

Hyperbolic Systems of Conservation Laws in One Space Dimension

III - Uniqueness and continuous dependence

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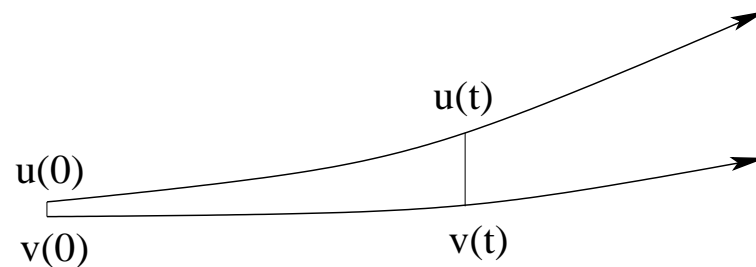
Continuous dependence on the initial data

$$u_t + f(u)_x = 0 \quad u(0, x) = \bar{u}(x)$$

Given two solutions u, v , estimate the difference $\|u(t) - v(t)\|_{L^1}$

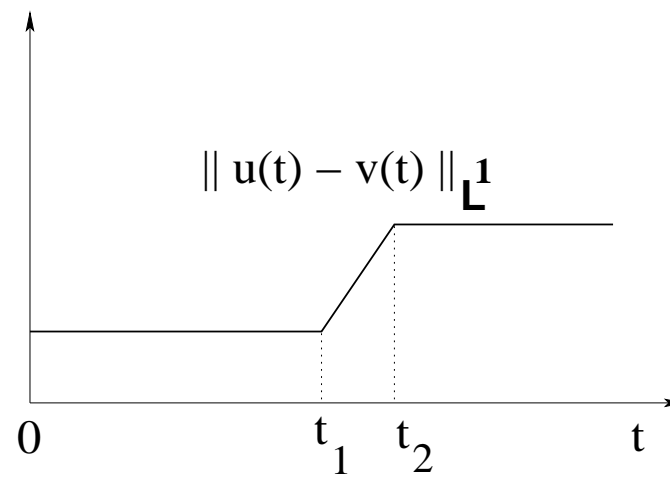
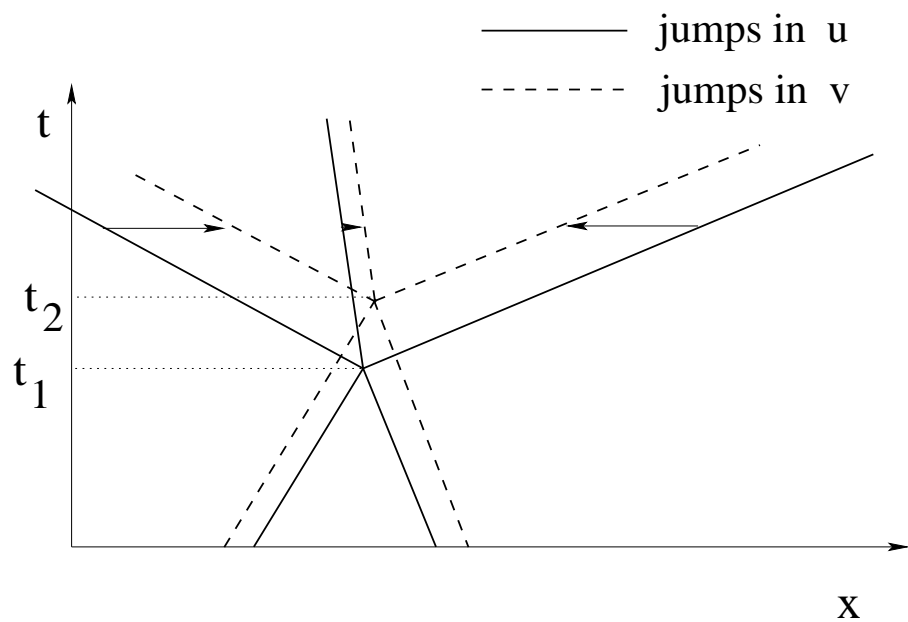
Standard approach: set $w = u - v$, show that

$$\frac{d}{dt} \|w(t)\| \leq C \|w(t)\| \quad \text{hence} \quad \|w(t)\| \leq e^{Ct} \|w(0)\|$$



Works for Lipschitz solutions, not in the presence of shocks

For two solutions u, v of a hyperbolic system containing shocks, the L^1 distance can increase rapidly during short time intervals



