Math 401 - Introduction to Real Analysis

Review 4

1 Functions

A function \( f : A \rightarrow B \) assigns to each element \( a \in A \) a unique element \( b = f(a) \in B \).

The set \( A \) is the domain of \( f \). The range of \( f \) is the set \( f(A) = \{ f(a) ; \ a \in A \} \).

We say that \( f : A \rightarrow B \) is one-to-one if distinct elements of \( A \) have distinct images:

\[
a \neq a' \implies f(a) \neq f(a').
\]

We say that \( f : A \rightarrow B \) is onto if \( f(A) = B \), i.e. if for every element \( b \in B \) one can find an element \( a \in A \) such that \( f(a) = b \).

If \( f : A \rightarrow B \) is one-to-one and onto, then we can define the inverse function \( f^{-1} : B \rightarrow A \) by setting

\[
f^{-1}(b) = a \quad \text{if and only if} \quad f(a) = b.
\]

Consider a function \( f : S \rightarrow \mathbb{R} \) taking values inside the set of real numbers.

- We say that \( f \) is bounded above if there exists a number \( B \) (an upper bound) such that \( f(x) \leq B \) for all \( x \in S \). This is the same as saying that the set \( f(S) \) is bounded above.

- We say that \( f \) is bounded below if there exists a number \( b \) (a lower bound) such that \( f(x) \geq b \) for all \( x \in S \). This is the same as saying that the set \( f(S) \) is bounded below.

- If \( f \) is bounded above and below, we simply say that \( f \) is bounded. This is the same as saying that the set \( f(S) \) is bounded.

2 Limits of functions

Consider an interval \((a, b)\) containing the point \( \xi \). Let \( f \) be a function defined at all points \( x \in (a, b) \), with the possible exception of \( x = \xi \).
Definition. We say that \( L \) is the limit of \( f(x) \) as \( x \) approaches \( \xi \), and write
\[
\lim_{x \to \xi} f(x) = L
\] (1)
if the following holds. Given any \( \varepsilon > 0 \) we can find \( \delta > 0 \) such that the inequality
\[
|f(x) - L| < \varepsilon
\]
is satisfied for every \( x \) such that \( 0 < |x - \xi| < \delta \).

The limit (1) describes a possible behavior of the function \( f \) near the point \( \xi \) (shown in Fig.1). Intuitively, this means:

\textit{as \( x \) gets closer and closer to \( \xi \), the values \( f(x) \) become closer and closer to \( L \).}

• According to the definition, to prove that the limit (1) holds, given any \( \varepsilon > 0 \) we need to study the inequalities
\[
L - \varepsilon < f(x) < L + \varepsilon
\]
and show that they are both satisfied whenever \( x \in (\xi - \delta, \xi + \delta) \), \( x \neq \xi \), for some \( \delta > 0 \) sufficiently small.

• The actual value \( f(\xi) \) is irrelevant in the definition of limit. In many interesting cases, the function \( f \) is not even defined at the point \( \xi \).

A very different possible behavior of a function \( f \) near a point \( \xi \) is shown in Fig. 2:

\textit{as \( x \) gets closer and closer to \( \xi \), the values \( f(x) \) become arbitrarily large (and positive).}

The next definition describes this behavior in a precise way.
**Definition.** We say that \( f(x) \) diverges to \(+\infty\) as \( x \) approaches \( \xi \), and write

\[
\lim_{x \to \xi} f(x) = +\infty
\]

if the following holds. Given any \( H > 0 \) we can find \( \delta > 0 \) such that the inequality

\[
f(x) > H
\]

is satisfied for every \( x \) such that \( 0 < |x - \xi| < \delta \).

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**Figure 2:** As \( x \) ranges in the interval \((\xi - \delta, \xi + \delta)\), the values \( f(x) \) are always \( > H \).

Limits can also describe the behavior of a function \( f \) for very large values of \( x \) (Fig.3).

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**Figure 3:** For every \( x > H \) one has \(|f(x) - L| < \varepsilon\).

**Definition.** We say that \( f(x) \) converges to \( L \) as \( x \) tends to \(+\infty\), and write

\[
\lim_{x \to +\infty} f(x) = L
\]
if the following holds. Given any \( \varepsilon > 0 \) we can find \( H > 0 \) such that the inequality
\[
|f(x) - L| < \varepsilon
\]
is satisfied for every \( x \) such that \( x > H \).

3 Continuous functions

**Definition.** A function \( f \) is continuous at a point \( \xi \) if
\[
\lim_{x \to \xi} f(x) = f(\xi).
\]  (4)

**Note:** this definition makes three requirements:

(i) The limit \( \lim_{x \to \xi} f(x) \) exists.

(ii) The function \( f \) is defined at the point \( x = \xi \).

(iii) The equality (4) holds.

More generally, we say that a function \( f \) is **continuous** if it is continuous at every point where it is defined.

**Properties of continuous functions**

Let \( f, g \) be continuous functions. Then

- for every constant \( \lambda \in \mathbb{R} \), the function \( \lambda f \) is continuous,
- the sum \( f + g \) is continuous,
- the product \( f \cdot g \) is continuous,
- the quotient \( f/g \) is continuous,
- the composition \((f \circ g)(x) = f(g(x))\) is continuous.

**Examples:** Every polynomial \( P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \) is a continuous function, defined for all \( x \in \mathbb{R} \).
Every rational function \( P(x)/Q(x) \) is a continuous function, defined at all points where \( Q(x) \neq 0 \).

**Theorem.** Let \( f : [a, b] \to \mathbb{R} \) be a continuous function defined on a closed bounded interval. Then the image \( f([a, b]) = [m, M] \) is a closed bounded interval. Namely:

(i) There exists a point \( x_M \in [a, b] \) where \( f \) attains its maximum: \( f(x_M) = M = \max_{x \in [a, b]} f(x) \).

(ii) There exists a point \( x_m \in [a, b] \) where \( f \) attains its minimum: \( f(x_m) = m = \min_{x \in [a, b]} f(x) \).

(iii) As \( x \) ranges over \( [a, b] \), the function \( f \) attains all values between its minimum and its maximum.