Math 411 - Ordinary Differential Equations

Review Notes - 2

1 - ODE's in the plane

An **autonomous** system of two ODEs has the form

$$\begin{cases} x' = f(x,y)\,, \\ y' = g(x,y). \end{cases} \tag{1}$$

We regard $(x(t), y(t))$ as the position at time t of a point moving in the plane, so that the vector $(x', y') = (f, g)$ determines its velocity. Here "autonomous" means that the functions f, g do not depend explicitly on time t.

If $t \mapsto (x(t),\, y(t))$ is a solution defined on a maximal interval (α, ω), then the set of points

$$\mathcal{O} \;=\; \Big\{(x(t), y(t))\,;\quad t \in (\alpha, \omega)\Big\} \;\subset\; \mathbb{R}^2$$

is called an **orbit**. A **phase plane diagram** for (1) is obtained by drawing orbits and equilibrium points, and marking the direction of motion along the orbits. Two methods:

• Reduce the system of two ODEs to one single scalar equation

$$\frac{dy}{dx} \;=\; \frac{dy/dt}{dx/dt} \;=\; \frac{g(x,y)}{f(x,y)}.$$

If this equation turns out to be linear, or separable, an explicit solution can be found.

• Start by drawing **null-clines**, i.e. curves in the x-y plane where

- either $f(x,y) = 0$, so that the speed of the point (x,y) is a vertical vector: $(x', y') = (0, g(x,y))$.

- or $g(x,y) = 0$, so that the speed of the point (x,y) is a horizontal vector: $(x', y') = (f(x,y),\, 0)$.

Then sketch the trajectories of the ODE, keeping in mind the sign of f, g in the various regions.

2 - Hamiltonian systems

The system (1) is **hamiltonian** if it can be written in the form

$$\begin{cases} x' \;=\; \dfrac{\partial H(x,y)}{\partial y}\,, \\[2mm] y' \;=\; -\dfrac{\partial H(x,y)}{\partial x}\,. \end{cases} \tag{2}$$

for some function $H(x, y)$. This is possible provided that

$$\frac{\partial}{\partial x} f(x, y) \;=\; -\frac{\partial}{\partial y} g(x, y). \tag{3}$$

If the identity (3) holds at every point (x, y), to find a function $H(x, y)$ such that $\frac{\partial H(x,y)}{\partial y} = f(x, y)$ and $\frac{\partial H(x,y)}{\partial x} = -g(x, y)$ we proceed in two steps:

1. Regarding x as a constant, we find an antiderivative of the function $y \mapsto f(x, y)$, in the form

$$H(x, y) \;=\; \int f(x, y)\, dy \;+\; k(x).$$

This guarantees that $\frac{\partial H(x,y)}{\partial y} = f(x, y)$.

2. We then determine $k(x)$ so that H satisfies the additional relation $\frac{\partial H(x,y)}{\partial x} = -g(x, y)$.

For the Hamiltonian system (2), the function H is constant along every solution. Indeed, by the chain rule

$$\frac{d}{dt} H(x(t), y(t)) \;=\; \frac{\partial H}{\partial x}\, x'(t) + \frac{\partial H}{\partial y}\, y'(t) \;=\; \frac{\partial H}{\partial x} \frac{\partial H}{\partial y} + \frac{\partial H}{\partial y} \left(-\frac{\partial H}{\partial x} \right) \;=\; 0\,.$$

The orbits of (2) are thus contained in level sets of H, i.e. sets where $H(x, y) = \text{constant}$.

3 - Phase plane diagrams for linear systems

Consider the linear homogeneous system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4}$$

Depending on the eigenvalues λ_1, λ_2 of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, various cases arise.

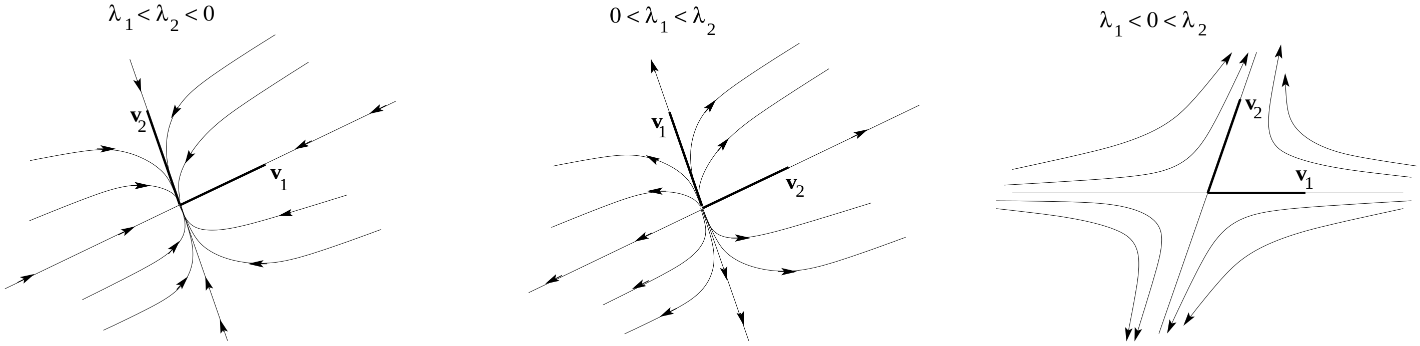
We first assume that the eigenvalues λ_1, λ_2 are real and distinct. Let $\mathbf{v}_1, \mathbf{v}_2$ be corresponding eigenvectors. The general solution is thus

