Math 411 - Ordinary Differential Equations

Review Notes - 1

1 - Basic Theory

A first order ordinary differential equation has the form

$$x' = f(t, x). \tag{1.1}$$

Here $x' = dx/dt$. Given an initial data

$$x(t_0) = x_0, \tag{1.2}$$

by a solution of the initial value problem we mean a *continuously differentiable* function $t \mapsto x(t)$ defined on some interval $t \in (a, b)$ with $a < t_0 < b$, which satisfies the equation (1.1) at every t, and the initial condition (1.2).

Local existence: If f is *continuous*, then there exists $\delta > 0$ and a solution $x(t)$ of the initial value problem (1.1)-(1.2), defined for $t \in (t_0 - \delta\,,\ t_0 + \delta)$.

Uniqueness: If, in addition, f has a *continuous partial derivative* $\partial f/\partial x$, then the solution of (1)-(2) is unique.

Global existence: If f is defined for all $t, x \in I\!R$, and if we can find continuous functions p, q such that $|f(t, x)| \leq p(t)|x| + q(t)$, then the solution of (1.1)-(1.2) is defined for all $t \in (-\infty, +\infty)$.

2 - Explicit Solutions

- The O.D.E. (1.1) is *separable* if it can be written as a product

$$x' \;=\; g(x)h(t).$$

In this case one can find the general solution by integrating:

$$\int \frac{1}{g(x)}\, dx \;=\; \int h(t)\, dt + C$$

where C is an arbitrary constant.

- The O.D.E. (1.1) is *linear* if it has the form

$$x' = p(t)x + q(t).$$

In this case, the explicit solution of (1.1)-(1.2) is

$$x(t) \;=\; e^{\int_{t_0}^{t} p(s)\,ds} x_0 + \int_{t_0}^{t} e^{\int_{s}^{t} p(\tau)\,d\tau}\, q(s)\; ds\,.$$

3 - Phase Line Diagrams

Consider the O.D.E.

$$x' = f(x) \tag{3.1}$$

where f is continuously differentiable and does not depend on t. We think $x(t)$ as the position of a point moving on the line, and $x'(t)$ as its velocity. The *phase line diagram* is obtained by

- marking the *equilibrium points* where $f = 0$,

- adding arrows marking the direction of motion.

Notation: We denote by $x(t) = \phi(t, x_1)$ the solution to the initial value problem

$$x' = f(x) \qquad\qquad x(0) = x_1\,.$$

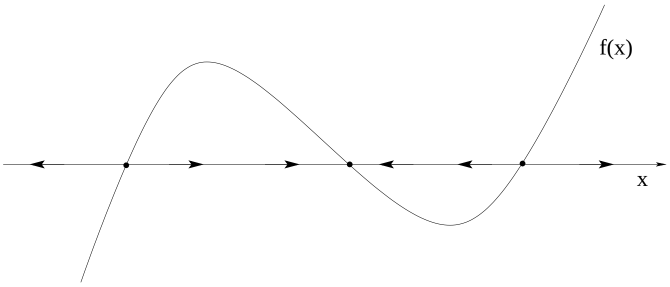
Definition (stable equilibrium). Let x_0 be an equilibrium point, so the $f(x_0) = 0$. We say that x_0 is *stable* if for every $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\text{if} \qquad |x_1 - x_0| < \delta \qquad \text{then} \qquad \big|\phi(t, x_1) - x_0\big| < \varepsilon \quad \text{for all} \;\; t \geq 0.$$

We say that x_0 is *asymptotically stable* if, in addition,

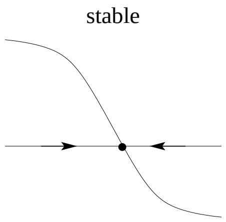
$$\lim_{t\to\infty} \phi(t, x_1) \;=\; x_0\,.$$

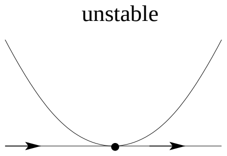
If the equilibrium point x_0 is not stable, we say it is *unstable.*
Intuitive meaning: x_0 is a stable equilibrium $\Longleftrightarrow$ every solution which starts close to x_0 remains close to x_0 at all positive times $t \in [0, +\infty)$.

