Differential Inclusions

Old and New

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Differential Inclusions

\[ \dot{x} \in F(x) \]

1 - Classical solutions for a continuous, non-convex valued right hand side 
(\textit{with Zipeng Wang})

2 - Estimate on trajectories constrained to a compact, convex domain 
(\textit{with Giancarlo Facchi})

3 - Solutions for an upper semicontinuous, non-convex valued right hand side 
(\textit{with Russel DeForest})
Continuously Differentiable Solutions

\[ \dot{x} \in F(x) \quad x(0) = \bar{x} \in \mathbb{R}^n \]

**Theorem (Filippov, 1963)** If \( F \) is Lipschitz continuous w.r.t. the Hausdorff metric, then the Cauchy problem admits a \( C^1 \) solution.
Assume: \(d_H(F(x), F(y)) \leq L|x - y|\), \(F(x) \subseteq B(0, M)\)

Construct Euler polygonal approximations, with step size \(1/\nu\). Set

\[t_i = i/\nu, \quad \dot{x}_\nu(t) = v_i, \quad t \in [t_i, t_{i+1}]\]

choose \(v_i \in F(x_i)\)

\[|v_i - v_{i-1}| \leq L|x_i - x_{i-1}| \leq LM(t_i - t_{i-1})\]

As \(\nu \to \infty\), taking a subsequence, the piecewise constant derivatives \(\dot{x}_\nu\) converge to a Lipschitz continuous function \(t \mapsto \dot{x}(t)\), with Lipschitz constant \(LM\).
Classical solutions may not exist if $F$ is continuous (not Lipschitz), with non-convex values

\[ \dot{x} \in F(t) \quad \quad x(0) = 0 \in \mathbb{R}^2, \quad \quad t \in [0, 1] \]

\[ F(t) \doteq \left\{ (\cos \theta, \sin \theta) ; \; \theta \in \left[ \frac{1}{t}, \frac{1}{t} + 2\pi - t \right] \right\} \]

\[ F(0) \doteq \left\{ (v_1, v_2) : \; v_1^2 + v_2^2 = 1 \right\} \]

The multifunction $t \mapsto F(t)$ is continuous but has no continuous selection
Existence of Classical Solutions

\[ \dot{x} \in F(x) \quad x(0) = 0 \in \mathbb{R}^n \]

**Theorem (Filippov, 1963)** Let \( F \) be continuous with compact, convex values, then the Cauchy problem has a \( C^1 \) solution.

**Proof:** Take a continuous selection \( f(x) \in F(x) \). Then solve \( \dot{x} = f(x) \).

**Theorem (A.B. - Z. Wang, 2008)** Let \( F \) be continuous with compact, totally disconnected values. Then the Cauchy problem has a \( C^1 \) solution.
Lemma. Let $t \mapsto G(t)$ be a Hausdorff continuous multifunction with compact, totally disconnected values. Then $G$ admits a continuous selection.

Proof of Theorem. Choose $\varepsilon_\nu \to 0$. Take $C^1$ approximate solutions

$$
\dot{x}_\nu(t) = F(x(t - \varepsilon_\nu)) \quad \quad \quad x(t) = x_0 \quad \text{for} \quad t \leq 0.
$$

Claim: the derivatives $\dot{x}_\nu$ admit a uniformly convergent subsequence.
Assume that a sequence of $C^1$ approximate solutions $x_{\nu}$ has highly oscillating derivatives $\dot{x}_{\nu}$.

Then, taking subsequences

$$x_{\nu}(\tau) \to x(\tau), \quad t_{\nu}^- \to \tau, \quad t_{\nu}^+ \to \tau$$

$$\dot{x}_{\nu}(t_{\nu}^-) \to v^- \in F(x(\tau)) \quad \dot{x}_{\nu}(t_{\nu}^+) \to v^+ \in F(x(\tau))$$

$$\implies F(x(\tau)) \text{ contains an arc joining the two values } v^-, v^+$$

CONTRADICTION!
Optimal Control problems - Regularity of the Value Function

minimize \[ \int_{0}^{T} L(t, x(t), u(t)) \, dt \]

subject to \[ x(0) = x_0, \quad \dot{x} = f(t, x, u), \quad x(t) \in \Omega \quad t \in [0, T] \]

\[ V(x_0) = \text{minimum cost, starting at } x_0 \]
Estimates for solutions constrained to a closed domain

\[ \dot{x} \in F(x) \quad x(t) \in \Omega \]