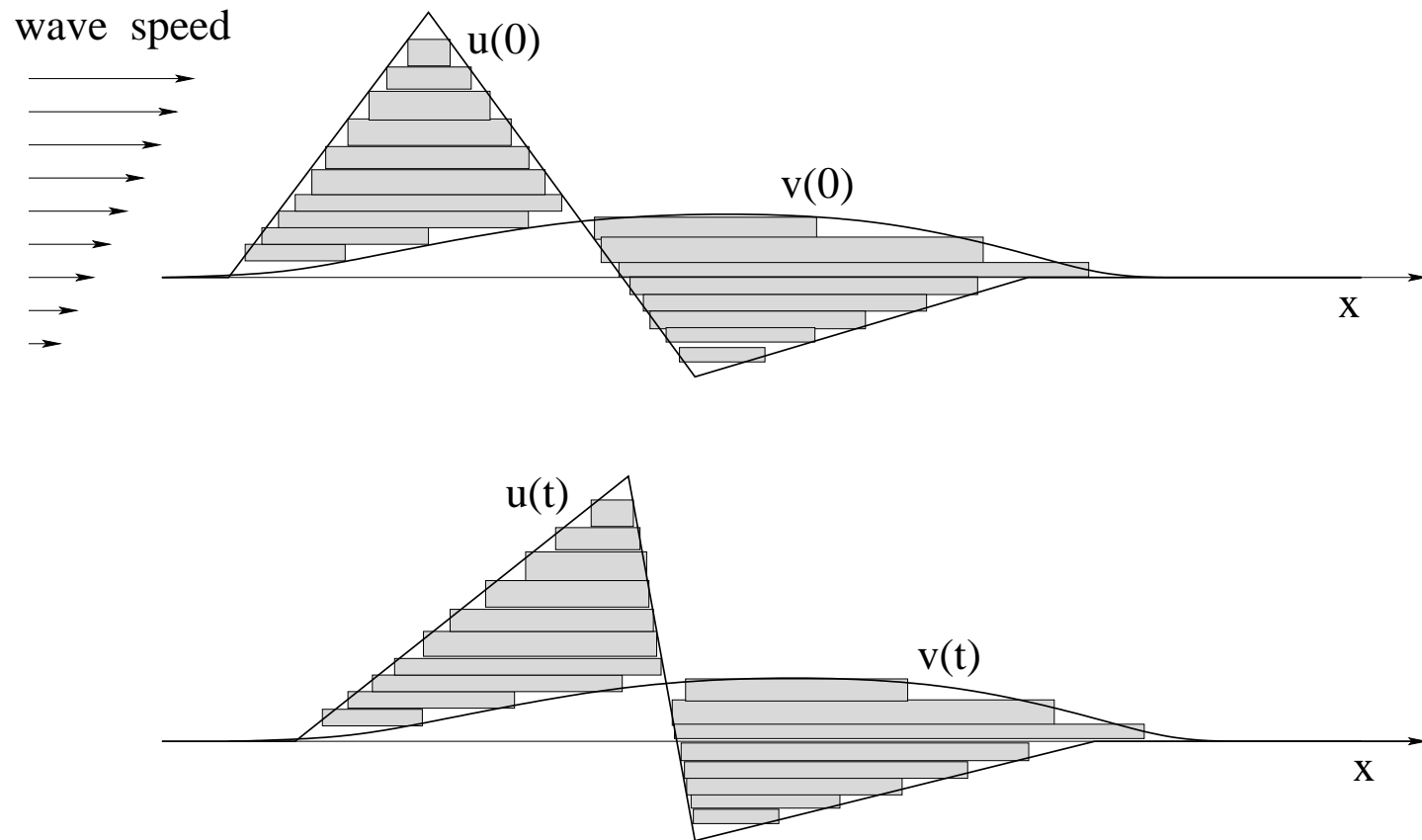
Scalar conservation law

$$u_t + f(u)_x = 0$$

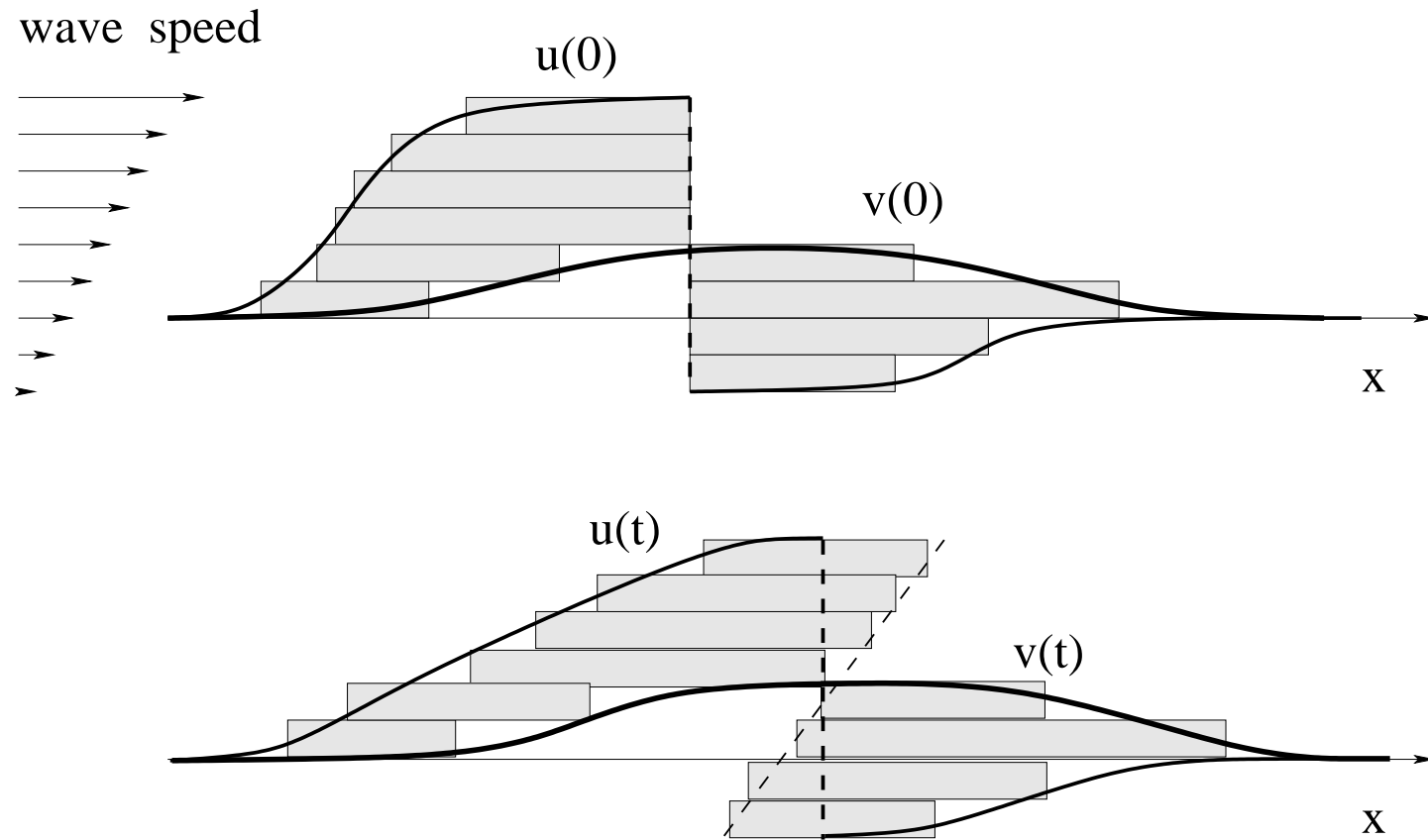
Contraction property (A.I. Volpert, 1967, S. Kruzhkov, 1970)

$$\|u(t) - v(t)\|_{L^1} \leq \|u(0) - v(0)\|_{L^1} \quad \text{for all } t \geq 0$$

