Research Themes on Traffic Flow on Networks

(Alberto Bressan, Aug. 20, 2013)

Models of traffic flow can be of two main types [1, 8, 11].

1 - Microscopic particle models, describing the position and velocity of each single car. If there are \( N \) cars on a road, one thus needs to write a system of \( N \) differential equations, one for each car. These ODEs specify how each driver adjusts his velocity depending on the distance and the velocity of the vehicle ahead.

2 - Macroscopic models, describing the evolution of the vehicle density (i.e. the number of cars per unit length of the road). Since the total number of cars is conserved, these models consist of one or more PDEs, usually in the form of a conservation law.

Particle models are easier to simulate numerically. On the other hand, macroscopic models are mathematically more interesting and yield a better qualitative understanding of traffic patterns.

Having written down a set of mathematical equations, the next steps of a mathematical analysis

(i) The first and most fundamental concern is making sure that the model “well posed”. This means that, given any initial configuration, the equations determine a unique solution for all future times.

(ii) Study the traffic behavior. How do traffic waves propagate along a highway? When can queues arise? Can shocks be produced, i.e. can the traffic density change suddenly at specific locations? Needless to say, the model should yield realistic predictions, and solutions should never exhibit absurd features, such as vehicles moving backwards or car density with negative values. Next, one can study how this evolution changes depending on parameters (i.e. speed limits, drivers’ behavior, cars’ performance, etc...).

(iii) At a further stage, one can analyze an optimization problems, i.e. how to optimally adjust certain parameters in order to minimize a cost criterion. What if the scheduling of departures are scheduled by a central planner, or each driver chooses his own departure time and route to destination?

In connection with the macroscopic PDE model, these ideas will be expanded in the sections below.

1 Conservation law models on a network of roads

The simplest and most popular macroscopic model for traffic flow, due to Lighthill-Witham-Richards [13, 14], takes the form of a single conservation law
\[ \partial_t \rho + \partial_x [\rho v(\rho)] = 0. \] (1)

Here \( t \) is time, \( x \) is the space variable denoting points along a highway, while \( \rho(t, x) \) is the density of cars (= number of cars per mile of highway) at time \( t \) at location \( x \). Moreover, \( v(\rho) \) describes the velocity of cars, which we assume is a decreasing function of the density \( \rho \). Notice that the product \( \rho v(\rho) \) is the flux of cars (= number of cars that cross a point \( x \) on the highway per unit time).

The conservation law (1) describes the evolution of traffic on a network of roads, one also needs to model behavior at intersections. In addition to a conservation law of the form (1) describing the traffic density on each road, one needs a set of equations relating the density on different roads, near the intersection. These equations need to take into account:

- The conservation of the total number of cars (in all, there are as many cars arriving to the intersection as cars leaving the intersection).
- Drivers’ preferences (i.e., assign the percentage of drivers arriving from a given road who will turn left or right at the intersection).
- Priority relations among incoming roads (assuming there is a crossroad light, among all roads leading to the intersection one needs to specify which one gets green light for a longer time).

An important feature of network models is the possibility of queues forming at an intersection and propagating backwards along one or more of the incoming roads.

At present, the mathematical theory for the conservation equation (1) is well established, including the existence, uniqueness, and qualitative properties of its solutions [2, 9, 11, 15]. However, many fundamental issues remain still open for conservation laws on networks. In particular, in a neighborhood of an intersection, the uniqueness and continuous dependence of solutions remains poorly understood.

## 2 Global optima and Nash equilibria

One can study vehicle flow also from a different point of view, regarding traffic patterns as the outcome of the decision problem. We assume that each individual driver has a cost \( \varphi(\tau^d) \) for early departure and an additional cost \( \psi(\tau^a) \) for late arrival. On a general network of
roads, the arrival time $\tau^a$ is determined by (i) the departure time $\tau^d$, (ii) the route taken to reach destination, and (iii) the overall traffic pattern, which of course depends globally on the decisions of all other drivers. The objective of minimizing the total cost $\varphi(\tau^d) + \varphi(\tau^a)$ leads to two distinct mathematical problems.

(P1) - Global Optimization Problem. Find departure times and routes to destinations in order to minimize the sum of all costs to all drivers.

(P2) - Nash Equilibrium Problem. Find departure times and routes to destinations in such a way that no driver can lower his own cost $\varphi(\tau^d) + \varphi(\tau^a)$ by changing his departure time or switching to a different route.

Note that (P1) is relevant in the case of a central planner who can decide the departure time and the route of every car. On the other hand, (P2) models the more realistic situation where each driver is free to choose his own departure time and route, in order to minimize his own personal cost. The existence, uniqueness, and characterization of solutions to the above problems has been recently studied both in the case of a single road, and in [5] for a network of roads, under very simplified assumptions on the dynamics at intersections.

The extension of these results to general models one road networks is currently an active research topic. In particular, it remains yet to be established whether a global optimum and a Nash equilibrium solution exists in cases where queues can propagate backwards along roads leading to a congested intersection.

Another challenging open problem is the dynamic stability of Nash equilibria. Assume that, day after day, each driver can change his own departure time seeking to lower the sum of his departure and arrival cost. As a result, the distribution of departures as well as the overall traffic pattern will change each day. This leads to a dynamical system on the space of all departure distributions, having Nash equilibria as steady states. It is natural to expect that, after several days, the departure distribution should converge to a Nash equilibrium. To study this problem, for a single group of drivers traveling on a single road, two specific models were introduced in [6]. Surprisingly, numerical simulations suggest that the unique Nash equilibrium is unstable, while the orbits approach a chaotic attractor. No theoretical analysis has yet proved or disproved this conjecture.
References