\( \Omega \subset \mathbb{R}^n \) closed convex domain, \( T_\Omega(x) \) = tangent cone at \( x \)

\( F \) Lipschitz continuous multifunction, containing an inward-pointing vector

\[ F(x) \cap \text{int } T_\Omega(x) \neq \emptyset \quad \text{for all } x \in \Omega \]

**Problem:** Let \( x_0 \in \Omega \) and \( x^* : [0, T] \mapsto \mathbb{R}^n \) be an \( F \)-trajectory, possibly violating the constraint \( x \in \Omega \).

Find a second trajectory \( x(\cdot) \) such that

\[ x(0) = x_0 \quad \dot{x}(t) \in F(x(t)) \quad x(t) \in \Omega \quad t \in [0, T] \]

with \( \|x - x^*\|_{W^{1,1}} \) as small as possible.
Basic case: $\Omega$ has smooth boundary

Then one can achieve $\|x - x^*\|_{W^{1,1}} \leq C\varepsilon$

(H. Frankowska, F. Rampazzo, J.D.E. 2000)

Harder case: $\Omega$ is a cone
Linear estimate fails!

Example (A.B., P.Bettiol, R.Vinter, 2009)

\[ \| x - x^* \|_{W^{1,1}} \geq C_0 \varepsilon |\ln \varepsilon| \]

for all \( x : [0, T] \mapsto \Omega \)
**Theorem (A.B., G.Facchi, 2010).** Assume

- $\Omega \subset \mathbb{R}^n$ is a compact, convex domain
- $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a Lipschitz continuous, compact valued multifunction, with
  \[ \text{co}F(x) \cap \text{int} T_\Omega(x) \neq \emptyset \quad \text{for all} \quad x \in \Omega \]

Given any $F$-trajectory $x^* : [0, T] \mapsto \mathbb{R}^n$ and any initial point $x_0 \in \Omega$, set

\[ \varepsilon \doteq |x_0 - x^*(0)| + \max_{t \in [0, T]} d(x^*(t), \Omega) \]

Then there exists a second $F$-trajectory $x : [0, T] \mapsto \Omega$ with $x(0) = x_0$ and

\[ \|x - x^*\|_{C^0([0,T])} \leq C\varepsilon, \quad \|\dot{x} - \dot{x}^*\|_{L^1([0,T])} \leq C\varepsilon(1 + |\ln \varepsilon|) \]
**Basic case:** $F(x)$ is a fixed compact, convex set, $\Omega$ is a closed convex cone.

Choose $a \in F \cap \text{int} \, \Omega$. Choose a constant $C$ large enough, and define

$$x(t) \equiv \begin{cases} 
x_0 + ta & t \in [0, C\varepsilon] 
x_0 + C\varepsilon a + \left(1 - \frac{C\varepsilon}{t}\right)(x^*(t) - x^*(0)) & t \in [C\varepsilon, T]
\end{cases}$$
Extension to the general case

1. The case where the velocity sets $F(x)$ depend Lipschitz continuously on $x$ is handled using a standard Gronwall type estimate.

2. The convexity assumption on the velocity sets $F(x)$ is removed using Lyapunov’s theorem on the range of a vector measure.

3. By a covering argument, one can extend the result to the case where $\Omega$ is an arbitrary compact convex domain.
Graph Completions

\[ \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) \dot{u}_i(t) \quad x(0) = x_0 \]

\[ u : [0, T] \mapsto \mathbb{R}^m \text{ possibly discontinuous} \]
Definition 1. A graph-completion of a BV function $u : [0, T] \mapsto \mathbb{R}^m$ is a Lipschitz continuous path $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_m) : [0, S] \mapsto [0, T] \times \mathbb{R}^m$ such that

- (i) $\gamma(0) = (0, u(0))$, $\gamma(S) = (T, u(T))$,
- (ii) $\gamma_0(s_1) \leq \gamma_0(s_2)$ for all $0 \leq s_1 < s_2 \leq S$,
- (iii) for each $t \in [0, T]$ there exists some $s$ such that $\gamma(s) = (t, u(t))$. 
\[
\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) \dot{u}_i(t) \quad \quad x(0) = x_0
\]