The L^1 distance between continuous solutions remains constant in time



The L^1 distance decreases when a shock in one solution crosses the graph of the other solution



Linear Hyperbolic Systems

$$u_t + Au_x = 0 \quad u \in \mathbb{R}^n$$

$$\|u\|_{L^1} \doteq \int |u(x)| dx$$

Left, right eigenvectors: $l_i A = \lambda_i l_i \quad A r_i = \lambda_i r_i$

Equivalent norm: $\|u\|_A \doteq \sum_{i=1}^n \|l_i \cdot u\|_{L^1}$

u, v solutions $\implies w = u - v$ satisfies $w_t + Aw_x = 0$

$$\|w(t)\|_A = \|w(0)\|_A \quad \text{for all } t \in \mathbb{R}$$

Continuous dependence on the initial data

$$u_t + f(u)_x = 0 \quad u(0, x) = \bar{u}(x)$$

Consider a positively invariant domain of integrable functions with small total variation (here cl means closure) :

$$\mathcal{D} \doteq cl \left\{ u \in \mathbf{L}^1(\mathbb{R}; \mathbb{R}^n); u \text{ is piecewise constant, } V(u) + C_0 \cdot Q(u) \leq \delta_0 \right\}$$

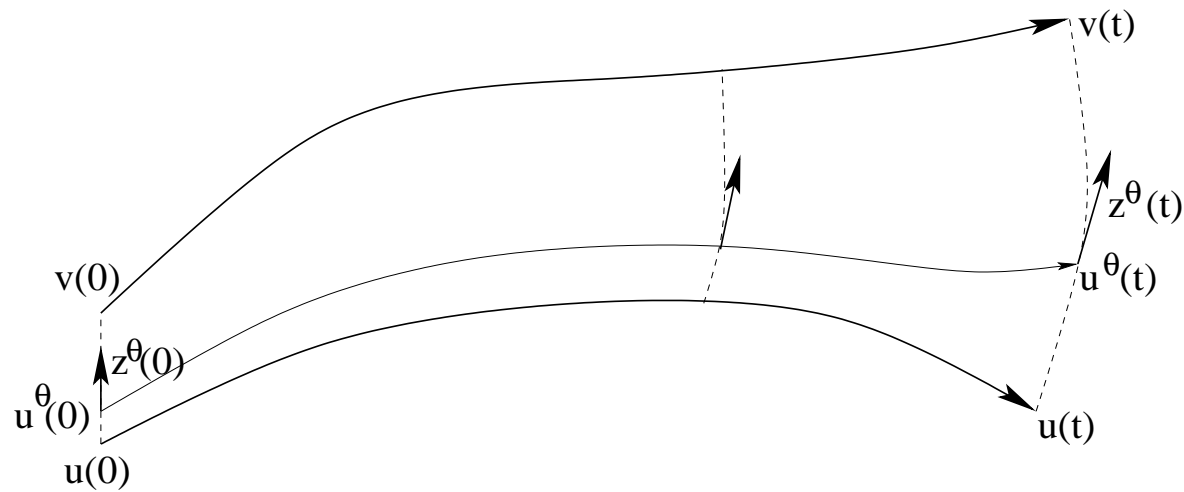
Theorem. For every $\bar{u} \in \mathcal{D}$, front tracking approximations converge to a unique limit solution $u : [0, \infty[\mapsto \mathcal{D}$.

The map $(\bar{u}, t) \mapsto u(t, \cdot) \doteq S_t \bar{u}$ is a uniformly Lipschitz semigroup, i.e.:

$$S_0 \bar{u} = \bar{u}, \quad S_s(S_t \bar{u}) = S_{s+t} \bar{u}$$

$$\|S_t \bar{u} - S_s \bar{v}\|_{\mathbf{L}^1} \leq L \cdot (\|\bar{u} - \bar{v}\|_{\mathbf{L}^1} + |t - s|) \quad \text{for all } \bar{u}, \bar{v} \in \mathcal{D}, \quad s, t \geq 0$$

Proof: by a homotopy method



2×2 systems : A. Bressan and R. M. Colombo, 1994

$n \times n$ systems : A. Bressan, G. Crasta, B. Piccoli, 1996

Alternative method: a non-linear distance functional

(A. Bressan, T.P.Liu, T. Yang, 1998)

Construct a functional $\Phi : \mathcal{D} \times \mathcal{D} \mapsto \mathbb{R}_+$ such that

- Φ is equivalent to the L^1 distance

$$\frac{1}{C} \cdot \|v - u\|_{L^1} \leq \Phi(u, v) \leq C \cdot \|v - u\|_{L^1}$$

- Φ is non-increasing in time, along couples of solutions

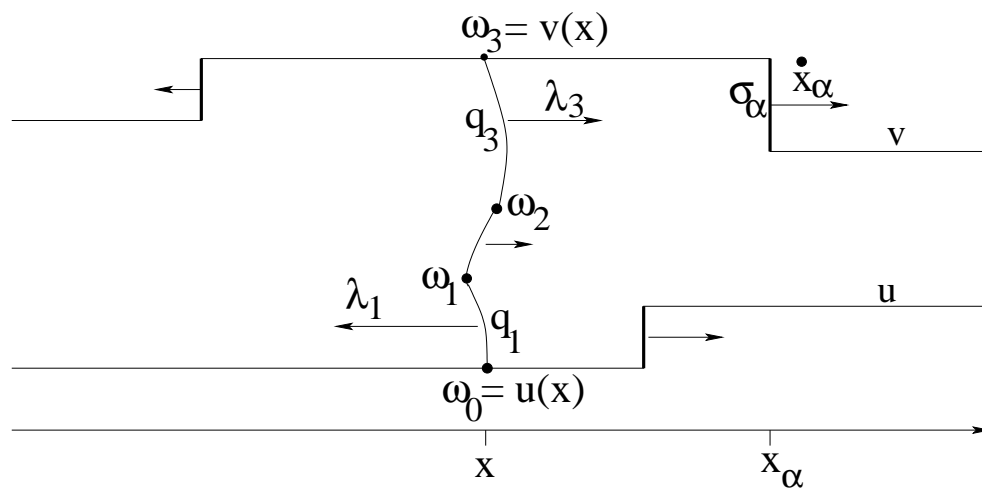
$$\Phi(u(t), v(t)) \leq \Phi(u(s), v(s)) \quad 0 \leq s \leq t$$