$$c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2\,.$$

CASE 1 (stable node): $\lambda_1 < \lambda_2 < 0$. As $t \to +\infty$, all trajectories flow into the origin. The component along $\mathbf{v}_1$ decays faster, and trajectories are asymptotically tangent to $\mathbf{v}_2$.

CASE 2 (unstable node): $0 < \lambda_1 < \lambda_2$. As $t \to +\infty$, trajectories flow away from the origin, becoming arbitrarily large. For negative times, as $t \to -\infty$, the component along $\mathbf{v}_2$ decays faster, and trajectories are asymptotically tangent to $\mathbf{v}_1$.

CASE 3 (saddle): $\lambda_1 < 0 < \lambda_2$. The zero solution is unstable. As $t \to +\infty$ the component along $\mathbf{v}_1$ approaches zero, while the component along $\mathbf{v}_2$ becomes arbitrarily large. On the other hand, as $t \to -\infty$, the $\mathbf{v}_1$-component becomes large, while the $\mathbf{v}_2$ component approaches zero.

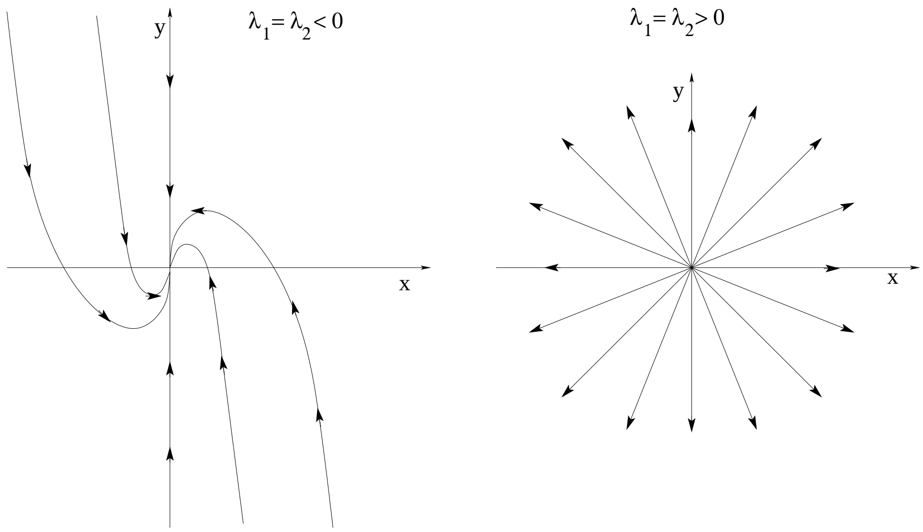


Left: a stable node. Middle: an unstable node. Right: a saddle.

CASE 4 (degenerate node): Assume that the matrix A has a double eigenvalue $\lambda \in \mathbb{R}$.

If $\lambda < 0$ then the origin is a **stable node**. If $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ is diagonal, then all trajectories are half lines emanating from the origin. If A is not diagonalizable (only one linearly independent eigenvector $\mathbf{v}_1$ can be found), then trajectories approach the origin tangent to $\mathbf{v}_1$.

If $\lambda > 0$ then the origin is an **unstable node**. The orbits are the same as in the stable case, reversing the time direction.



Left: a stable degenerate node (in the case of only one linearly independent eigenvector).
Right: an unstable degenerate node (in the case of two linearly independent eigenvectors).