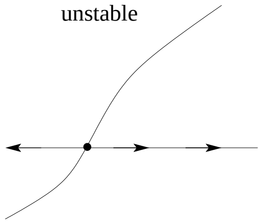


Theorem 1 (stable equilibrium). If $f(x_0) = 0$, $f(x) > 0$ for $x \in [x_0 - \delta,\, x_0)$ and $f(x) < 0$ for $x \in (x_0,\, x_0 + \delta]$, then x_0 is asymptotically stable.

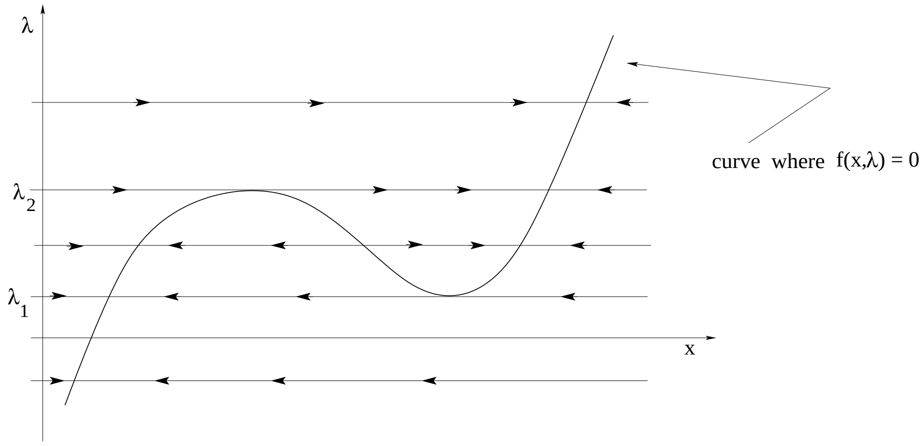
Theorem 2 (unstable equilibrium). If $f(x_0) = 0$ but $f(x) < 0$ for $x \in [x_0 - \delta,\, x_0)$, or $f(x) > 0$ for $x \in (x_0,\, x_0 + \delta]$, then x_0 is unstable.







For the O.D.E. $x' = f(x, \lambda)$ depending on an additional parameter λ, the phase line diagram can change type when the parameter λ crosses some particular values. In this case we say that a *bifurcation* occurs. In the picture below, this happens when λ crosses the values λ_1 and λ_2 .



4 - Some linear algebra

In a vector space, a family of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is *linearly independent* if the only way to obtain the zero vector as a linear combination

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m \;=\; 0$$

is to choose the coefficients $c_1 = c_2 = \cdots = c_m = 0$. On the other hand, if the zero vector can be obtained as a linear combination with coefficients $c_1, \ldots, c_m$ not all zero, then we say that the vectors are *linearly dependent.*

In $I\!R^n$, a set of n linearly independent vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ forms a *basis.* Every vector $\mathbf{v} \in I\!R^n$ can then be written as a linear combination:

$$\mathbf{v} \;=\; \sum_{i=1}^{n} c_i \mathbf{v}_i \,,$$

for some (unique) coefficients $c_1, c_2, \ldots, c_n \in I\!R$.

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent if and only if the matrix

$$M \;=\; \Big(\mathbf{v}_1 \Big| \mathbf{v}_2 \Big| \cdots \Big| \mathbf{v}_n \Big)$$

having $\mathbf{v}_1, \ldots \mathbf{v}_n$ as column vectors has non-zero determinant. This holds if and only if the matrix M is invertible.

Let A be a $n \times n$ matrix. If

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some number λ and some non-zero vector $\mathbf{x}$, we say that λ is an *eigenvalue* of A and $\mathbf{x}$ is a corresponding *eigenvector*. Equivalently, λ is an eigenvalue of A if and only if

$$p(\lambda) \;=\; \det(A - \lambda I) \;=\; 0.$$

Here I denotes the $n \times n$ identity matrix.