Construction of the distance functional

For each $x \in \mathbb{R}$, decompose the jump $u(x), v(x)$ as a concatenation of shocks, inserting intermediate states $u(x) = \omega_0, \omega_1, \dots, \omega_n = v(x)$

$$\omega_i = S_i(q_i)(\omega_{i-1}) \quad i = 1, \dots, n$$

$$v(x) = S_n(q_n(x)) \circ \dots \circ S_1(q_1(x))(u(x))$$



$q_1(x), \dots, q_n(x) =$ scalar components of the jump

Remark. In the linear case $u_t + Au_x = 0$

one has $q_i(x) = l_i \cdot (v(x) - u(x))$

$$\Phi(u, v) \doteq \sum_{i=1}^n \int_{-\infty}^{\infty} |q_i(x)| W_i(x) dx$$

Weight functions:

$$W_i(x) \doteq 1 + \kappa_1 \cdot [\text{total strength of waves in } u \text{ and in } v$$

which approach the i -wave $q_i(x)$]

$$+ \kappa_2 \cdot [\text{wave interaction potentials of } u \text{ and of } v]$$

$$\doteq 1 + \kappa_1 A_i(x) + \kappa_2 [Q(u) + Q(v)]$$

Uniqueness of Solutions

$$u_t + f(u)_x = 0 \qquad u(0, x) = \bar{u}(x)$$

Question: Do all entropy-admissible weak solutions coincide with the ones obtained by front tracking approximations ?

Key step: estimate the distance between a weak solution $t \mapsto u(t)$ and a trajectory $t \mapsto S_t \bar{u}$ of the semigroup obtained taking limits of front tracking approximations.

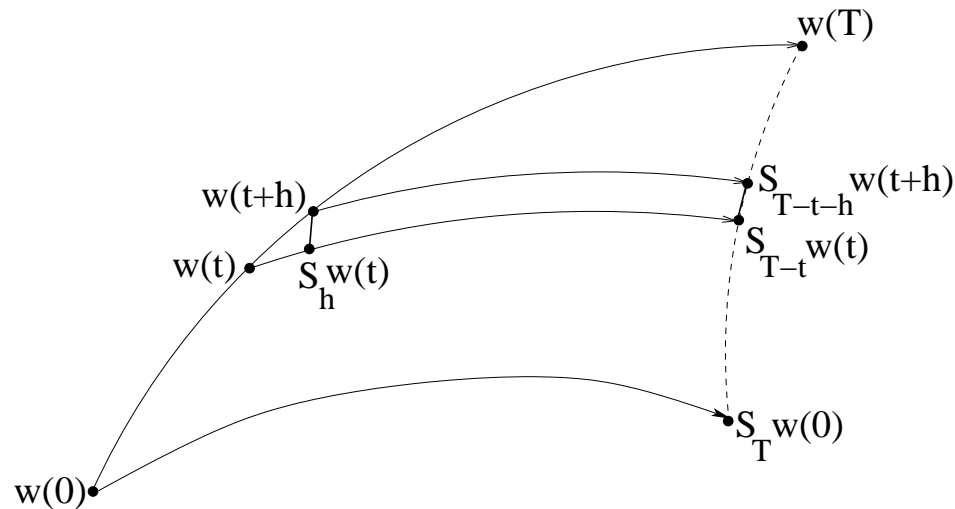
An error estimate

Theorem. Let $S : \mathcal{D} \times [0, \infty[\mapsto \mathcal{D}$ be a Lipschitz semigroup satisfying

