Hyperbolic Systems of Conservation Laws

in One Space Dimension

IV - Viscous approximations

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Vanishing Viscosity Approximations

Goal: show that *entropy admissible* weak solutions of the hyperbolic system

$$u_t + f(u)_x \;=\; 0$$

can be obtained as limits of solutions to the parabolic system

$$u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon\, u^\varepsilon_{xx}$$

letting the viscosity $\varepsilon \to 0+$

Three basic cases

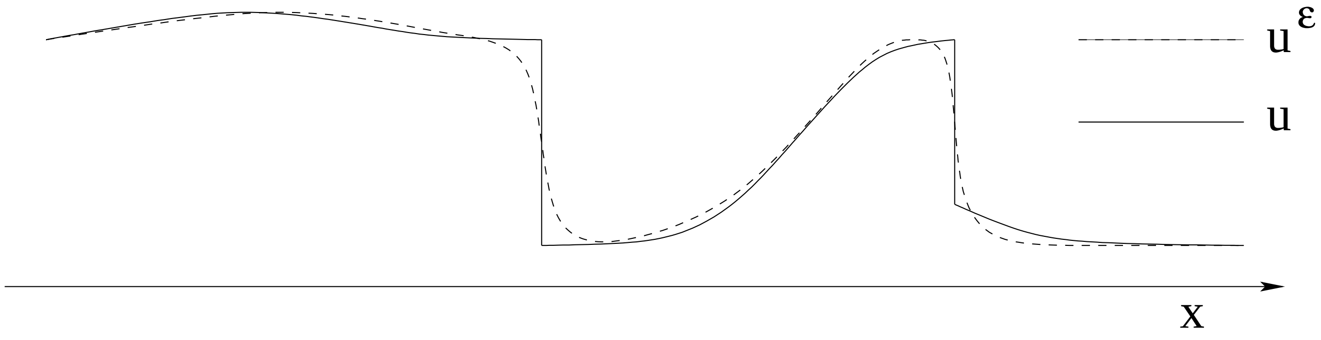
1. Smooth solutions. Let $u = u(t,x)$ be a smooth solution of the hyperbolic system

$$u_t + A(u)u_x = 0 \qquad u(0,x) = \bar{u}(x)$$

Then a standard perturbation argument yields: $u = \lim_{\varepsilon\to 0+} u^\varepsilon$, where

$$u^\varepsilon_t + A(u^\varepsilon)u^\varepsilon_x = \varepsilon u^\varepsilon_{xx} \qquad u^\varepsilon(0,x) = \bar{u}(x)$$

Remark. The conclusion is far from obvious if u contains jumps.



If u piecewise smooth, then *singular perturbation* techniques can be used (J.Goodman - Z.Xin, 1992)

2. Scalar conservation laws.

$$v_t + f'(v)v_x \;=\; \varepsilon v_{xx}$$

Total variation of the viscous solutions is non-increasing in time

$$(v_x)_t + [f'(v)v_x]_x \;=\; \varepsilon (v_x)_{xx}$$

Uniform BV bound $\Longrightarrow$ a subsequence converges strongly in $\mathbf{L}^1_{loc}$

(A. I. Volpert 1968, S. Kruzhkov 1970)

3. Linear Hyperbolic Systems.

$$u_t + Au_x \ = \ 0 \qquad\qquad u(0,x) \ = \ \bar{u}(x)$$

$$u^\varepsilon_t + Au^\varepsilon_x \ = \ \varepsilon u^\varepsilon_{xx} \qquad\qquad u(0,x) \ = \ \bar{u}(x)$$

$$l_i A = \lambda_i l_i \qquad\qquad A r_i = \lambda_i r_i \qquad\qquad i = 1, \ldots, n$$

Solve componentwise: $u_i \doteq l_i \cdot u$

$$(u_i)_t + \lambda_i (u_i)_x \ = \ 0 \qquad\qquad u_i(0,x) \ = \ l_i \cdot \bar{u}(x)$$

= limit of scalar viscous approximations $u_i^\varepsilon = l_i \cdot u^\varepsilon$

$$(u_i^\varepsilon)_t + \lambda_i (u_i^\varepsilon)_x \ = \ \varepsilon \, (u_i^\varepsilon)_{xx} \qquad\qquad u_i(0,x) \ = \ l_i \cdot \bar{u}(x)$$

Nonlinear Viscous Hyperbolic Systems

$$u_t + A(u)u_x \;=\; u_{xx}$$

BV bounds: estimating gradient components

Natural approach: decompose the gradient u_x along eigenvectors of $A(u)$

right, left eigenvectors: $r_i(u), \quad l_i(u),$ eigenvalues: $\lambda_i(u)$

normalization: $|r_i(u)| \;\equiv\; 1,$ $l_i \cdot r_j = \begin{cases} 1 & \text{if } \; i = j \\ 0 & \text{if } \; i \neq j \end{cases}$

$$u_x = \sum_i u_x^i r_i \qquad\qquad u_x^i \doteq l_i \cdot u_x$$

$$u_t + \sum_i \lambda_i u_x^i r_i \;=\; \sum_i (u_x^i r_i)_x$$

$$(u_x^i)_t + (\lambda_i u_x^i)_x - (u_x^i)_{xx} \;=\; \phi^i(u, u_x^1, \ldots, u_x^n) \quad \doteq \quad l_i \cdot \sum_{j<k} \lambda_k [r_k, r_j] u_x^j u_x^k$$

$$+\; l^i \cdot \left\{ 2 \sum_{j,k} (r_k \bullet r_j)(u_x^j)_x u_x^k + \sum_{j,k,\ell} \Big(r_\ell \bullet (r_k \bullet r_j) - (r_\ell \bullet r_k) \bullet r_j \Big) u_x^j u_x^k u_x^\ell \right\}$$