NOTE: even if the entries of the matrix A are all real numbers, its eigenvalues may be complex numbers. In this case, the corresponding eigenvectors will also have complex entries.

Eigenvectors corresponding to distinct eigenvalues are linearly independent. If the n-th degree polynomial $p(\lambda)$ has n distinct roots $\lambda_1, \lambda_2, \ldots, \lambda_n$, then one can find a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ consisting of eigenvectors for the matrix A. In this case, calling

$$P \;=\; \left(\mathbf{x}_1 \middle| \mathbf{x}_2 \middle| \cdots \middle| \mathbf{x}_n \right)$$

the $n \times n$ matrix whose columns are the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$, and denoting by P^{-1} its inverse, one has

$$P^{-1}AP \;=\; \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

On the other hand, if the characteristic polynomial $p(\lambda)$ has multiple roots, one may not be able to find n linearly independent eigenvectors. In this case, one can still find a basis of *generalized eigenvectors*, i.e. vectors $\mathbf{x}$ such that

$$(A - \lambda I)^k \mathbf{x} = 0 \qquad\qquad \text{for some} \quad k \geq 1.$$

Two basic identities for complex numbers:

$$i^2 \;=\; -1\,, \qquad\qquad e^{i\alpha} \;=\; \cos\alpha + i\sin\alpha\,.$$

5 - Systems of O.D.E's

A system of n ordinary differential equations on $\mathbb{R}^n$ can still be written as

$$\mathbf{x}' = f(t, \mathbf{x}), \tag{5.1}$$

but keeping in mind that now both $\mathbf{x} = (x_1, \ldots, x_n)$ and $f = (f_1, \ldots, f_n)$ are vectors in $\mathbb{R}^n$. This means

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} \;=\; \begin{pmatrix} f_1(t, x_1, \ldots, x_n) \\ f_2(t, x_1, \ldots, x_n) \\ \vdots \\ f_n(t, x_1, \ldots, x_n) \end{pmatrix}. \tag{5.2}$$

Given an initial condition

$$\mathbf{x}(t_0) = \mathbf{x}_0 \in I\!R^n, \tag{5.3}$$

by a solution of the initial value problem we mean a *continuously differentiable* function $t \mapsto \mathbf{x}(t) \in I\!R^n$ defined on some interval $t \in (a, b)$ with $a < t_0 < b$, which satisfies the systems of equations (5.2) at every t, and the initial condition (5.3).

Local existence: If f is *continuous*, then there exists $\delta > 0$ and a solution $\mathbf{x}(t)$ of the initial value problem (5.2)-(5.3), defined for $t \in (t_0 - \delta\,,\ t_0 + \delta)$.

Uniqueness: If, in addition, $f = (f_1, \ldots, f_n)$ has *continuous partial derivatives* $\partial f_i/\partial x_j$, then the solution of (5.2)-(5.3) is unique.

Global existence: If f is defined for all $t \in I\!R$, $\mathbf{x} \in I\!R^n$, and if we can find continuous functions p, q such that $|f(t, \mathbf{x})| \leq p(t)|\mathbf{x}| + q(t)$, then the solution of (5.2)-(5.3) is defined for all $t \in (-\infty, +\infty)$.

6 - Linear Systems of O.D.E's

A *linear homogeneous* system of O.D.E's has the form

$$\mathbf{x}' = A(t)\mathbf{x}\,. \tag{6.1}$$

This is a short-hand notation for

$$\begin{pmatrix} x_1' \\ x_2' \\ \\ \vdots \\ \\ x_n' \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \\ \vdots & \vdots & \ldots & \vdots \\ \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \\ \vdots \\ \\ x_n \end{pmatrix}. \tag{6.2}$$

We always assume that the entries of matrix $A(t)$ are continuous functions of time. This implies that every initial value problem has a unique solution, globally defined.

