Math 503 - Functional Analysis

Homework 9, due Monday April 1, 2013

1. On the Hilbert space $H = L^2([0, \infty])$, consider the operator $(\Lambda f)(x) = f(e^x)$.
   (i) Compute the norm of linear operator $\Lambda$.
   (ii) Find the kernel and the range of $\Lambda$.
   (iii) Compute the adjoint operator $\Lambda^*$.

2. Let $f \in L^2(\mathbb{R})$. Prove that there exists a unique even function $g_0$ such that
   \[ ||f - g_0||_{L^2} = \min_{g \in L^2(\mathbb{R}), \ g \ even} ||f - g||_{L^2}. \]
   Explicitly determine the function $g_0$.

3. (i) Find two bounded linear operators $\Lambda_1, \Lambda_2$ from $L^2([0, \infty])$ into itself such that $\Lambda_1 \circ \Lambda_2 = I$ (the identity operator), but $\Lambda_2 \circ \Lambda_1 \neq I$.
    (ii) Let $H$ be a real Hilbert space, and let $\Lambda, K : H \mapsto H$ be bounded linear operators, with $K$ compact. Show that in this case
    \[ \Lambda(I - K) = I \quad \text{if and only if} \quad (I - K)\Lambda = I. \]

4. On the Hilbert space $L^2(\mathbb{R})$, consider the linear operator defined by
   \[ (\Lambda f)(x) = 2f(3x + 1) \quad x \geq 0. \]
   (i) Is $\Lambda$ a bounded operator? Is it compact?
   (ii) Explicitly determine the adjoint operator $\Lambda^*$.
   (iii) Describe $\text{Ker}(\Lambda)$ and $\text{Ker}(\Lambda^*)$.

5. Consider a sequence of functions $f_n \in L^2([0, T])$. Assume that there exists a constant $C$ such that $||f_n||_{L^2} \leq C$ for all $n \geq 1$, and moreover
   \[ \lim_{n \to \infty} \int_0^b f_n(x) \, dx = \int_0^b f(x) \, dx \quad \text{for every} \ b \in [0, T]. \]
   Prove that, as $n \to \infty$, one has the weak convergence $f_n \rightharpoonup f$. 

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6 (extra credit). On the space $L^2([0,1])$, consider the two sequences of functions

$$f_n(x) = \sqrt{n} \cos nx, \quad f_n(x) = \begin{cases} n^{2/3} & \text{if } x \in [0, n^{-1}], \\ 0 & \text{if } x > n^{-1}. \end{cases}$$

(i) In both cases, prove that $\lim_{n \to \infty} \int_0^b f_n(x) \, dx = 0$ for every $b \in [0,1]$.
(ii) By taking linear combinations, show that $\lim_{n \to \infty} \int_0^1 f_n g \, dx = 0$ for every piecewise constant function $g$.
(iii) Is it true that $f_n \to 0$?

7 (extra credit). Let $K \in B(X; H)$ be a bounded linear operator from a Banach space $X$ into a Hilbert space $H$. Prove that the following conditions are equivalent.

(i) $K$ is compact.
(ii) For every $\varepsilon > 0$ there exists an operator $K_\varepsilon : X \mapsto H$ with finite dimensional range, such that $\|K_\varepsilon - K\| < \varepsilon$. 