$r_k \bullet r_j \doteq (Dr_j) r_k$ (directional derivative of r_j along r_k)

$[r_k, r_j] \;=\; (Dr_j) r_k - (Dr_k) r_j$ (Lie bracket)

$$\text{Tot.Var.}\{u\} \;=\; \|u_x\|_{\mathbf{L}^1} \;\leq\; \sum_{i=1}^n \|u_x^i\|_{\mathbf{L}^1}$$

Set $v^i \doteq u_x^i$. Seek an estimate on the $\mathbf{L}^1$-norm of solutions to

$$v_t^i + \left(\lambda_i(u) v^i\right)_x - v_{xx}^i \;\doteq\; \phi^i(u, v^1, \ldots, v^n)$$

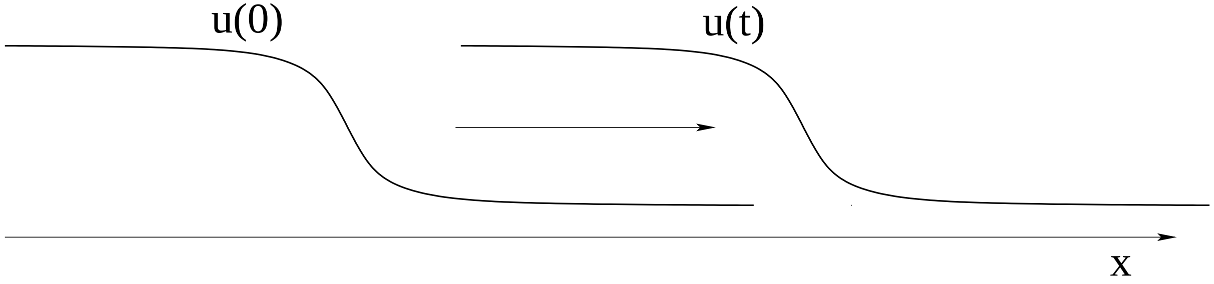
$$\|v^i(t)\|_{\mathbf{L}^1} \;\leq\; \|v^i(0)\|_{\mathbf{L}^1} + \int_0^t \|\phi^i(\tau)|_{\mathbf{L}^1}\, d\tau$$

To achieve uniform BV bounds, one needs: $\int_0^\infty \int |\phi^i|\, dx dt \;<\; \infty$

FALSE, in general !

Example: for a viscous travelling wave $u(t,x) = \bar u(x-\lambda t)$ source terms do NOT vanish: $\phi_i \neq 0$

$$\int_0^t \int |\phi^i(\tau,x)|\, dx d\tau \;=\; t\cdot \int |\phi^i(0,x)|\, dx \;\to\; \infty \qquad \text{as} \quad t\to\infty$$



Key idea: decompose u_x NOT along eigenvectors $r_1,\ldots,r_n$ of $A(u)$, but using a basis $\{\tilde r_1,\ldots \tilde r_n\}$ of **gradients of viscous travelling waves**.

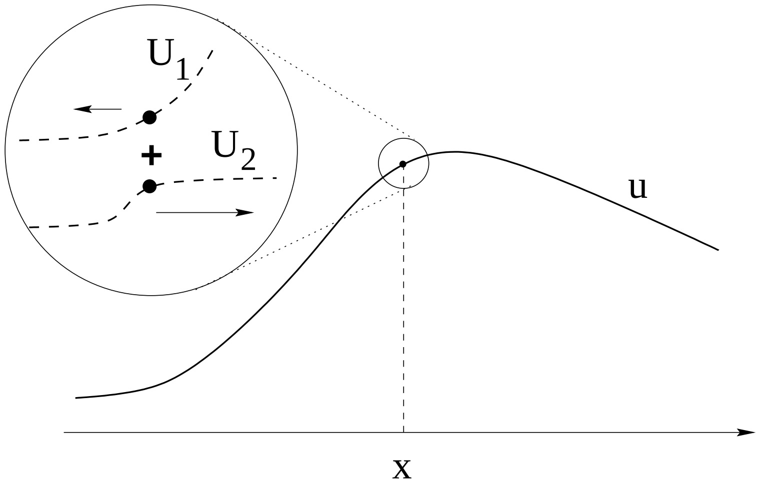
Viscous travelling waves

$$u_t + A(u)u_x \;=\; u_{xx}$$

travelling wave solution: $u(t,x) = U(x-\sigma t)$ $\qquad U'' = (A(U)-\sigma)U'$

Remark. In the scalar case $\sigma = -U_t/U_x$

Goal: decompose $u_x = \sum U_i'$ as sum of gradients of travelling waves



Given (u, u_x, u_{xx}) at a point x, find travelling wave profiles $U_1, \ldots, U_n$ such that

$$U_i'' = (A(U_i) - \sigma_i)U_i' \qquad U_i(x) = u(x) \qquad i = 1, \ldots, n$$

$$\sum_i U_i'(x) = u_x(x)\,, \qquad \sum_i U_i''(x) = u_{xx}(x)$$

Having fixed $u(x)$, one has

- $n + n$ equations
- $n^2 + n$ free parameters: $U_1'(x), \ldots, U_n'(x)$, $\sigma_1, \ldots, \sigma_n$

UNDER-DETERMINED !