Solutions form a vector space: If $\phi_1(t)$ and $\phi_2(t)$ are solutions of the system (6.1), then the linear combination

$$\phi(t) \;=\; c_1\phi_1(t) + c_2\phi_2(t)$$

is also a solution, for every choice of the constants c_1, c_2 (real, or complex numbers).

Solutions remain independent: Let $\phi_1, \phi_2, \ldots, \phi_n$ be solutions of (6.1). Assume that, at a certain time t_0, the vectors $\phi_1(t_0), \phi_2(t_0), \ldots, \phi_n(t_0)$ are linearly independent. Then, for every other time t, the vectors

$$\phi_1(t),\ \phi_2(t),\ \ldots\ ,\ \phi_n(t)$$

are still linearly independent.

If n linearly independent solutions are known, then we can solve every initial value problem: Let $\phi_1, \phi_2, \ldots, \phi_n$ be linearly independent solutions of (6.1). The matrix having these solutions as column vectors:

$$\Phi(t) \;=\; \Big(\phi_1(t)\Big|\phi_2(t)\Big|\cdots\Big|\phi_n(t)\Big)$$

is called a *matrix fundamental solution.* The assumption of linear independence implies that, for each time t, this matrix has an inverse $\Phi^{-1}(t)$.
Given any initial condition

$$\mathbf{x}(t_0) \;=\; \mathbf{x}_0 \;\in\; \mathbb{R}^n, \tag{6.3}$$

the solution of the initial value problem (6.1), (6.3) is explicitly given by

$$\mathbf{x}(t) \;=\; \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0\,. \tag{6.4}$$

An alternative, simpler way to solve the initial value problem (6.1), (6.3), without having to invert the matrix $\Phi(t_0)$, is to write a general linear combination

$$\mathbf{x}(t) = c_1\phi_1(t) + c_2\phi_2(t) + \cdots + c_n\phi_n(t),$$

determining the coefficients $c_1, c_2, \ldots, c_n$ so that $\mathbf{x}(t_0) = c_1\phi_1(t_0) + c_2\phi_2(t_0) + \cdots + c_n\phi_n(t_0) = \mathbf{x}_0$.

How to find special solutions: Consider the linear system of O.D.E's

$$x' = Ax$$

where $A(t) = A$ is a constant matrix with real-valued entries.

- **Single eigenvalue:** If $(A - \lambda I)\mathbf{v} = 0$, then $\phi(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- **Double eigenvalue:** If $(A - \lambda I)\mathbf{v}_1 = 0$ and $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$,
  then $\phi_1(t) = e^{\lambda t}\mathbf{v}_1$ and $\phi_2(t) = e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1)$ are solutions.
- **Complex eigenvalues:** If $\phi(t) = \phi_1(t) + i\phi_2(t)$ is a (complex-valued) solution, then its real and its imaginary part ϕ_1, ϕ_2 are both solutions.

Linear, non-homogeneous problems. Given any continuous vector valued function $\mathbf{b}(t) \in \mathbb{R}^n$, the solution to the initial value problem

$$\mathbf{x}'(t) \;=\; A(t)\mathbf{x}(t) + \mathbf{b}(t), \qquad\qquad \mathbf{x}(t_0) = \mathbf{x}_0$$

is given by

$$\mathbf{x}(t) \;=\; \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0 + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)\mathbf{b}(s)\,ds. \tag{6.5}$$

High order equations. The linear differential equation of order n

$$y^{(n)}(t) + p_{n-1}(t)y^{(n-1)}(t) + \cdots + p_1(t)y'(t) + p_0(t)y(t) \;=\; r(t) \tag{6.6}$$

can be rewritten as a first order system of n linear equations. Namely, setting

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \quad \ldots \quad , \; x_n = y^{(n-1)},$$

the equation (6.6) becomes

$$\begin{pmatrix} x_1' \\ x_2' \\ \\ \vdots \\ \\ x_{n-1}' \\ x_n' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \\ 0 & 0 & 0 & \cdots & 1 \\ -p_0(t) & -p_1(t) & -p_2(t) & \cdots & -p_{n-1}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \\ \vdots \\ \\ x_{n-1} \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \\ \vdots \\ \\ 0 \\ r(t) \end{pmatrix}. \tag{6.7}$$

