Math 401 - Introduction to Real Analysis

Review 5

1 Continuous functions on an interval

Intermediate Value Theorem. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a continuous function defined on a closed bounded interval. If \( f(a) \leq 0 \) and \( f(b) \geq 0 \), then there exists a point \( \xi \in [a, b] \) such that

\[
  f(\xi) = 0.
\]

Mean Value Theorem. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a continuous function, differentiable on \( (a, b) \). Then there exists a point \( \xi \in (a, b) \) such that

\[
  f'(\xi) = \frac{f(b) - f(a)}{b - a}
\]

[slope of tangent line at \( \xi \) = slope of secant line through \( a, b \)].

Theorem (uniform continuity). Let \( f : [a, b] \rightarrow \mathbb{R} \) be a continuous function. Then \( f \) is uniformly continuous. That means: given any \( \varepsilon > 0 \) we can find \( \delta > 0 \) such that

\[
  \text{if} \quad |x - x'| < \delta, \quad \text{then} \quad |f(x) - f(x')| < \varepsilon.
\]

Figure 1: Left: a continuous function \( f \) that changes sign over the interval \([a, b]\) must have a zero. Center: the function \( g \) has opposite signs at \( a \) and at \( b \) but does not have any zero. This can happen because \( g \) is not continuous on the whole interval \([a, b]\). Right: if \( f \) is differentiable on \([a, b]\), there exists a point \( \xi \) such that the slope of the tangent at \( \xi \) equals the slope of the secant line over \([a, b]\).
2 Taylor polynomials

Let $f$ be $n$ times differentiable on an interval $I$ containing the point $\xi$.

The **Taylor polynomial** of order $n$ for $f$ at the point $\xi$ is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(\xi)}{k!}(x-\xi)^k.$$  \hspace{1cm} (1)

Here $f^{(k)}(\xi) = k$-th derivative of $f$ computed at the point $\xi$.

The polynomial $P_n$ has two important properties:

(i) The derivatives of $f$ and $P_n$ at the point $x = \xi$ coincide, up to order $n$:

$$f(\xi) = P_n(\xi), \quad f'(\xi) = P'_n(\xi), \quad f''(\xi) = P''_n(\xi), \quad \ldots \quad f^{(n)}(\xi) = P^{(n)}_n(\xi).$$

(ii) For $x$ close to $\xi$, the value $P_n(x)$ is very close to $f(x)$. Indeed

$$\lim_{x \to \xi} |P_n(x) - f(x)| \cdot \frac{1}{|x - \xi|^n} = 0.$$  

In other words, multiplying the difference $|P_n(x) - f(x)|$ by the very large quantity $|x - \xi|^{-n}$, the product still approaches zero as $x \to \xi$.

**Taylor’s Theorem.** Let $f$ be $n$ times differentiable on an interval $I$ containing the point $\xi$. Then for every $x \in I$ one can find an intermediate point $\eta$ between $\xi$ and $x$ such that

$$f(x) = \underbrace{\sum_{k=0}^{n-1} \frac{f^{(k)}(\xi)}{k!}(x-\xi)^k}_{=P_{n-1}(x)} + \underbrace{\frac{f^{(n)}(\eta)}{n!}(x-\xi)^n}_{=E_n}.$$  \hspace{1cm} (2)

This formula allows us to estimate the error $E_n = f(x) - P_{n-1}(x)$ when we approximate the function $f$ with the polynomial $P_{n-1}$:

$$E_n = \frac{f^{(n)}(\eta)}{n!}(x-\xi)^n.$$  

Note that here the $n$-th derivative of $f$ is computed not at $\xi$ but at some intermediate point $\eta$, depending on $x$.

3 Power series

A **power series** is function of the form

$$f(x) = \sum_{n=0}^{\infty} a_n(x-\xi)^n.$$
Theorem. Every power series has a radius of convergence $r$ (possibly with $r = 0$ or $r = +\infty$), such that:

- If $|x - \xi| < r$, then the power series converges.
- If $|x - \xi| > r$, then the power series diverges.

On the interval $x \in (\xi - r, \xi + r)$ the function $f(x)$ is continuously differentiable. Its derivative is the sum of the series (obtained by differentiating each term)

$$f'(x) = \sum_{n=1}^{\infty} na_n(x - \xi)^{n-1}.$$

4 The Riemann integral

Let $f$ be a continuous function defined on an interval $[a, b]$. By a partition $\mathcal{P}$ of $[a, b]$ we mean a finite set of points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

The (lower) Riemann sum corresponding to the partition $\mathcal{P}$ is defined as

$$S(\mathcal{P}) = \sum_{i=1}^{n} (x_i - x_{i-1}) m_i, \quad m_i = \min \left\{ f(x); \ x \in [x_{i-1}, x_i] \right\}.$$

The Riemann integral of $f$ over the interval $[a, b]$ is defined as

$$\int_{a}^{b} f(x) \, dx = \sup_{\mathcal{P}} S(\mathcal{P}),$$

where the supremum is taken over all partitions of the interval $[a, b]$.

Figure 2: The Riemann sum corresponding to the partition $\mathcal{P}$ is the sum of the areas of the shaded rectangles.
Theorem (properties of the Riemann integral). Let \( f, g \) be continuous on \([a, b]\).

(i) If \( m \leq f(x) \leq M \), then \( m(b-a) \leq \int_a^b f(x) \, dx \leq M(b-a) \).

(ii) For every constant \( \lambda \) one has \( \int_a^b \lambda f(x) \, dx = \lambda \int_a^b f(x) \, dx \). Moreover
\[
\int_a^b (f + g)(x) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx .
\]

(iii) If \( a < c < b \), then
\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx .
\]

Fundamental Theorem of Calculus. Let \( f \) be continuous on the interval \([a, b]\), and define the integral function
\[
F(x) = \int_a^x f(t) \, dt .
\]
Then \( F \) is differentiable and satisfies \( F'(x) = f(x) \) at every point \( x \in (a, b) \).

A function \( F \) such that \( F' = f \) is called an antiderivative of \( f \).

- Riemann sums provide a way to define the integral \( \int_a^b f(x) \, dx \), and to approximately compute its value.

- The Fundamental Theorem of Calculus allows us to exactly compute the Riemann integral \( \int_a^b f(x) \, dx \), whenever an antiderivative of \( f \) is known.

Theorem. Let \( f, F : [a, b] \to \mathbb{R} \) be continuous functions such that \( F'(x) = f(x) \) for all \( x \in (a, b) \). Then we have
\[
\int_a^b f(x) \, dx = F(x) \bigg|_a^b = F(b) - F(a) .
\]

Substitution rule: Assume that the function \( \varphi : [a, b] \to \mathbb{R} \) has a continuous derivative. Let \( f \) be a continuous function, defined on the image \( \varphi([a, b]) \). Then
\[
\int_a^b f(\varphi(t)) \varphi'(t) \, dt = \int_{\varphi(a)}^{\varphi(b)} f(x) \, dx .
\]

Integration by parts. Let the functions \( F, G \) be differentiable on the interval \([a, b]\), with continuous derivatives. Then
\[
\int_a^b F'G \, dx = FG \bigg|_a^b - \int_a^b FG' \, dx .
\]