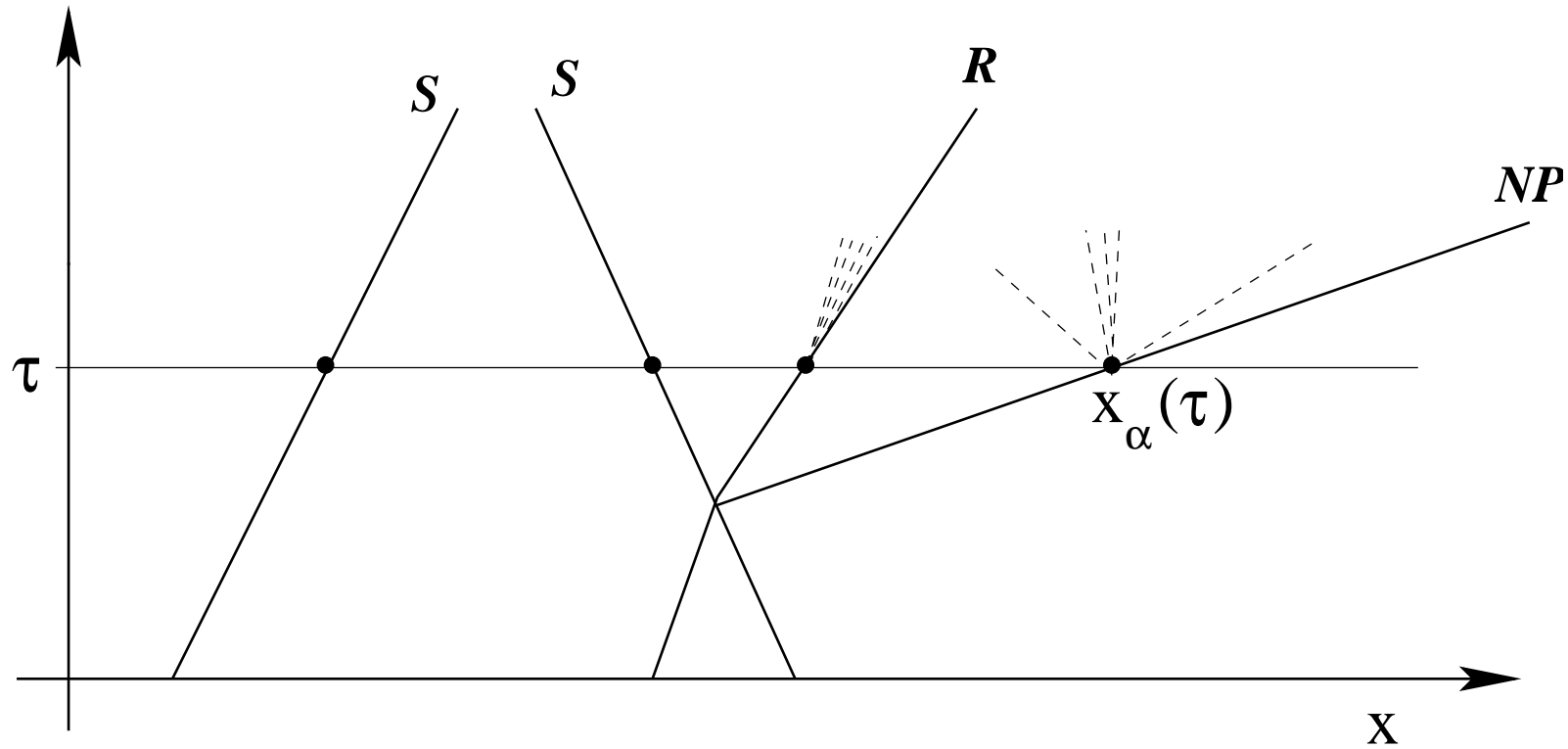
$$\|S_t u - S_s v\| \leq L \cdot \|u - v\| + L' \cdot |t - s|$$

Then, for every Lipschitz continuous map $w : [0, T] \mapsto \mathcal{D}$ one has

$$\begin{aligned} \|w(T) - S_T w(0)\| &\leq L \cdot \int_0^T \left\{ \liminf_{h \rightarrow 0^+} \frac{\|w(t+h) - S_h w(t)\|}{h} \right\} dt \\ &= L \cdot \int_0^T [\text{instantaneous error rate at time } t] dt \end{aligned}$$



Instantaneous error rate for a front tracking approximation



$$\frac{1}{h} \|w(\tau + h) - S_h w(\tau)\|_{L^1} = \mathcal{O}(1) \cdot \sum_{\alpha \in \mathcal{R}} |\sigma_\alpha|^2 + \mathcal{O}(1) \cdot \sum_{\alpha \in \mathcal{NP}} |\sigma_\alpha|$$

Characterization of semigroup trajectories

Question: among all weak solutions of

$$u_t + f(u)_x = 0$$

which ones are obtained as limits of front tracking approximations ?

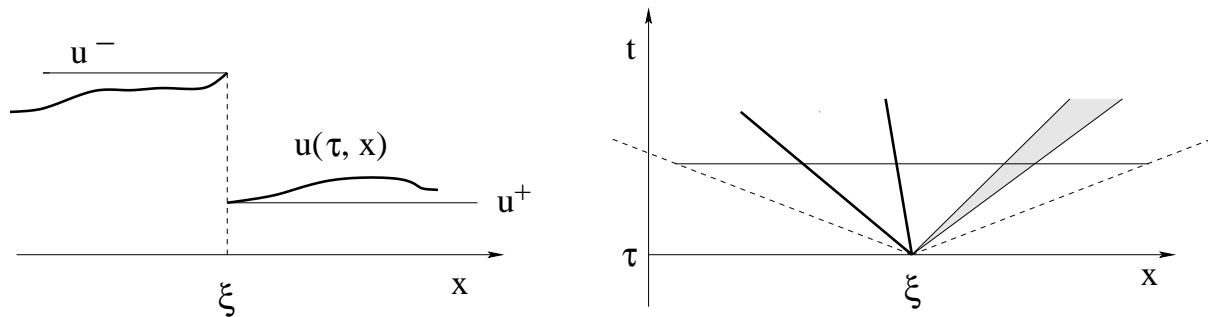
1. Comparison with solutions to a Riemann problem

Fix (τ, ξ) . Define $U^\# = U^\#_{(\tau, \xi)}$ as the solution of the Riemann problem

$$w_t + f(w)_x = 0, \quad w(\tau, x) = \begin{cases} u^+ \doteq u(\tau, \xi+) & \text{if } x > \xi \\ u^- \doteq u(\tau, \xi-) & \text{if } x < \xi \end{cases}$$

Then for every $\hat{\lambda} > 0$ we expect

$$\lim_{h \rightarrow 0^+} \frac{1}{h} \int_{\xi - h\hat{\lambda}}^{\xi + h\hat{\lambda}} \left| u(\tau + h, x) - U^\#_{(\tau, \xi)}(\tau + h, x) \right| dx = 0 \quad (E1)$$