Given a **graph completion** we can reparameterize the graph \( s \mapsto (t(s), u(s)) = \gamma(s) \) and solve the O.D.E. with measurable r.h.s.

\[
\frac{d}{ds} x(s) = f(x(s)) \frac{dt}{ds} + \sum_{i=1}^{m} g_i(x(s)) \frac{du_i}{ds} \quad \quad x(0) = x_0
\]


Extension to multi-dimensional domains
(A.B. - R. DeForest, 2010)

\[ u : \Omega \to \mathbb{R}^m \quad \Omega \subset \mathbb{R}^n \text{ bounded, open} \]

Goal: embed the graph of \( f \) in the graph of a multifunction \( F \) continuously parameterized by the variable \( s \in \Omega \)

One-dimensional case: \( \Omega = [0, T] \)
Given two continuous maps \( f, g : \overline{\Omega} \mapsto \mathbb{R}^m \), define an alternative distance:

\[
d^\diamondsuit(f, g) \doteq \inf_{\phi \in \text{Hom}(\Omega)} \sup_{x \in \Omega} \left\{ |x - \phi(x)| + |f(x) - g(\phi(x))| \right\}.
\]

\( \text{Hom}(\Omega) \doteq \) family of all homeomorphisms \( \phi : \overline{\Omega} \mapsto \overline{\Omega} \) which keep the boundary \( \partial \Omega \) fixed

The space \( C(\overline{\Omega}; \mathbb{R}^m) \) is not complete w.r.t. the metric \( d^\diamondsuit \).

To a Cauchy sequence of functions \( (f_k)_{k \geq 1} \), one can associate a unique multifunction \( F : \overline{\Omega} \mapsto \mathbb{R}^m \).

The graph of \( F \) can be parameterized by a continuous map \( \Phi : \overline{\Omega} \mapsto \overline{\Omega} \times \mathbb{R}^m \).
Definition. If \( f : \overline{\Omega} \mapsto \mathbb{R}^m \) is a function such that

\[
\text{graph}(f) \subseteq \text{graph}(F) \quad \text{and} \quad F(x) = \{f(x)\} \quad \text{for a.e. } x \in \Omega
\]

we regard \( F \) as a graph completion of \( f \).

- It is convenient to take Cauchy sequences (always w.r.t. the metric \( d^\Diamond \)) consisting of functions whose Sobolev norm \( \|f_k\|_{W^{1,p}(\Omega)} \) is uniformly bounded. This yields the notion of \( W^{1,p} \)-graph completions.

- Working in \( W^{1,1} \) we obtain graph completions of bounded variation. In the one-dimensional case, these are equivalent to the graph completions introduced by Bressan-Rampazzo (1988).
Upper Semicontinuous Differential Inclusions

\( x \mapsto F(x) \) multifunction with compact values. The Cauchy problem

\[
\dot{x} \in F(x), \quad x(0) = 0 \in \mathbb{R}^n
\]

admits a Carathéodory solution in the following cases:

(i) \( F \) is upper semicontinuous, with convex values

(ii) \( F \) is lower semicontinuous, possibly with non-convex values.
\[
\dot{x} \in F(x), \quad x(0) = 0 \in \mathbb{R}^n
\]

If $F$ upper semicontinuous with non-convex values, solutions may not exist

\[
\dot{x} \in F(x) = \begin{cases} 
\{-1\} & \text{if } x > 0, \\
\{-1, 1\} & \text{if } x = 0, \\
\{1\} & \text{if } x < 0.
\end{cases}
\]

**Definition.** $F : \Omega \mapsto \mathbb{R}^n$ is Cellina-approximable if for every $\varepsilon > 0$ there exists a continuous function $f_\varepsilon : \Omega \mapsto \mathbb{R}^n$ such that

\[
\text{graph}(f_\varepsilon) \subset B(\text{graph}(F), \varepsilon)
\]
If the multifunction $G$ is upper semicontinuous, with convex values, and $\phi : \mathbb{R}^n \mapsto \mathbb{R}^n$ is continuous, then
\[ x \mapsto F(x) \doteq \phi(G(x)) \] is Cellina approximable