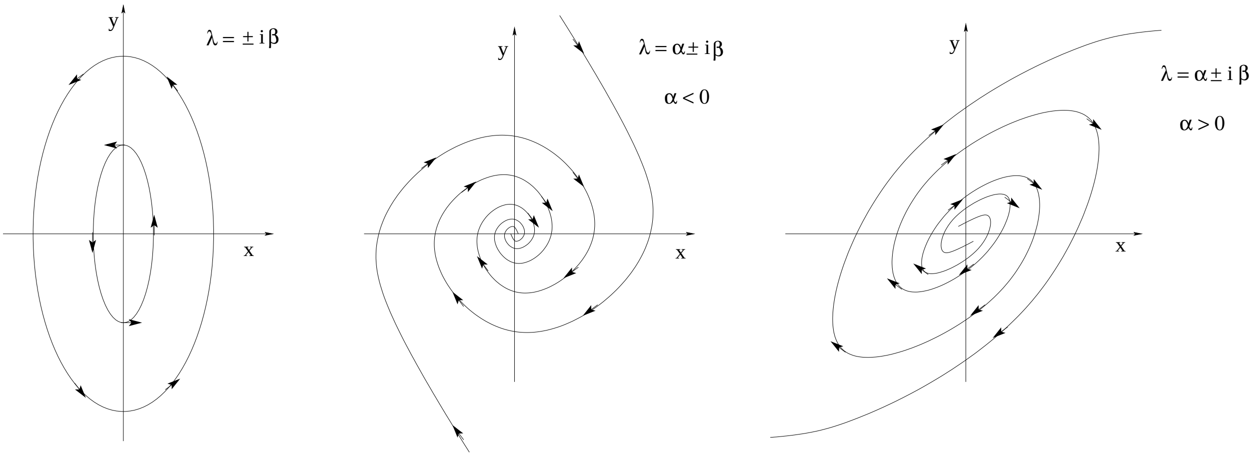
Next, assume that the matrix A has complex eigenvalues: $\lambda = \alpha \pm i\beta$, with $\beta \neq 0$.

CASE 5 (center): If $\alpha = 0$, solutions are periodic. Trajectories are ellipses (or circumferences) centered at the origin.

CASE 6 (stable spiral point): If $\alpha < 0$, trajectories are spirals converging to the origin as

$t \to +\infty$.

CASE 7 (unstable spiral point): If $\alpha > 0$, trajectories are spirals moving away from the origin as time increases.



Left: a center. Middle: a stable spiral point. Right: an unstable spiral point.

4 - Stability for nonlinear systems

Given the differential equation on $\mathbb{R}^n$

$$x' = f(x), \tag{5}$$

we denote by $x(t) = \phi(t, y)$ the solution to (5) which starts at the point $y \in \mathbb{R}^n$:

$$x(0) = y\,. \tag{6}$$

The function ϕ satisfies the **semigroup property**

$$\phi(t + \tau,\, y) \;=\; \phi(t\,,\; \phi(\tau, y)) \qquad\qquad \text{for every } t, \tau \geq 0,\, y \in \mathbb{R}^n.$$

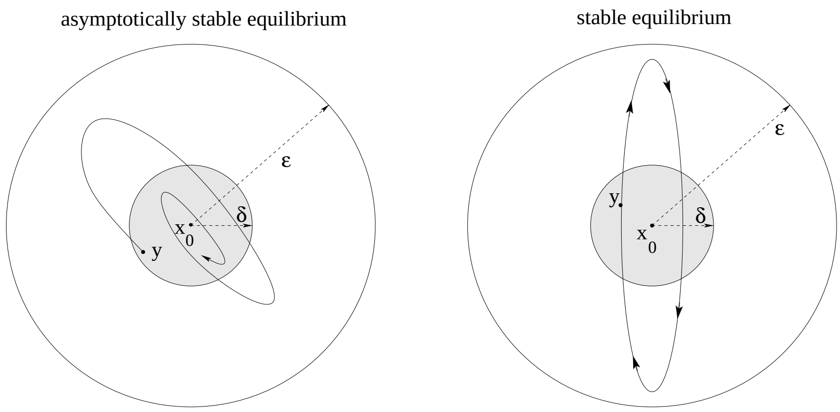
We say that $x_0 \in \mathbb{R}^n$ is an **equilibrium point** if $f(x_0) = 0$.

The point x_0 is a **stable equilibrium** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that:

$$\text{if } \; |y - x_0| < \delta \qquad \text{then} \qquad \big|\phi(t, y) - x_0\big| < \varepsilon \quad \text{for all } \; t \geq 0.$$

The point x_0 is an **asymptotically stable equilibrium** if, in addition, for $|y - x_0| < \delta$ one has

$$\lim_{t \to +\infty} \; \phi(t, y) \;=\; x_0\,.$$



THE METHOD OF LYAPUNOV FUNCTIONS

Let x_0 be an equilibrium point for the differential equation (5).

