Hyperbolic Systems of Conservation Laws

in One Space Dimension

I - Basic concepts

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The Scalar Conservation Law

\[ u_t + f(u)_x = 0 \]

\[ u : \text{conserved quantity} \quad f(u) : \text{flux} \]

\[
\frac{d}{dt} \int_a^b u(t, x) \, dx = \int_a^b u_t(t, x) \, dx = -\int_a^b f(u(t, x))_x \, dx
\]

\[ = f(u(t, a)) - f(u(t, b)) = [\text{inflow at } a] - [\text{outflow at } b]. \]
Example: Traffic Flow

\( \rho = \text{density of cars} \)

\[
\frac{d}{dt} \int_a^b \rho(t, x) \, dx = \text{[flux of cars entering at } a] - \text{[flux of cars exiting at } b]\]

**flux:**  \( f(t, x) = \text{[number of cars crossing the point } x \text{ per unit time]} \)

\( = \text{[density] } \times \text{[velocity]} \)

\[
\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} [\rho \, v(\rho)] = 0
\]
Weak solutions

conservation equation: \[ u_t + f(u)_x = 0 \]

quasilinear form: \[ u_t + a(u)u_x = 0 \quad a(u) = f'(u) \]

Conservation equation remains meaningful for \( u = u(t, x) \) discontinuous, in distributional sense:
\[
\int \int \{u\phi_t + f(u)\phi_x\} \, dx \, dt = 0 \quad \text{for all} \quad \phi \in C^1_c
\]

Need only: \( u, f(u) \) locally integrable
Convergence

\[ u_t + f(u)_x = 0 \]

Assume: \( u_n \) is a solution for \( n \geq 1 \),

\[ u_n \rightarrow u \quad f(u_n) \rightarrow f(u) \quad \text{in} \quad L^1_{loc} \]

then

\[
\int \int \{u \phi_t + f(u) \phi_x\} \, dx \, dt = \lim_{n \to \infty} \int \int \{u_n \phi_t + f(u_n) \phi_x\} \, dx \, dt = 0
\]

for all \( \phi \in C^1_c \). Hence \( u \) is a weak solution as well.
Systems of Conservation Laws

\[
\begin{align*}
\frac{\partial}{\partial t} u_1 + \frac{\partial}{\partial x} f_1(u_1, \ldots, u_n) &= 0, \\
&\quad \ldots \ldots \\
\frac{\partial}{\partial t} u_n + \frac{\partial}{\partial x} f_n(u_1, \ldots, u_n) &= 0.
\end{align*}
\]

\[u_t + f(u)_x = 0\]

\[u = (u_1, \ldots, u_n) \in \mathbb{R}^n \quad \text{conserved quantities}\]

\[f = (f_1, \ldots, f_n) : \mathbb{R}^n \mapsto \mathbb{R}^n \quad \text{fluxes}\]
Euler equations of gas dynamics (1755)

\[
\begin{align*}
\rho_t + (\rho v)_x &= 0 \quad \text{(conservation of mass)} \\
(\rho v)_t + (\rho v^2 + p)_x &= 0 \quad \text{(conservation of momentum)} \\
(\rho E)_t + (\rho E v + p v)_x &= 0 \quad \text{(conservation of energy)}
\end{align*}
\]

\(\rho\) = mass density \quad \(v\) = velocity

\(E = e + \frac{v^2}{2}\) = energy density per unit mass (internal + kinetic)

\(p = p(\rho, e)\) \quad \text{constitutive relation}
Hyperbolic Systems

\[ u_t + f(u)_x = 0 \quad \text{for} \quad u = u(t, x) \in \mathbb{R}^n \]

\[ u_t + A(u)u_x = 0 \quad A(u) = Df(u) \]

The system is **strictly hyperbolic** if each \( n \times n \) matrix \( A(u) \) has real distinct eigenvalues

\[ \lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u) \]

right eigenvectors \( r_1(u), \ldots, r_n(u) \) (column vectors)
left eigenvectors \( l_1(u), \ldots, l_n(u) \) (row vectors)

\[ Ar_i = \lambda_i r_i \quad l_i A = \lambda_i l_i \]