For each given state $u \in I\!\!R^n$ and $i = 1, \ldots, n$, we need to select a 2-parameter family of travelling waves through u

Basic technique: **center manifold theorem**

second order O.D.E. describing travelling waves: $U'' = (A(U) - \sigma) U'$

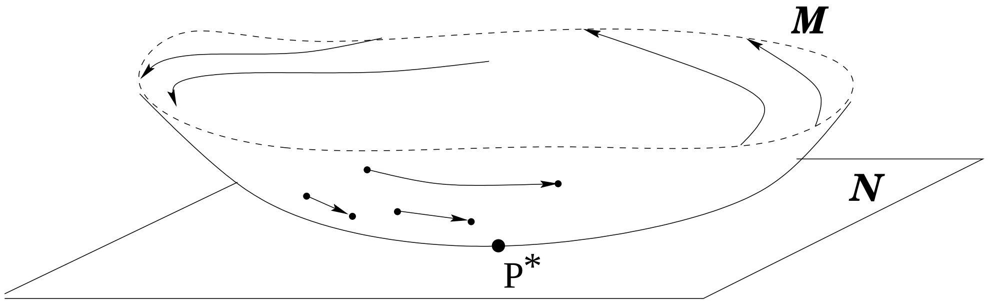
equivalent first order system:
$$\begin{cases} \dot{u} &= v \\ \dot{v} &= (A(u) - \sigma) v \\ \dot{\sigma} &= 0 \end{cases}$$

Linearize at the equilibrium point $P^* = (u^*, 0, \lambda_i(u^*))$

$$\begin{pmatrix} \dot{u} \\ \\ \dot{v} \\ \\ \dot{\sigma} \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ \\ 0 & A(u^*) - \lambda_i(u^*) I & 0 \\ \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ \\ v \\ \\ \sigma \end{pmatrix} \in \mathbb{R}^{n+n+1}$$

Center subspace (= invariant subspace corresponding to eigenvalues with zero real part) has dimension $n + 2$

$\Longrightarrow$ the nonlinear system has a center manifold of dimension $n + 2$, tangent to the center subspace at P^*



$$\sum_i U_i'(x) = u_x(x)\,, \qquad\qquad \sum_i U_i''(x) = u_{xx}(x)$$

Choose travelling waves U_i corresponding to trajectories on the center manifold $\Longrightarrow \quad n+n$ equations for $2n$ variables

$$u_x(x) = \sum_i U_i'(x) = \sum_i v^i \tilde{r}_i(u, v^i, \sigma_i) \qquad\qquad |\tilde{r}_i| \equiv 1$$

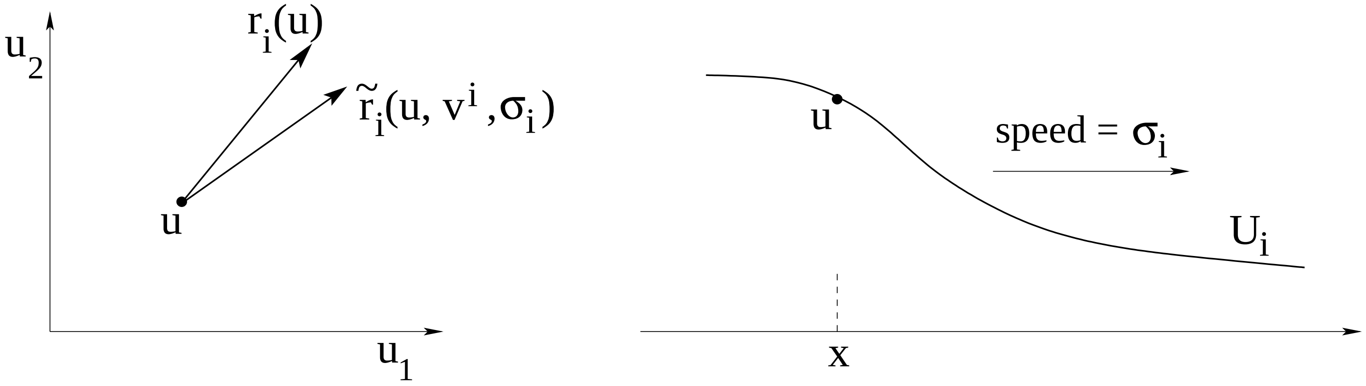
v^i = signed strength, $\qquad \sigma_i$ = wave speed

Coordinates on the Center Manifold

Given $(u, v^i, \sigma_i) \in I\!R^n \times I\!R \times I\!R$ with $u \approx u^*$, $v^i \approx 0$, $\sigma_i \approx \lambda_i(u^*)$

there exists a unique travelling profile U_i on the center manifold,

with speed σ_i and signed strength v^i



$$U_i(x) = u, \qquad U_i'(x) = v^i \tilde{r}_i \qquad |\tilde{r}_i| = 1, \qquad U_i'' = (A(U_i) - \sigma_i) U_i'$$

A key identity

Generalized eigenvector: $\tilde r_i(u, v^i, \sigma_i)$ $\qquad |\tilde r_i| = 1$

Generalized eigenvalue: $\tilde\lambda_i(u, v^i, \sigma_i) \doteq \langle \tilde r_i\,,\ A(u)\tilde r_i\rangle$

$$
\begin{array}{lrcl}
\text{yields} & (A(u)-\tilde\lambda_i)\tilde r_i & = & v^i\big(\tilde r_{i,u}\tilde r_i + \tilde r_{i,v}(\tilde\lambda_i - \sigma_i)\big) \\
\text{intead of} & (A(u)-\lambda_i) r_i & = & 0
\end{array}
$$