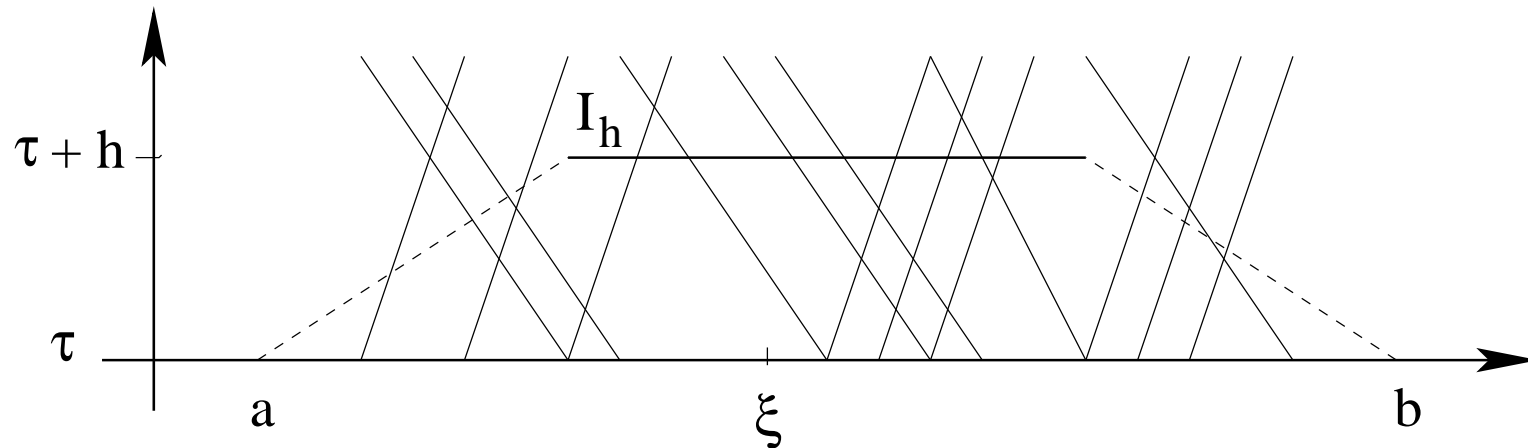
2. Comparison with solutions to a linear hyperbolic problem

Fix (τ, ξ) . Choose $\hat{\lambda} > 0$ larger than all wave speeds.

Define $U^b = U^b_{(\tau, \xi)}$ as the solution of the linear Cauchy problem

$$w_t + \tilde{A}w_x = 0 \quad w(\tau, x) = u(\tau, x)$$

with “frozen” coefficients: $\tilde{A} \doteq A(u(\tau, \xi))$



Then, for $a < \xi < b$ and $h > 0$ we expect

$$\frac{1}{h} \int_{a+\hat{\lambda}h}^{b-\hat{\lambda}h} |u(\tau+h, x) - U^b(\tau+h, x)| dx = \mathcal{O}(1) \cdot \left(\text{Tot.Var.} \{u(\tau, \cdot);]a, b[\} \right)^2 \quad (E2)$$

Motivation: $u_t = -A(u)u_x$, $w_t = -\tilde{A}w_x$, $u(\tau) = w(\tau)$. Hence

$$\int_{a+\hat{\lambda}h}^{b-\hat{\lambda}h} \left| u(\tau+h, x) - U^b(\tau+h, x) \right| dx \approx \int_{\tau}^{\tau+h} \int_{J(t)} \left| A(u(t, x))u_x - A(u(\tau, \xi))w_x \right| dx dt$$

$$J(t) \doteq]a + (t - \tau)\hat{\lambda}, b - (t - \tau)\hat{\lambda}[$$

$$\int_{J(t)} \left(|u_x(t, x)| + |w_x(t, x)| \right) dx = \mathcal{O}(1) \cdot \text{Tot.Var.}\{u(\tau, \cdot);]a, b[\}$$

$$\sup_{\tau < t < \tau+h, x \in J(t)} \left| A(u(t, x)) - A(u(\tau, \xi)) \right| = \mathcal{O}(1) \cdot \text{Tot.Var.}\{u(\tau, \cdot);]a, b[\}$$

Theorem (A.Bressan, *Archive Rat. Mech. Anal.* 1994).

Let $u : [0, T] \mapsto \mathcal{D}$ be Lipschitz continuous w.r.t. the L^1 distance.

Then u is a weak solution to the system of conservation laws

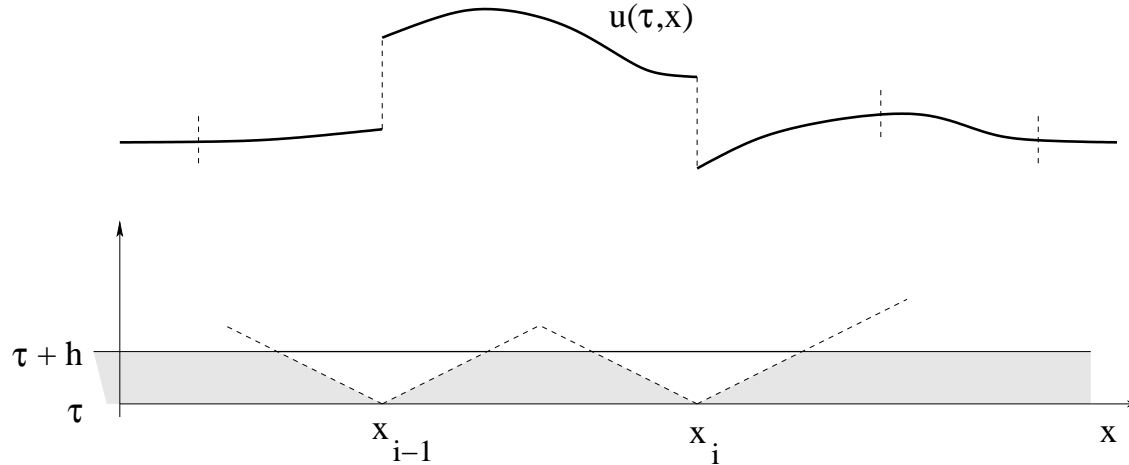
$$u_t + f(u)_x = 0$$

obtained as limit of front tracking approximations if and only if the estimates (E1)-(E2) are satisfied for a.e. $\tau \in [0, T]$, at every $\xi \in \mathbb{R}$.

$$\text{Indeed: (E1) + (E2) } \implies \lim_{h \rightarrow 0^+} \frac{\|u(\tau + h) - S_h u(\tau)\|_{L^1}}{h} = 0$$