A continuously differentiable function $V = V(x)$ defined for x in a neighborhood of x_0 is a **Lyapunov function** if

$$V(x_0) = 0 \quad \text{and} \quad V(x) > 0 \quad \text{for every} \quad x \neq x_0\,, \tag{L1}$$

$$\nabla V(x) \cdot f(x) \leq 0 \qquad \text{at every point} \quad x \in I\!R^n. \tag{L2}$$

Because of (L2), for every solution of the differential equation (5) we have

$$\frac{d}{dt}V(x(t)) \;=\; \nabla V(x(t)) \cdot x'(t) \;=\; \nabla V(x(t)) \cdot f(x(t)) \;\leq 0.$$

Hence $V(x(t))$ is non-increasing in time.

In case where (L2) is replaced by the stronger condition

$$\nabla V(x) \cdot f(x) < 0 \qquad \text{at every point} \quad x \neq x_0. \tag{L2+}$$

then we say that V is a **strict Lyapunov function**. In this case, $V(x(t))$ is strictly decreasing along solutions of the differential equation (except when $x(t) = x_0$).

• *If a Lyapunov function exists, then x_0 is a stable equilibrium point.*

• *If a strict Lyapunov function exists, then x_0 is an asymptotically stable equilibrium point.*

• (LaSalle) *If a Lyapunov function V exists, and for every initial point $y \neq x_0$ the function $t \mapsto V(\phi(t, y))$ is not a constant, then x_0 is an asymptotically stable equilibrium point.*

There are no general rules for constructing a Lyapunov function. Some hints:

– If the ODE models a physical system, try with V = total energy of the system.

– For the planar system (1), if (x_0, y_0) are the coordinates of an equilibrium point, try with $V(x, y) = a(x - x_0)^2 + b(y - y_0)^2$, with suitable coefficients $a, b > 0$.

THE METHOD OF LINEARIZATION

Let x_0 be an equilibrium point for the differential equation (5). Compute the $n \times n$ Jacobian matrix of f at the point x_0 :

$$A \;=\; Df(x_0) \;=\; \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

- *If all the eigenvalues of A have strictly negative real part, then x_0 is an asymptotically stable equilibrium.*

- *If at least one of the eigenvalues of A has strictly positive real part, then x_0 is an unstable equilibrium point.*

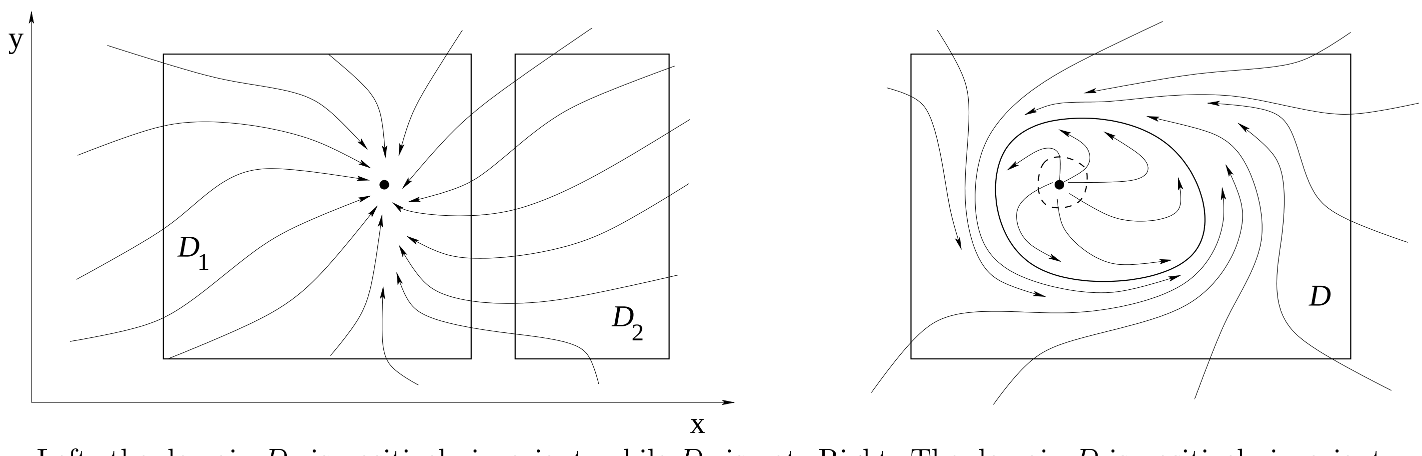
This method does not provide information if the eigenvalues of A have zero real part.