Evolution of Gradient Components

$$u_t = u_{xx} - A(u)u_x \qquad\qquad (u, u_x, u_{xx}) \longleftrightarrow (u, u_x,\ u_t)$$

Given u, u_x, u_{xx} small, there exists a unique decomposition

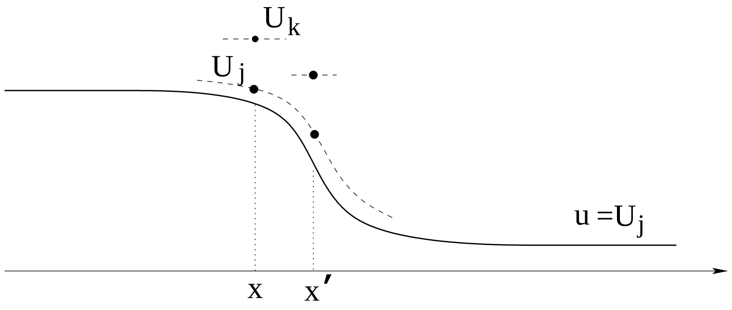
$$u_x = \sum v^i \tilde r_i(u, v^i, \sigma_i) \qquad\qquad u_t = \sum w^i \tilde r_i(u, v^i, \sigma_i) \qquad\qquad \sigma_i \approx -\frac{w^i}{v^i}$$

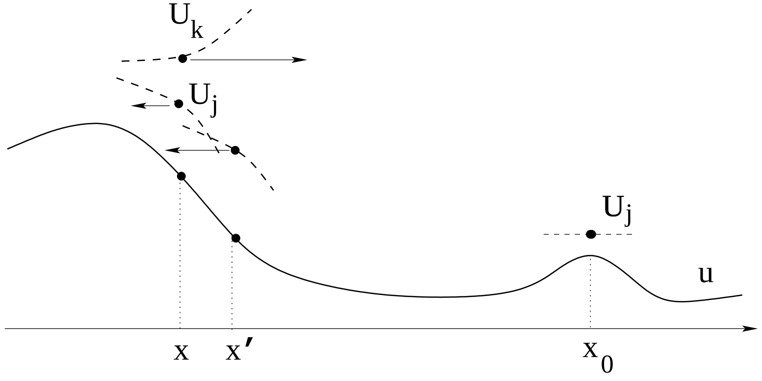
After N^{N^N} computations one finds

$$\begin{cases} v^i_t + (\tilde\lambda_i v^i)_x - v^i_{xx} & = \phi_i \\ w^i_t + (\tilde\lambda_i w^i)_x - w^i_{xx} & = \psi_i \end{cases}$$

$$\begin{aligned} \phi_i,\ \psi_i = \ & \mathcal{O}(1)\cdot \sum |w^j + \sigma_j v^j| \cdot [|v^j w^j| + |v^j_x| + |w^j_x|] && \textbf{(wrong speed)} \\ & + \mathcal{O}(1)\cdot \sum |w^j_x v^j - v^j_x w^j| && \textbf{(change in strength)} \\ & + \mathcal{O}(1)\cdot \sum \left| v^j \left(\tfrac{w^j}{v^j}\right)_x \right|^2 && \textbf{(change in speed)} \\ & + \mathcal{O}(1)\cdot \sum_{j\neq k} [|v^j v^k| + |v^j_x v^k| + |v^j w^k| + |v^j_x w^k| + |w^j w^k|] \\ & && \textbf{(interaction of waves of different families)} \end{aligned}$$

Motivation of the Source Terms





Wrong speed: Near a point x_0 where $u_x = 0$, the speed of a travelling wave $\sigma = -u_t/u_x \to \infty$. A cut-off function is used.

Change in wave speed, or strength: The viscous travelling j-wave that best approximates u near a point x is not the same at a nearby point x'

Transversal wave interactions: At a point x, waves of distinct families $j \neq k$ are present

KEY ESTIMATES

$$\begin{cases} v^i_t + (\tilde{\lambda}_i v^i)_x - v^i_{xx} & = \phi_i \\ w^i_t + (\tilde{\lambda}_i w^i)_x - w^i_{xx} & = \psi_i \end{cases}$$

To prove that $\int_0^\infty \|\phi_i(\tau)\|_{\mathbf{L}^1}\, d\tau < \infty, \qquad \int_0^\infty \|\psi_i(\tau)\|_{\mathbf{L}^1}\, d\tau < \infty,$
we construct suitable Lyapunov functionals $\Psi(u) \geq 0$ such that

$$\|\phi_i(t)\|_{\mathbf{L}^1}, \|\psi_i(t)\|_{\mathbf{L}^1} \ \leq \ -\frac{d}{dt}\Psi(u(t))$$

[$\mathbf{L}^1$ norm of source terms] $\leq$ [rate of decrease of the functional]

Wrong speed $\Longrightarrow$ Energy estimates for the heat equation

Change in wave strength $\Longrightarrow$ Area functional

Change in wave speed $\Longrightarrow$ Curve length functional

Interaction of waves of different families $\Longrightarrow$ Wave interaction potential

Lyapunov functionals for a pair of linear parabolic equations (controlling the interactions of waves of distinct families)