Hence, by the error estimate: $u(t) = S_t u(0)$

Proof. Insert points x_i such that $\text{Tot.Var.}\{u(\tau);]x_{i-1}, x_i[\} < \varepsilon$



$$\begin{aligned} \frac{1}{h} \int_{-\infty}^{\infty} |u(\tau + h, x) - S_h u(\tau)(x)| dx &= \sum_i \frac{1}{h} \int_{x_i - \hat{\lambda}h}^{x_i + \hat{\lambda}h} |\dots| dx + \sum_i \frac{1}{h} \int_{x_{i-1} + \hat{\lambda}h}^{x_i - \hat{\lambda}h} |\dots| dx \\ &= \sum_i A_i + \sum_i B_i \end{aligned}$$

estimate (E1) implies $A_i \rightarrow 0$ as $h \rightarrow 0$

estimate (E2) implies $B_i \leq \varepsilon \cdot \text{Tot.Var.}\{u(\tau);]x_{i-1}, x_i[\}$

$$\sum_i B_i \leq \varepsilon \cdot \text{Tot.Var.}\{u(\tau); \mathbb{R}\} = \mathcal{O}(\varepsilon), \quad \text{with } \varepsilon > 0 \text{ arbitrary.}$$

Uniqueness of weak solutions

$$u_t + f(u)_x = 0 \qquad u(0, x) = \bar{u}(x)$$

- introduce a suitable set of admissibility + regularity assumptions
- show that these assumptions imply the estimates (E1)+(E2)

$$\implies u(t) = S_t \bar{u} \quad \text{for all } t \geq 0$$

A set of Assumptions

(A1) (Conservation Equations)

$u : [0, T] \mapsto \mathcal{D}$ is continuous w.r.t. the L^1 distance.

The initial condition $u(0, x) = \bar{u}(x)$ holds.

Moreover, u is a weak solution:

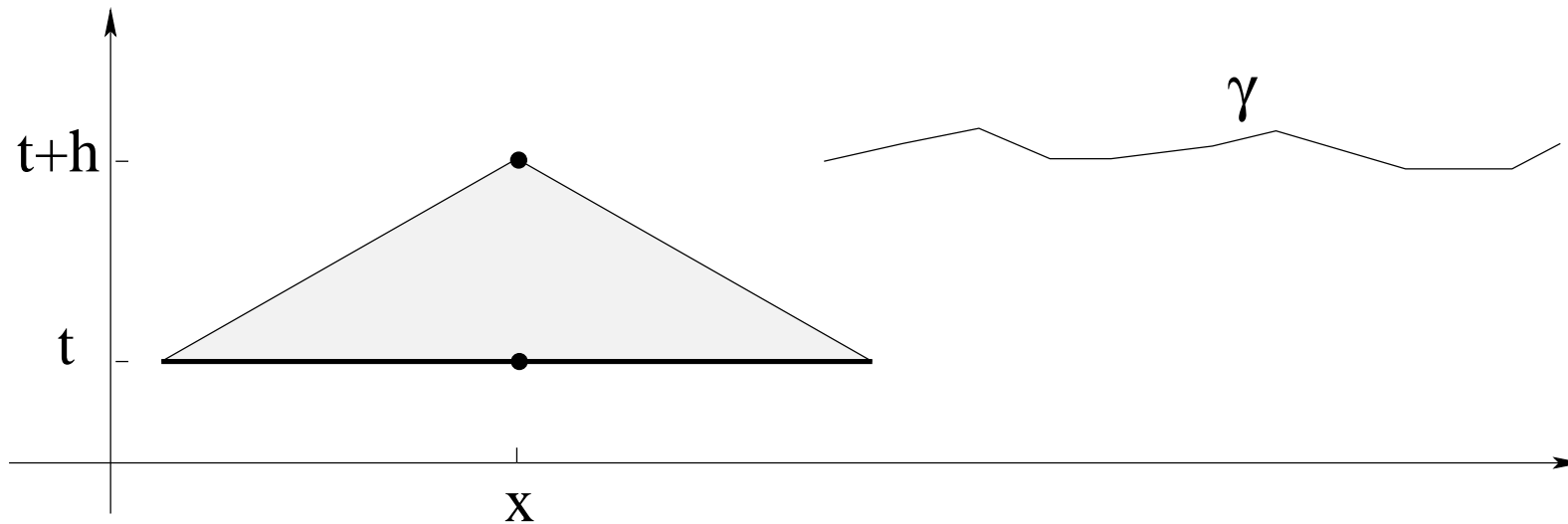
$$\iint \{u\varphi_t + f(u)\varphi_x\} dxdt = 0 \quad \text{for all } \varphi \in \mathcal{C}_c^1$$

(A2) (Admissibility Conditions) u satisfies the Lax admissibility conditions at each point (τ, ξ) of approximate jump.

(A3) (Tame Oscillation Condition) For some constants $C, \hat{\lambda}$ the following holds. For every point $x \in \mathbb{R}$ and every $t, h > 0$ one has

$$|u(t+h, x) - u(t, x)| \leq C \cdot \text{Tot.Var.}\{u(t, \cdot); [x - \hat{\lambda}h, x + \hat{\lambda}h]\}$$

(A4) (Bounded Variation Condition) There exists $\delta > 0$ such that, for every space-like curve $\{t = \tau(x)\}$ with $|d\tau/dx| \leq \delta$ a.e., the function $x \mapsto u(\tau(x), x)$ has locally bounded variation.



$$u_t + f(u)_x = 0 \qquad u(0, x) = \bar{u}(x)$$

Theorem. *Let the system be strictly hyperbolic, with each characteristic field either linearly degenerate or genuinely nonlinear.*

- *Every weak solution $u = u(t, x)$ obtained as limit of front tracking approximations satisfies all conditions (A1)–(A4).*
- *If $u : [0, T] \mapsto \mathcal{D}$ satisfies (A1), (A2), (A3), then $u(t) = S_t \bar{u}$ for $t \geq 0$. The same is true if u satisfies (A1), (A2), (A4).*

