \in \overline{R^\Gamma(t)} \right\} \)

Set of times where the constraint is saturated

\[
S \doteq \left\{ t \geq 0 ; \ \text{meas}\left( \Gamma \cap \overline{R^\Gamma(t)} \right) = \sigma t \right\}
\]

**Boundary arcs:** \( \Gamma_S \doteq \{ x \in \Gamma ; \ T^\Gamma(x) \in S \} \)
constructed along the advancing fire front

**Free arcs:** \( \Gamma_F \doteq \{ x \in \Gamma ; \ T^\Gamma(x) \notin S \} \)
constructed away from the fire front
Optimality conditions, minimizing the value of burned area

1. A free arc $\Gamma$. The curvature must be proportional to the local value of the land

$$r(s) = \text{radius of curvature} \quad \alpha = \text{land value}$$

$$r(s) \cdot \alpha(\Gamma(s)) = \text{const.}$$

2. A single boundary arc $\Gamma$. The wall is constructed at maximum speed $\sigma$, always remaining at the edge of the burned set

$$\sigma \sin \beta = \max_{y \in F(x)} n \cdot y$$
3. Two or more boundary arcs: \( \Gamma_1, \ldots, \Gamma_\nu \), constructed simultaneously for \( t \in [a, b] \)

Sum of construction speeds \( \leq \sigma \)

At which speed should each wall be constructed?

- recast the problem in the standard setting of optimal control
- apply the Pontryagin Maximum Principle
Two boundary arcs originating at the same point are not optimal

Non-parallel junctions between a free arc and a boundary arc are not optimal
circle + two spirals

(is better than  two spirals only)
Further classification

- blocking arcs $\Gamma^b \doteq \Gamma \cap \partial R^{\Gamma}_\infty$
- delaying arcs $\Gamma^d$

Necessary conditions for the optimality of delaying arcs
Sufficient conditions for optimality?

**Standard Isotropic Problem:**
- Fire starts on the unit disc, propagating with unit speed in all directions.
- Barrier can be constructed at speed $\sigma > 2$.
- Minimize the total burned area.

![Diagram of barrier and fire front](image)

**Theorem (A.B. - T.Wang, 2010)**

The barrier consisting of

\[ \text{circle} + \text{two logarithmic spirals} \]

is optimal among all simple closed curves enclosing $R_0$.
original curve

star shaped

symmetric, nondecreasing rearrangement
Polar coordinate representation: \( \theta \mapsto r(\theta) \) non-decreasing, for \( \theta \in [0, \pi] \)

Admissibility constraint: \[
m_1 \left( \{ x \in \Gamma ; \ |x| \leq 1 + t \} \right) \leq \sigma t
\]

\(< \sigma t \quad \implies \quad \text{circumferences} \)

\(= \sigma t \quad \implies \quad \text{logarithmic spirals} \)

\(\text{joining tangentially}\)

Assume: only blocking arcs, no delaying arcs.

\[
\begin{align*}
\text{minimize total burned area:} & \quad m_2\left(R^\Gamma_{\infty}\right) \\
\text{subject to} & \quad m_1\left(\Gamma \cup \overline{R^\Gamma(t)}\right) \leq \sigma t \quad \text{for all} \quad t \geq 0
\end{align*}
\]
Approximate the barrier with a polygonal:

- fix an angle $\theta = 2\pi / n$
- assign radii $r_k = r(k\theta)$, $k = 1, \ldots, n$
- starting with an admissible polygonal, search for a local minimizer subject to admissibility constraints
- double number of nodes (replace $n$ by $2n$), repeat local minimization \ldots
1. The isotropic case

\[ F(x) = R_0 = B_1 \text{ (unit disc)}, \quad \sigma = 4. \text{ Minimize: total burned area.} \]
2. A non-isotropic case

\[ F = \left\{ (\lambda x, \lambda y); \ (x - 3)^2 + y^2 \leq 1, \ \lambda \in [0, 1] \right\} \]

Choose: \( \sigma = 4.1, \ R_0 = \text{unit disc} \).

Some open problems

1 \textbf{(Isotropic blocking problem)} On the whole plane, assume:

- fire propagates with unit speed in all directions
- barrier is constructed at speed $\sigma$

\textbf{Conjecture 1:}
A blocking strategy exists if and only if the wall construction speed is $\sigma > 2$.

2 \textbf{(Sufficient optimality conditions)} Not one single example is known where a blocking strategy can be proved to be optimal.

\textbf{Conjecture 2:}
The “circle + two spirals” strategy is optimal for the isotropic problem.

\textbf{Basic difficulty: delaying arcs}
3 (Existence of optimal strategies). Determine whether an optimal strategy exists, in the general case where the velocity sets satisfy

\[ 0 \in F(x), \quad \text{but without assuming} \quad B(0, \rho) \subset F(x) \]

so that fire propagation speed is not uniformly positive in all directions.

4 (Regularity). Assume: the initial set \( R_0 \) has a smooth boundary and the cost functions \( \alpha, \beta \) are smooth.

- What is the regularity of an optimal strategy?
- Does it produce a finite number of piecewise \( C^1 \) arcs?
- Is the optimal barrier connected?
- Is it ever useful to construct purely delaying arcs?