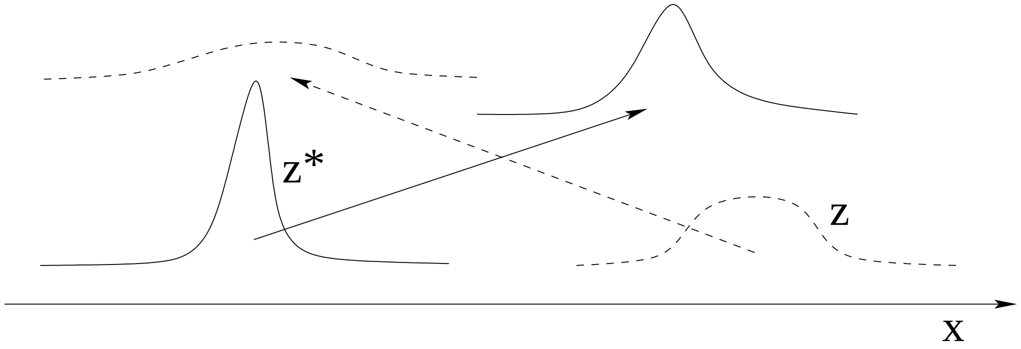
$$\begin{cases} z_t + [\lambda(t,x) z]_x - z_{xx} & = 0 \\ z^*_t + [\lambda^*(t,x) z^*]_x - z^*_{xx} & = 0 \end{cases}$$

assume: $\inf_{t,x} \lambda^*(t,x) - \sup_{t,x} \lambda(t,x) \ \geq \ c \ > \ 0$

$z \doteq$ density of waves with (slow) speed λ

$z^* \doteq$ density of waves with (fast) speed λ^*

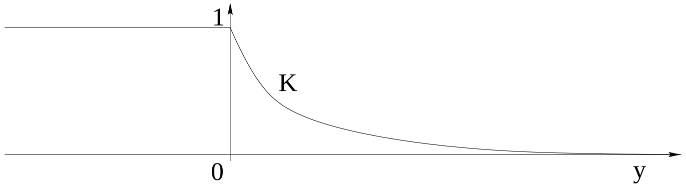
Instantaneous amount of interaction: $I(t) \ \doteq \ \int |z(t,x)| \cdot |z^*(t,x)|\, dx$



A Potential for Transversal Wave Interactions

$$Q(z,z^*) \doteq \iint_{\mathbb{R}^2} K(x_2-x_1)|z(x_1)|\,|z^*(x_2)|\,dx_1dx_2$$

$$K(y) \doteq \begin{cases} e^{-cy/2} & \text{if} \quad y>0 \\ 1 & \text{if} \quad y\le 0 \end{cases}$$



$$\begin{aligned}\int_0^\infty I(t)\,dt &= \int_0^\infty \int_{I\!\!R} |z(t,x)|\,|z^*(t,x)|\,dxdt \;\leq\; -\int_0^\infty \left[\frac{d}{dt}\frac{Q(t)}{c}\right] dt\\ &\leq \frac{Q(0)}{c} \;\leq\; \frac{1}{c}\int_{I\!\!R} |z(0,x)|\,dx \cdot \int_{I\!\!R} |z^*(0,x)|\,dx\end{aligned}$$

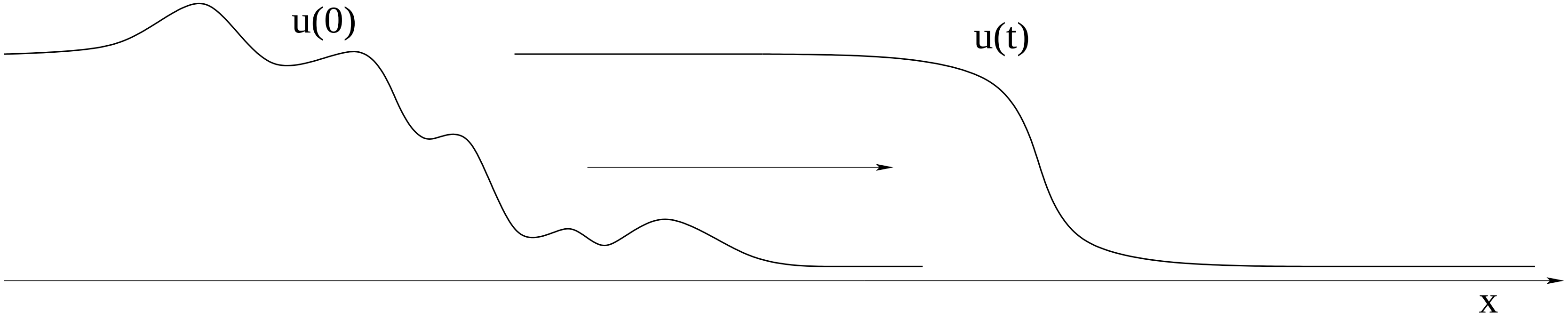
This functional controls the source terms

$$\mathcal{O}(1)\cdot\sum_{j\neq k}\left[|v^j v^k| + |v^j_x v^k| + |v^j w^k| + |v^j_x w^k| + |w^j w^k|\right]$$

(interaction of waves of different families)

Lyapunov functionals for a scalar viscous conservation law (controlling the interactions of waves of the same family)

$$u_t + f(u)_x = u_{xx} \qquad\qquad u \in I\!\!R$$



As $t \to \infty$, the solution profile approaches a viscous travelling wave

Seek: Lyapunov functionals describing how far u is from a viscous travelling wave profile