5 - Invariant domains

Let $x(t)$ be a solution of the differential equation (5), defined for all $t \in [0, +\infty)$.
Its ω-**limit set** is the set

$$\big\{z \in \mathbb{R}^n\,;\quad \text{there exists a sequence}\quad t_k \to +\infty \quad \text{such that}\quad x(t_k) \to z\big\}.$$

Note: if $\lim_{t\to\infty} x(t) = x_0$, then the ω-limit set is simply $\{x_0\}$.



Left: the domain D_1 is positively invariant, while D_2 is not. Right: The domain D is positively invariant. By removing a neighborhood of the strictly unstable equilibrium point, we obtain a domain which is still positively invariant but does not contain any equilibrium point. Hence it must contain a cycle.

A domain $\mathcal{D} \subset I\!R^2$ is **positively invariant** for the differential equation (5) if

$$y \in \mathcal{D} \qquad \text{implies} \qquad \phi(t, y) \in \mathcal{D} \quad \text{for all } t \geq 0.$$

In other words, a solution that starts in $\mathcal{D}$ remains in $\mathcal{D}$ for all times $t \geq 0$. The domain $\mathcal{D}$ is positively invariant provided that the velocity vector $x' = f(x)$ is tangent, or points toward the interior of $\mathcal{D}$, at every point x on the boundary of $\mathcal{D}$.

6 - Periodic solutions

We now look again at ODEs in the plane. These are written as

$$\begin{cases} x' = f(x, y)\,, \\ y' = g(x, y). \end{cases} \tag{1}$$

Existence of periodic orbits.

A nontrivial periodic orbit is called a **cycle**.

• (Poincaré-Bendixson) *Let $\mathcal{D} \subset I\!R^2$ be a closed, bounded, positively invariant set. Then $\mathcal{D}$ contains at least one equilibrium point or a cycle.*

• *In addition, assume that all equilibrium points inside $\mathcal{D}$ are strictly unstable, i.e. at these equilibrium points the Jacobian matrix $\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$ has eigenvalues with strictly positive real parts. Then $\mathcal{D}$ contains a cycle.*

Note: inside the region bounded by a periodic orbit, one can also find at least one equilibrium point.

Non-existence of periodic orbits.

• If the system (1) has no equilibrium points, then it cannot have any cycle.

• If $f(x, y) \geq 0$ for all x, y then there exists no cycle. Same conclusion if $f(x, y) \leq 0$ for all x, y, or if $g(x, y) \geq 0$ for all x, y, or if $g(x, y) \leq 0$ for all x, y.

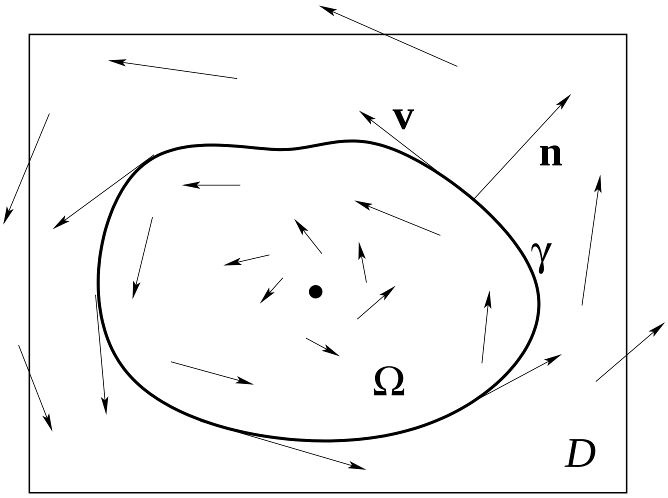
• (Bendixson-Dulac) *Let $\mathcal{D} \subseteq I\!R^2$ be a convex domain in the x-y plane. Assume that we can find a function $\alpha(x, y)$ such that the vector field $\mathbf{v} = \begin{pmatrix} \alpha(x,y) f(x,y) \\ \alpha(x,y) g(x,y) \end{pmatrix}$ satisfies*

$$\textit{div}\, \mathbf{v} \;=\; f_x + g_y \;>\; 0 \qquad \textit{at every point} \;\; (x, y) \in \Omega.$$

Then the domain $\mathcal{D}$ cannot contain any periodic orbit.

Indeed, if there exists a closed orbit γ entirely contained in the domain $\mathcal{D}$, call Ω the domain having γ as boundary. Then the divergence theorem gives a contradiction:

$$0 \;<\; \int_\Omega \text{div}\ \mathbf{v} \;=\; \int_\gamma \mathbf{v} \cdot \mathbf{n} \;=\; 0.$$



Applying the divergence theorem: the vector **v** is tangent to the cycle γ, while **n** is the outer unit normal.