Choose bases so that \( l_i \cdot r_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \)
Scalar Equation with Linear Flux

\[ u_t + f(u)_x = 0 \quad f(u) = \lambda u + c \]

\[ u_t + \lambda u_x = 0 \quad u(0, x) = \phi(x) \]

Explicit solution: \[ u(t, x) = \phi(x - \lambda t) \]

traveling wave with speed \[ f'(u) = \lambda \]
A Linear Hyperbolic System

\[ u_t + Au_x = 0 \quad u(0, x) = \phi(x) \]

\[ \lambda_1 < \cdots < \lambda_n \quad \text{eigenvalues} \]
\[ \left\{ \begin{array}{c}
  r_1, \ldots, r_n \\
l_1, \ldots, r_n
\end{array} \right\} \quad \text{right eigenvectors} \quad \text{left eigenvectors} \]

Explicit solution: linear superposition of travelling waves

\[ u(t, x) = \sum_i \phi_i(x - \lambda_i t) r_i \quad \phi_i(s) = l_i \cdot \phi(s) \]
Nonlinear Effects

\[ u_t + A(u)u_x = 0 \]

eigenvalues depend on \( u \quad \implies \quad \text{waves change shape} \)
eigenvectors depend on $u$ $\implies$ nontrivial wave interactions

linear

nonlinear
Loss of Regularity

\[ u_t + f(u)_x = 0 \quad u(0, x) = \phi(x) \]

Method of characteristics yields:
\[ u(t, x_0 + t f'(\phi(x_0))) = \phi(x_0) \]

characteristic speed = \( f'(u) \)

Global solutions only in a space of discontinuous functions

\[ u(t, \cdot) \in BV \]
Wave Interactions

\[ u_t = -A(u)u_x \]

\( \lambda_i(u) = i\text{-th eigenvalue} \)

\( l_i(u), r_i(u) = i\text{-th eigenvectors} \)

\( u_x^i \triangleq l_i \cdot u_x = [i\text{-th component of } u_x] = [\text{density of } i\text{-waves in } u] \)

\[ u_x = \sum_{i=1}^{n} u_x^i r_i(u) \quad u_t = -\sum_{i=1}^{n} \lambda_i(u) u_x^i r_i(u) \]

differentiate first equation w.r.t. \( t \), second one w.r.t. \( x \)

\( \Rightarrow \) evolution equation for scalar components \( u_x^i \)

\[ (u_x^i)_t + (\lambda_i u_x^i)_x = \sum_{j>k} (\lambda_j - \lambda_k)(l_i \cdot [r_j, r_k]) u_x^j u_x^k \]
source terms: \( (\lambda_j - \lambda_k)(l_i \cdot [r_j, r_k])u_x^j u_x^k \)

= amount of \( i \)-waves produced by the interaction of \( j \)-waves with \( k \)-waves

\( \lambda_j - \lambda_k \) = [difference in speed]
= [rate at which \( j \)-waves and \( k \)-waves cross each other]

\( u_x^j u_x^k \) = [density of \( j \)-waves] \( \times \) [density of \( k \)-waves]

\[ [r_j, r_k] = (Dr_k)r_j - (Dr_j)r_k \] (Lie bracket)

= [directional derivative of \( r_k \) in the direction of \( r_j \)]
= [directional derivative of \( r_j \) in the direction of \( r_k \)]

\( l_i \cdot [r_j, r_k] \) = \( i \)-th component of the Lie bracket \([r_j, r_k]\) along the basis of eigenvectors \( \{r_1, \ldots, r_n\} \)
Shock solutions

\[ u_t + f(u)_x = 0 \]

\[ u(t, x) = \begin{cases} u^- & \text{if} \quad x < \lambda t \\ u^+ & \text{if} \quad x > \lambda t \end{cases} \]

is a weak solution if and only if

\[ \lambda \cdot [u^+ - u^-] = f(u^+) - f(u^-) \quad \text{Rankine - Hugoniot equations} \]

\[ [\text{speed of the shock}] \times [\text{jump in the state}] = [\text{jump in the flux}] \]
Derivation of the Rankine - Hugoniot Equations