Variable transformation

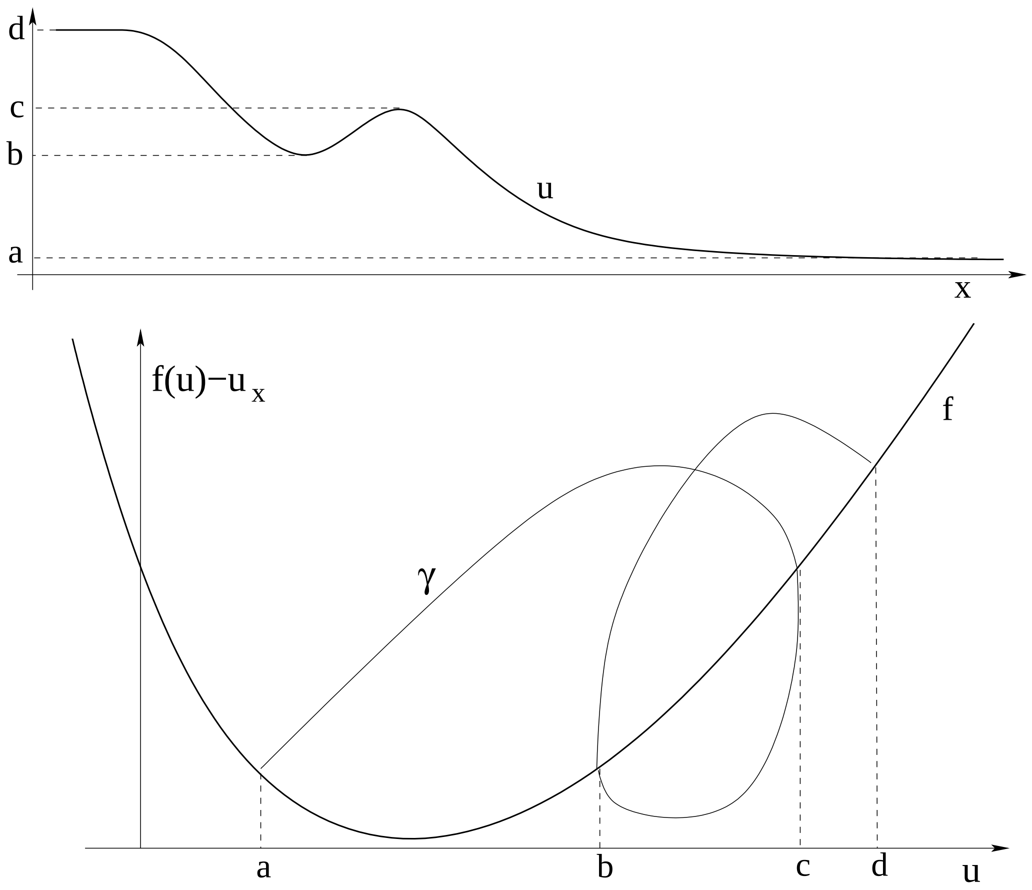
Given $u = u(x)$, consider the curve

$$\gamma \doteq \begin{pmatrix} u \\ f(u) - u_x \end{pmatrix} \;=\; \begin{pmatrix} \text{conserved quantity} \\ \text{flux} \end{pmatrix}$$

Remark: u is a travelling wave profile $\iff$ γ is a segment

Proof. $$-\frac{u_t}{u_x} \;=\; \frac{f(u)_x - u_{xx}}{u_x} \;=\; \text{constant} \;=\; \text{[wave speed]}$$

if and only if $$\frac{d}{du}\big[f(u) - u_x\big] \;=\; \big[f(u) - u_x\big]_x \cdot \frac{1}{u_x} \;=\; \text{constant}$$



$$\gamma \doteq \begin{pmatrix} u \\ f(u) - u_x \end{pmatrix} = \begin{pmatrix} \text{conserved quantity} \\ \text{flux} \end{pmatrix}$$

$$u_t + f(u)_x = u_{xx} \qquad \Longrightarrow \qquad \gamma_t + f'(u)\gamma_x = \gamma_{xx}$$

Curve Length

$$\gamma \doteq \begin{pmatrix} u \\ f(u) - u_x \end{pmatrix} \qquad \gamma_x = \begin{pmatrix} v \\ w \end{pmatrix} \doteq \begin{pmatrix} u_x \\ -u_t \end{pmatrix}$$

Length $$L(\gamma) \doteq \int |\gamma_x| \, dx = \int \sqrt{v^2 + w^2} \, dx$$

$$\gamma_t + f'(u)\gamma_x = \gamma_{xx} \qquad \Longrightarrow \qquad \frac{dL}{dt} \leq 0$$

$$\begin{aligned} -\frac{d}{dt}L(\gamma(t)) &= \int_{I\!R} \frac{|v|\,\big[(w/v)_x\big]^2}{(1+(w/v)^2)^{3/2}}\;dx \\ &\geq \frac{1}{(1+\delta^2)^{3/2}} \int_{|w/v|\leq\delta} |v|\,\big[(w/v)_x\big]^2\;dx \end{aligned}$$