**Note:** this should remove *topological obstructions* to the existence of solutions

**Conjecture.** Let $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a bounded upper semicontinuous multifunction with compact values, which is Cellina approximable. Then the Cauchy problem
\[ \dot{x} \in F(x), \quad x(0) = 0 \in \mathbb{R}^n \]
has at least one solution.

\[ \phi(s) \doteq s \cos \frac{1}{s} \quad \text{if} \quad s \neq 0, \quad \phi(0) = 0 \]

\[ F(x_1, x_2) \doteq \begin{cases} 
(0, -1) & \text{if} \quad x_2 > \phi(x_1), \\
(0, 1) & \text{if} \quad x_2 < \phi(x_1), \\
(y_1, y_2); \quad y_1 \geq 0, \quad y_1^2 + y_2^2 = 1 & \text{if} \quad x_2 = \phi(x_1).
\end{cases} \]
What goes wrong: $F$ is discontinuous on a curve of infinite length.
Need: additional regularity.

**Definition.** $F : \Omega \mapsto \mathbb{R}^n$ is **Cellina approximable** if for every $\varepsilon > 0$ there exists a smooth function $f_\varepsilon : \Omega \mapsto \mathbb{R}^n$ such that

$$
\text{graph}(f_\varepsilon) \subset B(\text{graph}(F), \varepsilon)
$$

and $\|f_\varepsilon\|_{W^{1,p}}$ remains bounded uniformly w.r.t. $\varepsilon$.
\[ \dot{x} \in F(x) \quad x(0) = 0 \]

**Theorem (A.B. - R.DeForest, 2010).** Let \( F : \mathbb{R}^2 \mapsto \mathbb{R}^2 \) be a bounded, upper semicontinuous multifunction with compact, possibly non-convex values. Assume that \( F \) is Cellina \( W^{1,1} \)-approximable. Then the Cauchy problem admits a Caratheodory solution, defined for all times \( t \in \mathbb{R} \).

Proof. (i) Can assume \( F(x) \subseteq \{ v \in \mathbb{R}^2 ; \ |v| = 1 \} \).

(ii) Take a sequence of continuous approximate selections \( f_k \), solve
\[ \dot{x}_k(t) = f_k(x_k(t)) \quad x_k(0) = 0 \]

(iii) As \( k \to \infty \), take a convergent subsequence \( x_k(t) \mapsto x(t) \). The arc-length reparameterization \( s \mapsto x(s) \) of this limit trajectory is the desired solution.
\[ \dot{x}_k(t) = f_k(x_k(t)) \quad \text{and} \quad x_k(0) = 0 \]

The assumption $F$ is $W^{1,1}$ - Cellina selectionable implies the uniform BV bound

\[ \int_Q |Df_k| dx \leq C \]

On a fixed interval $[0, T]$, taking a subsequence this guarantees that the trajectories $x_k(\cdot)$ converge to a well defined curve $x(\cdot)$, not reduced to a point.
Lemma. Let \( f : Q \mapsto \mathbb{R}^2 \) be a smooth vector field defined on the unit square, with \( |f(x)| = 1 \) for every \( x \).
Then every trajectory of the ODE \( \dot{x} = f(x) \) starting inside \( Q \) reaches the boundary of \( Q \) within time
\[
T = 4 + \frac{1}{2} \| Df \|_{L^1(Q)}
\]
Indeed, the total variation of \( f \) on \( Q \) can be bounded below in terms of the length of a trajectory remaining inside \( Q \).
Open problem. Let $F : \mathbb{IR}^n \mapsto \mathbb{IR}^n$ be a bounded upper semicontinuous multifunction with compact values, which is $W^{1,p}$ - Cellina approximable.

For which values of $p$ does this imply that the Cauchy problem
\[
\dot{x} \in F(x), \quad x(0) = 0 \in \mathbb{IR}^n
\]
always has a solution?

When $n = 2$, the choice $p = 1$ works

When $n = 3$, the choice $p = 1$ does not work

In general, can one take $p = n - 1$?

(Motivation: the space $W^{1,p}(\mathbb{IR}^n)$ should not contain functions which have jumps on a set of infinite 1-dimensional measure - i.e. on a curve with infinite length)