\[
\int \int \{u\phi_t + f(u)\phi_x\} \, dxdt = 0 \quad \text{for all} \quad \phi \in \mathcal{C}^1_c
\]

\[
v \equiv (u\phi, f(u)\phi)
\]

\[
0 = \int \int_{\Omega^+ \cup \Omega^-} \text{div} \, v \, dxdt = \int_{\partial \Omega^+} n^+ \cdot v \, ds + \int_{\partial \Omega^-} n^- \cdot v \, ds
\]

\[
= \int \left[ \lambda u^+ - f(u^+) \right] \phi(t, \lambda t) \, dt + \int \left[ -\lambda u^- + f(u^-) \right] \phi(t, \lambda t) \, dt
\]

\[
= \int \left[ \lambda (u^+ - u^-) - (f(u^+) - f(u^-)) \right] \phi(t, \lambda t) \, dt.
\]
Alternative formulation:

\[ \lambda (u^+ - u^-) = f(u^+) - f(u^-) = \int_0^1 Df(\theta u^+ + (1 - \theta)u^-) \cdot (u^+ - u^-) \, d\theta \]

\[ = A(u^+, u^-) \cdot (u^+ - u^-) \]

\[ A(u, v) \doteq \int_0^1 Df(\theta u + (1 - \theta)v) \, d\theta = \text{ [averaged Jacobian matrix]} \]

The Rankine-Hugoniot conditions hold if and only if

\[ \lambda (u^+ - u^-) = A(u^+, u^-) (u^+ - u^-) \]

- The jump \( u^+ - u^- \) is an eigenvector of the averaged matrix \( A(u^+, u^-) \)
- The speed \( \lambda \) coincides with the corresponding eigenvalue
scalar conservation law: \[ u_t + f(u)_x = 0 \]

\[ \lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{1}{u^+ - u^-} \int_{u^-}^{u^+} f'(s) \, ds \]

[speed of the shock] = [slope of secant line through \( u^-, u^+ \) on the graph of \( f \)]

= [average of the characteristic speeds between \( u^- \) and \( u^+ \)]
Points of Approximate Jump

The function $u = u(t, x)$ has an **approximate jump** at a point $(\tau, \xi)$ if there exists states $u^- \neq u^+$ and a speed $\lambda$ such that, calling

$$U(t, x) = \begin{cases} u^- & \text{if } x < \lambda t, \\ u^+ & \text{if } x > \lambda t, \end{cases}$$

there holds

$$\lim_{\rho \to 0^+} \frac{1}{\rho^2} \int_{\tau - \rho}^{\tau + \rho} \int_{\xi - \rho}^{\xi + \rho} \left| u(t, x) - U(t - \tau, x - \xi) \right| \, dx \, dt = 0 \quad (2)$$

**Theorem.** If $u$ is a weak solution to the system of conservation laws $u_t + f(u)_x = 0$ then the Rankine-Hugoniot equations hold at each point of approximate jump.
Construction of Shock Curves

**Problem:** Given $u^- \in \mathbb{R}^n$, find the states $u^+ \in \mathbb{R}^n$ which, for some speed $\lambda$, satisfy the Rankine - Hugoniot equations

$$\lambda(u^+ - u^-) = f(u^+) - f(u^-) = A(u^-, u^+)(u^+ - u^-)$$

**Alternative formulation:** Fix $i \in \{1, \ldots, n\}$. The jump $u^+ - u^-$ is a (right) $i$-eigenvector of the averaged matrix $A(u^-, u^+)$ if and only if it is orthogonal to all (left) eigenvectors $l_j(u^-, u^+)$ of $A(u^-, u^+)$, for $j \neq i$

$$l_j(u^-, u^+)'(u^+ - u^-) = 0 \quad \text{for all } j \neq i \quad (RH_i)$$

Implicit function theorem $\implies$ for each $i$ there exists a curve

$$s \mapsto S_i(s)(u^-) \text{ of points that satisfy } (RH_i)$$
Weak solutions can be non-unique

Example: a Cauchy problem for Burgers’ equation

\[ u_t + (u^2/2)_x = 0 \quad \quad u(0, x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases} \]