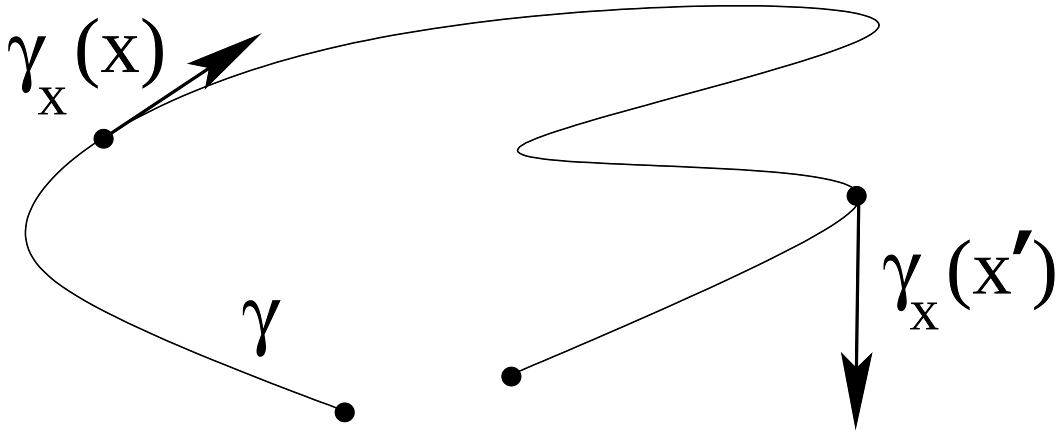
$$\begin{aligned} \int_0^\infty \int_{|w/v|\leq\delta} |v|\,\big[(w/v)_x\big]^2\,dx\,dt &= \mathcal{O}(1)\cdot\int_0^\infty \left|\frac{d}{dt}L(\gamma(t))\right|\,dt \\ &= \mathcal{O}(1)\cdot L(\gamma(0)) \end{aligned}$$

This functional controls the source terms

$$\mathcal{O}(1)\cdot\left|v^j\left(\frac{w^j}{v^j}\right)_x\right|^2 \qquad\qquad \textbf{(change in speed)}$$

Area Swept by a Curve

$$\gamma_t + f'(u)\gamma_x = \gamma_{xx}$$

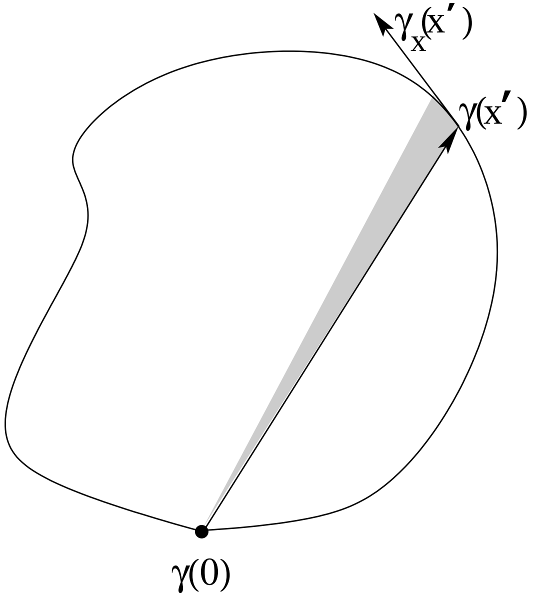


Area functional $$Q(\gamma) \doteq \frac{1}{2}\iint_{x<x'} \left|\gamma_x(x) \wedge \gamma_x(x')\right| dx\, dx'$$

Remark: for a simple closed curve: $\gamma : [0, 1] \mapsto \mathbb{R}^2$

$$\frac{1}{2}\iint_{x<x'} \gamma_x(x) \wedge \gamma_x(x')\, dx\, dx' \;=\; \frac{1}{2}\int_0^{x'} [\gamma(x') - \gamma(0)] \wedge \gamma_x(x')\, dx'$$

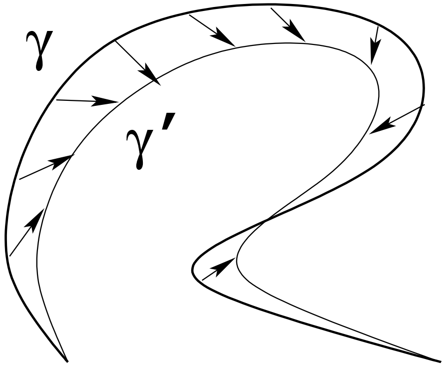
= Area of region enclosed by the curve

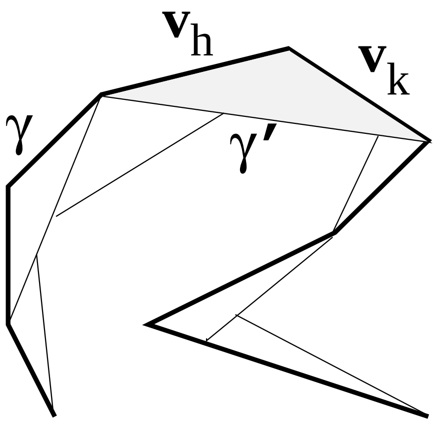


Area functional $Q(\gamma) \doteq \frac{1}{2} \iint_{x<x'} \left| \gamma_x(x) \wedge \gamma_x(x') \right| dx\, dx'$

