1. Let \((v_k)_{k \geq 1}\) be a sequence of unit vectors in a real Hilbert space \(H\).
Let \((\alpha_k)_{k \geq 1}\) be a sequence of real numbers.

(i) If \(\sum_{k=1}^{\infty} |\alpha_k| < \infty\), prove that the series \(\sum_{k=1}^{\infty} \alpha_k v_k\) converges.

(ii) Assume that the sequence \((e_k)_{k \geq 1}\) is orthonormal. In this case, prove that the series \(\sum_{k=1}^{\infty} \alpha_k e_k\) converges if and only if \(\sum_{k=1}^{\infty} |\alpha_k|^2 < \infty\).

2. On the Hilbert space \(H = L^2(\mathbb{R})\), consider the linear operator \(\Lambda : H \mapsto H\) defined by \((\Lambda f)(x) = f(|x|)\).

(i) Find the operator norm of \(\Lambda\).

(ii) Find the kernel and the range of \(\Lambda\).

(iii) Compute the adjoint operator \(\Lambda^*\).

3. Let \(S\) be a convex set. We say that \(x \in S\) is an extreme point of \(S\) if \(x\) cannot be expressed as a convex combination of distinct points of \(S\). In other words,
\[x \neq \theta x_1 + (1-\theta) x_2\]
whenever \(0 < \theta < 1\), \(x_1, x_2 \in S\), \(x_1 \neq x_2\).

Prove the following.

(i) If \(S\) is the closed unit ball in a Hilbert space \(H\), then every point \(x \in S\) with \(\|x\| = 1\) is an extreme point of \(S\). This is true, in particular, for the space \(H = L^2([0, 1])\).

(ii) On the other hand, consider the unit ball in \(L^1([0, 1])\), i.e.
\[B = \left\{ f : [0, 1] \mapsto \mathbb{R} : \int_0^1 |f(t)| \, dt \leq 1 \right\}.

Prove that \(B\) does not contain any extreme point.

4. Let \(Q = [0, 1] \times [0, 1]\) be the unit square. Within the space \(L^2(Q)\), consider the subspace of all functions depending only on the variable \(y\):
\[U = \left\{ u \in L^2(Q) : u(x, y) = \varphi(y) \text{ for some function } \varphi : [0, 1] \mapsto \mathbb{R} \text{ and a.e. } (x, y) \in Q \right\}.

(i) Find the orthogonal subspace \(W = U^\perp\).
(ii) Given any \( f \in L^2(Q) \), determine the function \( g \in U \) such that
\[
\| f - g \|_{L^2(Q)} = \min_{u \in U} \| f - u \|_{L^2(Q)}.
\]

5. Let \( H \) be an infinite dimensional, separable Hilbert space over the reals, and let \( \ell^2 \) be the space of all sequences of real numbers \( a = (a_1, a_2, \ldots) \) such that \( \| a \|_{\ell^2} = (\sum_{k=1}^{\infty} a_k^2)^{1/2} < \infty \). Construct a linear bijection \( \Lambda : H \to \ell^2 \) which preserves distances, i.e. such that
\[
\| \Lambda x \|_{\ell^2} = \| x \|_H \quad \text{for all } x \in H.
\]

6 (extra credit). Let \( H \) be a Hilbert space and let any vector \( x \in H \) be given, with \( \| x \| \leq 1 \). Construct a sequence of vectors \( x_n \) with \( \| x_n \| = 1 \) for every \( n \geq 1 \), such that the weak convergence holds: \( x_n \rightharpoonup x \).