1. Let $X$ be a normed space over the real numbers, and let $V \subset X$ be a finite dimensional subspace.
   (i) For any $x \in X$, prove that there exists $v^* \in V$ such that
   \[
   \|x - v^*\| = d(x, V) = \inf_{v \in V} \|x - v\|
   \]
   Hint: show that the infimum is attained on the compact set \( \{v \in V; \|v\| \leq 2\|x\|\} \).
   
   (ii) If $V \neq X$, prove that there exists a vector $z$ such that
   \[
   \|z\| = 1, \quad d(z, V) = \min_{v \in V} \|z - v\| = 1.
   \]
   (Note: in an Euclidean space, this vector $z$ would be orthogonal to $V$.)
   
   (iii) If $X$ has dimension $\geq n$, prove that there exist $2n$ vectors $z_1, \ldots, z_{2n}$ such that $\|z_i\| = 1$, $\|z_i - z_j\| \geq 1$ for all $i \neq j$.

2. Let $\phi : \mathbb{R}^2 \mapsto \mathbb{R}$ be a linear functional, say $\phi(x_1, x_2) = ax_1 + bx_2$, with operator norm
   \[\|\phi\| = \sup_{\|x\| \leq 1} |\phi(x)|.\]
   Give a direct proof that
   (i) If $\mathbb{R}^2$ is endowed with the norm $\|x\|_1 = |x_1| + |x_2|$, then the corresponding operator norm (1) is $\|\phi\|_\infty = \max\{|a|, |b|\}$.
   (ii) If $\mathbb{R}^2$ is endowed with the norm $\|x\|_\infty = \max\{|x_1|, |x_2|\}$, then the corresponding norm (1) is $\|\phi\|_1 = |a| + |b|$.
   (iii) If $\mathbb{R}^2$ is endowed with the norm $\|x\|_2 = (|x_1|^2 + |x_2|^2)^{1/2}$, with $1 < p < \infty$, then the corresponding norm (1) is $\|\phi\|_2 = (|a|^2 + |b|^2)^{1/2}$. 

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3. Let $C^0([0,1])$ be the Banach space of all real valued continuous functions $f : [0,1] \mapsto \mathbb{R}$, with norm $\|f\| = \max_{x \in [0,1]} |f(x)|$.
   (i) Show that $X = \{f \in C^0([0,1]) ; f(0) = 0\}$ is a closed subspace of $C^0$, hence it is a Banach space.
   (ii) Prove that the map $f \mapsto \Lambda f = \int_0^1 f(x) \, dx$ is a continuous linear functional on $X$. Compute its norm $\|\Lambda\| = \sup_{\|f\| \leq 1} |\Lambda f|$. Is this supremum over the closed unit ball actually attained as a maximum?

4. Let $\Omega \subset \mathbb{R}^N$ be an open set. Define the compact sets
   $$A_k = \{x \in \Omega ; |x| \leq k, \, d(x, \partial \Omega) \geq \frac{1}{k}\}.$$  
   Let $X$ be the space of all continuous functions $f : \Omega \mapsto \mathbb{R}$, with the seminorms
   $$p_k(f) = \max_{x \in A_k} |f(x)|$$  
   and the distance
   $$d(f,g) = \sum_{k \geq 1} 2^{-k} \frac{p_k(f-g)}{1 + p_k(f-g)}.$$  
   Given a sequence of continuous functions $(f_n)_{n \geq 1}$, prove that
   $$\lim_{n \to \infty} d(f_n, f) \to 0$$  
   if and only if the functions $f_n$ converge to $f$ uniformly on compact sets. Namely
   $$\lim_{n \to \infty} \max_{x \in K} |f_n(x) - f(x)| = 0$$  
   for every compact set $K \subset \Omega$.

5 (extra credit). Show that the concept of Lebesgue measure cannot be extended to infinite dimensional spaces. More precisely, let $X$ be an infinite dimensional Banach space. Prove that there there cannot be a measure $\mu$, defined on the sigma-algebra of Borel subsets of $X$, with the following properties:
   (i) $\mu(\Omega) > 0$ for every nonempty open set $\Omega \subset X$,
   (ii) $\mu$ is translation-invariant: $\mu(x+S) = \mu(S)$ for every $x \in X$ and $S \subset X$,
   (iii) there exists a nonempty open set $\Omega_0$ such that $\mu(\Omega_0) < \infty$.
   Hint: if $B_\varepsilon$ is an open ball of radius $\varepsilon > 0$ and $\mu(B_\varepsilon) = \eta > 0$, using problem 1. give a lower bound on $\mu(B_{4\varepsilon})$. 