1 Subsequences

Starting with a sequence \((x_1, x_2, x_3, x_4, \ldots)\), by choosing an increasing set of natural numbers \(n_1 < n_2 < n_3 < \cdots\), we obtain the subsequence \((x_{n_1}, x_{n_2}, x_{n_3}, \ldots)\).

Examples: \((x_2, x_4, x_6, \ldots)\) is a subsequence, 
\((x_1, x_3, x_5, x_7, \ldots)\) is another subsequence, 
\((x_1, x_2, x_4, x_8, x_{16}, \ldots, x_{2^j}, \cdots)\) is yet another subsequence.

![Figure 1: Start with the sequence \((x_1, x_2, x_3, x_4, \ldots)\), by choosing \(\{n_1, n_2, n_3, \cdots\}\) to be the set of prime numbers we obtain the subsequence \((x_2, x_3, x_5, x_7, x_{11}, x_{13}, x_{17}, x_{19}, \cdots)\).]

**Theorem.** From every sequence one can extract a monotone (either increasing, or decreasing) subsequence.

**Theorem (Bolzano-Weierstrass).** From every bounded sequence one can extract a convergent subsequence.

2 Cauchy sequences

A sequence \((x_n)\) converges to a limit \(L\) if

\[
\text{for any } \varepsilon > 0 \text{ we can find } N_\varepsilon \text{ such that } |x_n - L| < \varepsilon \text{ for all } n > N_\varepsilon. \quad (1)
\]

A sequence \((x_n)\) is a Cauchy sequence if

\[
\text{for any } \varepsilon > 0 \text{ we can find } N_\varepsilon \text{ such that } |x_n - x_m| < \varepsilon \text{ for all } m, n > N_\varepsilon. \quad (2)
\]
Note:

- In the definition of limit, (1) says that, as \( n \to \infty \), the values \( x_n \) get closer and closer to \( L \).
- In the definition of Cauchy sequence, (2) says that, as \( m, n \to \infty \), the values \( x_m, x_n \) get closer and closer to each other.

**Theorem.** A sequence \((x_n)\) is a Cauchy sequence if and only if it converges to some limit \( L \).

3 Series

A **series** is a sum of infinitely many terms: \( \sum_{n=1}^{\infty} a_n \).

- The **partial sums** of this series are computed as
  \[
  s_N = a_1 + a_2 + \cdots + a_N = \sum_{n=1}^{N} a_n.
  \]

- We say that the series **converges to** \( S \) (or equivalently that \( S \) is the **sum of the series**) if \( S \) is the limit of the partial sums: \( \lim_{N \to \infty} s_N = S \). We then write \( \sum_{n=1}^{\infty} a_n = S \).
- If \( \lim_{N \to \infty} s_N = +\infty \), then we say that the series **diverges**, and write \( \sum_{n=1}^{\infty} a_n = +\infty \).

**Some important series:**

(i) A convergent series: if \( 0 \leq \lambda < 1 \), then
   \[
   \sum_{n=0}^{\infty} \lambda^n = \lim_{N \to \infty} (1 + \lambda + \lambda^2 + \cdots + \lambda^N) = \lim_{N \to \infty} \frac{1 - \lambda^{N+1}}{1 - \lambda} = \frac{1}{1 - \lambda}.
   \]

(ii) If \( \alpha > 1 \), then the series \( \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \) converges.

(iii) A divergent series: \( \sum_{n=1}^{\infty} \frac{1}{n} = +\infty \).

By a **comparison** with these series, we can prove the convergence or divergence of other series.
• If the series \( \sum_{n=1}^{\infty} a_n \) converges, then for any constant \( C \) the series \( \sum_{n=1}^{\infty} Ca_n \) converges as well.

• If the series \( \sum_{n=1}^{\infty} a_n \) diverges, then for any constant \( C \neq 0 \) the series \( \sum_{n=1}^{\infty} Ca_n \) diverges as well.

• If the series \( \sum_{n=1}^{\infty} b_n \) converges, and \( |a_n| \leq b_n \) for every \( n \), then the series \( \sum_{n=1}^{\infty} a_n \) converges as well.

• If \( \sum_{n=1}^{\infty} b_n = +\infty \) and \( a_n \geq b_n \) for every \( n \), then also \( \sum_{n=1}^{\infty} a_n = +\infty \).

**Ratio test:**

- If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \), then the series \( \sum_{n=1}^{\infty} a_n \) converges.

- If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1 \), then the series \( \sum_{n=1}^{\infty} a_n \) diverges.

**Further properties of series:**

- Consider two series which converge: \( \sum_{n=1}^{\infty} a_n = A \), \( \sum_{n=1}^{\infty} b_n = B \).

  Then for any numbers \( \lambda, \mu \) one has the convergence

  \[
  \sum_{n=1}^{\infty} (\lambda a_n + \mu b_n) = \lambda A + \mu B.
  \]

- If the series \( \sum_{n=1}^{\infty} a_n \) converges, then \( \lim_{n \to \infty} a_n = 0 \).

  (Note that the converse is not true: the series \( \sum_{n=1}^{\infty} \frac{1}{n} \) does not converge.)

- **(Series with alternating signs).** If the sequence \( (a_n) \) is decreases to zero, then the series \( \sum_{n=0}^{\infty} (-1)^n a_n \) converges.