If γ evolves in the direction of curvature, then

Q controls the area swept by the curve γ: $|dA| \leq -dQ$





$$Q(\gamma) \;=\; \frac{1}{2} \sum_{i<j} |\mathbf{v}_i \wedge \mathbf{v}_j| \qquad\qquad |dA| \;=\; \frac{1}{2} |\mathbf{v}_h \wedge \mathbf{v}_k| \leq -dQ$$

$$\gamma_t + f'(u)\gamma_x \;=\; \gamma_{xx} \qquad\qquad \gamma_x \;=\; \begin{pmatrix} v \\ w \end{pmatrix} \doteq \begin{pmatrix} u_x \\ -u_t \end{pmatrix}$$

$$-\frac{dQ}{dt} \;\geq\; \left|\frac{dA}{dt}\right| \;=\; \int |\gamma_t \wedge \gamma_x|\, dx \;=\; \int |\gamma_{xx} \wedge \gamma_x|\, dx \;=\; \int |v_x w - v w_x|\, dx$$

$$\int_0^\infty \int |v_x w - v w_x|\, dx\, dt \;\leq\; \int_0^\infty \left|\frac{dQ(\gamma(t))}{dt}\right| dt \;\leq\; Q(\gamma(0))$$

This functional controls the source terms

$\mathcal{O}(1) \cdot |v_x^j w^j - v^j w_x^j|$ **(change in strength)**

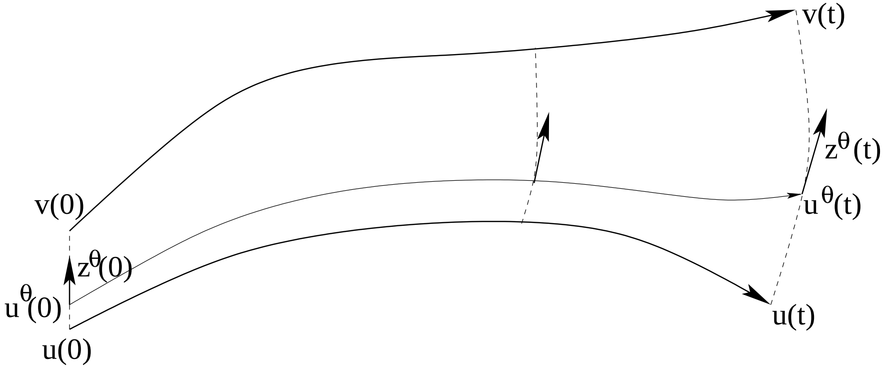
Linearized evolution equation for a first order perturbation

$$u_t + A(u)u_x = u_{xx} \qquad\qquad u^\epsilon = u + \epsilon z + O(\epsilon^2)$$

$$z_t + [DA(u)\cdot z]u_x + A(u)z_x \;=\; z_{xx}$$

Basic estimate: $\qquad \|z(t)\|_{\mathbf{L}^1} \leq C\,\|z(0)\|_{\mathbf{L}^1}$

homotopy argument $\Longrightarrow$ continuous dependence on initial data



Rescaling

$u^\varepsilon(t,x)$ is a solution to the Cauchy problem with small viscosity

$$u^\varepsilon_t + A(u^\varepsilon)u^\varepsilon_x \;=\; \varepsilon\, u^\varepsilon_{xx} \qquad\qquad u^\varepsilon(0,x) \;=\; \bar u(x)$$

if and only if

$u(t,x) \doteq u^\varepsilon(\varepsilon t, \varepsilon x)$ is a solution to the system with unit viscosity

$$u_t + A(u)u_x = u_{xx} \qquad\qquad u(0,x) = \bar u(\varepsilon x)$$

Uniform BV bounds + stability estimates for u

$\Longrightarrow$ uniform BV bounds + stability estimates for u^ε, for all $\varepsilon > 0$

Theorem (S. Bianchini, A. Bressan, *Annals of Math.* 2005). Consider a strictly hyperbolic system with viscosity

$$u_t + A(u)u_x = \varepsilon\, u_{xx} \qquad\qquad u(0,x) = \bar u(x)\,. \qquad\qquad (CP)$$

If Tot.Var.$\{\bar u\}$ is sufficiently small, then (CP) admits a unique solution $u^\varepsilon(t,\cdot) = S_t^\varepsilon \bar u$, defined for all $t \geq 0$. Moreover, we have the estimates

$$\text{Tot.Var.}\{S_t^\varepsilon \bar u\} \;\leq\; C\,\text{Tot.Var.}\{\bar u\}\,, \qquad\qquad \textbf{(BV bounds)}$$

$$\|S_t^\varepsilon \bar u - S_t^\varepsilon \bar v\|_{\mathbf{L}^1} \;\leq\; L\,\|\bar u - \bar v\|_{\mathbf{L}^1} \qquad\qquad \textbf{(}\mathbf{L}^1\textbf{ stability)}$$

(Convergence) As $\varepsilon \to 0$, the solutions u^ε converge to the trajectories of a semigroup S such that

$$\|S_t \bar u - S_t \bar v\|_{\mathbf{L}^1} \;\leq\; L\,\|\bar u - \bar v\|_{\mathbf{L}^1} \qquad\qquad t \geq 0\,.$$

These vanishing viscosity limits can be regarded as the unique **viscosity solutions** of the hyperbolic Cauchy problem

$$u_t + A(u)u_x = 0 \qquad\qquad u(0,x) = \bar{u}(x)\,.$$

- In the conservative case $A(u) = Df(u)$, the viscosity solutions are weak solutions of

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \bar{u}(x)$$

satisfying the Liu entropy conditions.

Extensions ?

- vanishing viscosity limits for hyperbolic balance laws, with dissipative sources (C.Christoforou, *J. Differential Equations* 2006)

- vanishing viscosity limits for hyperbolic systems on half line, with boundary conditions (F.Ancona, S.Bianchini, to appear ?)

- general viscosity matrix: $u_t + f(u)_x \; = \; \varepsilon(B(u)u_x)_x$

(main open problem)