7 - The Matrix Exponential Function

Given an $n \times n$ matrix A, we seek a matrix-valued function $\Psi(t)$ such that

$$\Psi'(t) = A\,\Psi(t), \qquad\qquad \Psi(0) = I,$$

where I is the $n \times n$ identity matrix. As in the scalar case, the solution can be found as

$$\Psi(t) = e^{At}, \qquad \text{where we define} \qquad e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}\,.$$

Here the series represents an infinite sum of matrices. Each entry converges absolutely for every fixed time t.

Properties of the matrix exponential. Let A, B be $n \times n$ constant matrices. Then

- $\frac{d}{dt}\, e^{At} = Ae^{At}$.
- $e^{As}e^{At} = e^{A(s+t)}$. Each matrix e^{At} is invertible and $[e^{At}]^{-1} = e^{-At}$.
- If $AB = BA$ then $e^A B = Be^A$ and $e^A e^B = e^{(A+B)}$.
- If P is an invertible matrix, then $e^{PBP^{-1}} = Pe^B P^{-1}$.

Diagonal matrices:

$$\text{If} \quad B = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \qquad \text{then} \qquad e^{Bt} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{pmatrix}.$$

Jordan block:

$$\text{If}\quad B = \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\qquad\text{then}\qquad e^{Bt} = e^{\lambda t}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}.$$

General matrix: To compute e^{At}, find any matrix fundamental solution $\Phi(t)$. Then $e^{At} = \Phi(t)\Phi^{-1}(0)$.

Alternative method: find an invertible matrix P such that $B = P^{-1}AP$ is diagonal (or in Jordan block form). This is the case if the columns of P are linearly independent (generalized) eigenvectors of A. Then use the identity $e^{At} = e^{P(Bt)P^{-1}} = Pe^{Bt}P^{-1}$, where e^{Bt} is easy to compute (because B is diagonal, or a Jordan block).

Using the matrix exponential, the solution to the linear system

$$\mathbf{x}'(t) = A\,\mathbf{x}(t), \tag{7.1}$$

with initial data $\mathbf{x}(0) = \mathbf{x}_0$, can be explicitly written as $\mathbf{x}(t) = e^{At}\mathbf{x}_0$.

Stability, instability: If all the eigenvalues of the matrix A have strictly negative real part, then the equilibrium solution $\mathbf{y}(t) \equiv 0$ of (7.1) is asymptotically stable. If at least one eigenvalue has strictly positive real part, then this equilibrium solution is unstable.

8 - Floquet theory

We consider a linear homogeneous system

$$\mathbf{x}'(t) \;=\; A(t)\,\mathbf{x}(t), \tag{8.1}$$

assuming that the matrix A is *periodic* of period ω, namely $A(t+\omega) = A(t)$ for every $t \in \mathbb{R}$.

In general, a matrix fundamental solution $\Phi(t)$ will not be periodic. However, one can always find a constant matrix B and a periodic function $P(t)$ such that

$$\Phi(t) = P(t)e^{Bt}. \tag{8.2}$$

Indeed, consider the matrix $C = \Phi^{-1}(0)\Phi(\omega)$, and find another matrix B such that $e^{B\omega} = C$. Then the function $P(t) = \Phi(t)e^{-Bt}$ is periodic and the representation (8.2) holds.

The eigenvalues of the matrix $C = \Phi^{-1}(0)\Phi(\omega)$ are called the *Floquet multipliers* of the periodic system (8.1). They do not depend on the choice of the fundamental matrix solution Φ.

Stability, instability: The equilibrium solution $\mathbf{y}(t) \equiv 0$ of the periodic system (8.1) is

- asymptotically stable if all Floquet multipliers satisfy $|\mu_i| < 1$, $i = 1, \ldots, n$.
- unstable if one of the Floquet multipliers satisfies $|\mu_{i_0}| > 1$.