Each \( \alpha \in [0, 1] \) yields a weak solution

\[ u_\alpha(t, x) = \begin{cases} 0 & \text{if } x < \alpha t/2, \\ \alpha & \text{if } \alpha t/2 \leq x < (1 + \alpha)t/2, \\ 1 & \text{if } x \geq (1 + \alpha)t/2. \end{cases} \]
Admissibility Conditions on Shocks

For physical systems:

a concave entropy should not decrease

For general hyperbolic systems:

require stability w.r.t. small perturbations
Stability Conditions: the Scalar Case

Perturb the shock with left and right states $u^-, u^+$ by inserting an intermediate state $u^* \in [u^-, u^+]$

Initial shock is stable $\iff$

\[
\frac{f(u^*) - f(u^-)}{u^* - u^-} \geq \frac{f(u^+) - f(u^*)}{u^+ - u^*}
\]
speed of a shock = slope of a secant line to the graph of $f$

Stability conditions:

- when $u^- < u^+$ the graph of $f$ should remain above the secant line
- when $u^- > u^+$, the graph of $f$ should remain below the secant line
General Stability Conditions

Scalar case: stability holds if and only if

\[
\frac{f(u^*) - f(u^-)}{u^* - u^-} \geq \frac{f(u^+) - f(u^-)}{u^+ - u^-}
\]

Vector valued case: \( u^+ = S_i(\sigma)(u^-) \) for some \( \sigma \in \mathbb{R} \).

Admissibility Condition (T.P.Liu, 1976). The speed \( \lambda(\sigma) \) of the shock joining \( u^- \) with \( u^+ \) must be less or equal to the speed of every smaller shock, joining \( u^- \) with an intermediate state \( u^* = S_i(s)(u^-) \), \( s \in [0, \sigma] \).

\[
\lambda(u^-, u^+) \leq \lambda(u^-, u^*)
\]
**Admissibility Condition (P. Lax, 1957)** A shock connecting the states $u^-, u^+$, travelling with speed $\lambda = \lambda_i(u^-, u^+)$ is *admissible* if

$$\lambda_i(u^-) \geq \lambda_i(u^-, u^+) \geq \lambda_i(u^+)$$

$Liu$ condition $\implies$ $Lax$ condition
Entropy - Entropy Flux

\[ u_t + f(u)_x = 0 \]

**Definition.** A function \( \eta : \mathbb{R}^n \to \mathbb{R} \) is called an *entropy*, with entropy flux \( q : \mathbb{R}^n \to \mathbb{R} \) if

\[ D\eta(u) \cdot Df(u) = Dq(u) \]

For *smooth* solutions \( u = u(t, x) \), this implies

\[ \eta(u)_t + q(u)_x = D\eta \cdot u_t + Dq \cdot u_x = -D\eta \cdot Df \cdot u_x + Dq \cdot u_x = 0 \]

\[ \implies \eta(u) \text{ is an additional conserved quantity, with flux } q(u) \]
Entropy Admissibility Condition

A weak solution $u$ of the hyperbolic system $u_t + f(u)_x = 0$ is **entropy admissible** if

$$[\eta(u)]_t + [q(u)]_x \leq 0$$

in the sense of distributions, for every pair $(\eta, q)$, where $\eta$ is a **convex entropy** and $q$ is the corresponding entropy flux.

$$\int \int \{\eta(u)\varphi_t + q(u)\varphi_x\} \, dxdt \geq 0 \quad \varphi \in C^1_c, \quad \varphi \geq 0$$

- smooth solutions conserve all entropies

- solutions with shocks are admissible if they dissipate all convex entropies
Existence of entropy - entropy flux pairs

\[ D\eta(u) \cdot Df(u) = Dq(u) \]

\[
\begin{pmatrix}
\frac{\partial \eta}{\partial u_1} & \cdots & \frac{\partial \eta}{\partial u_n}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_n} \\
\frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_n}
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial q}{\partial u_1} & \cdots & \frac{\partial q}{\partial u_n}
\end{pmatrix}
\]

- a system of \( n \) equations for 2 unknown functions: \( \eta(u), q(u) \)

- over-determined if \( n > 2 \)

- however, some physical systems (described by several conservation laws) are endowed with natural entropies