(A.Bressan, P.Goatin, *J. Diff. Equat.* 1999)

(A.Bressan, M.Lewicka, *Discr. Cont. Dynam. Syst.* 2000)

\implies Solutions obtained by the Glimm scheme are unique, and coincide with limits of front tracking approximations

Extensions:

- well posedness of the Cauchy problem for hyperbolic balance laws, with Lipschitz continuous sources

(D.Amadori, L.Gosse, G.Guerra, *Arch. Rational Mech. Anal.* 2002)

- well posedness of the initial-boundary value problem for hyperbolic systems of conservations laws on the half line

(C.Donadello, A.Marson, *No.D.E.A.* 2008)

Solutions with large data

$$u_t + f(u)_x = 0 \quad u(0, x) = \bar{u}(x)$$

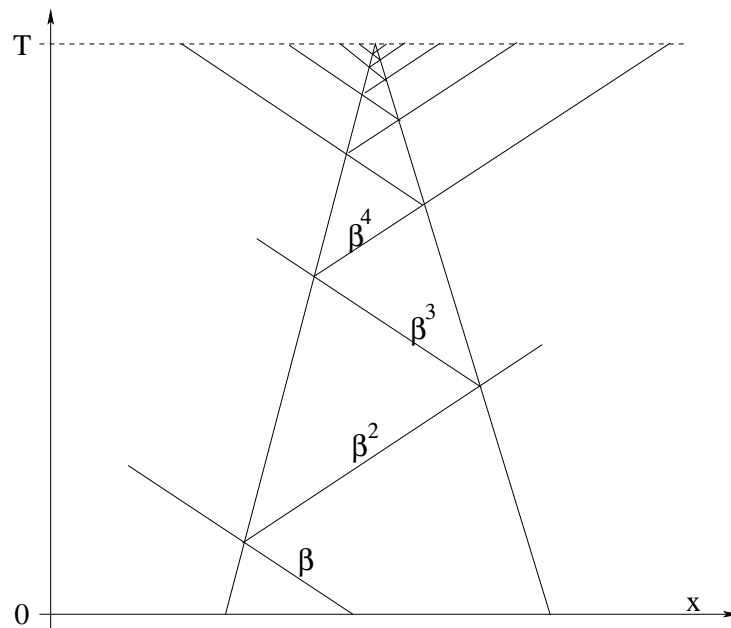
$\bar{u} \in BV$, possibly with large total variation

- main difficulty: derive a-priori BV bounds, valid for all times (largely open problem)
- global BV bounds are known for special systems (Temple class...) or special initial data (small BV perturbations of Riemann data...)
- uniqueness and continuous dependence hold as long as the total variation remains bounded (M.Lewicka, 2001–2005)
- for some (non-physical) strictly hyperbolic systems, the total variation can blow up in finite time (K. Jenssen, 2000)

An example of finite time blow up

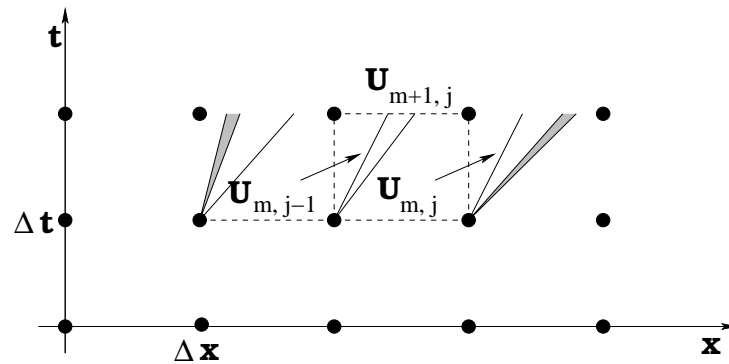
$$U_t + F(U)_x = 0$$

$$U = (u, v, w) \quad F(U) = F(u, v, w) = \begin{pmatrix} uv + w \\ g(v) \\ u(1 - v^2) - vw \end{pmatrix}$$

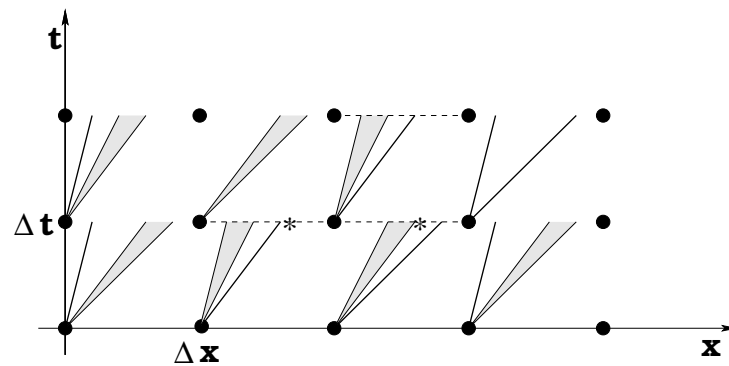


Discrete approximation schemes

Godunov (upwind) scheme: $U_{m+1,j} = U_{m,j} - \frac{\Delta t}{\Delta x}(f(U_{m,j}) - f(U_{m,j-1}))$



In the Glimm scheme, restart by **averaging** instead of **random sampling**



BV bounds, convergence, for finite difference schemes

For Godunov's finite difference approximations:

- for the 2×2 system of isentropic gas dynamics, a subsequence converges to a weak solution (X.Ding, G.Q.Chen, P.Luo, *Comm. Math. Phys.* 1989)
- for $n \times n$ systems where the shock curves are straight lines, one has BV bounds and convergence to a unique limit (A.B., K.Jenssen, *Chinese Ann. Math.* 2000)
- even for small initial data, the total variation can become arbitrarily large (A.B., K.Jenssen, P.Baiti, *Comm. Pure Appl. Math.* 2006)
- major open question: convergence of finite difference approximations, with small BV data, for general hyperbolic systems ???

(cannot be answered by seeking a priori BV bounds, since these do not hold, in general)