Hyperbolic Systems of Conservation Laws

in One Space Dimension

I - Basic concepts

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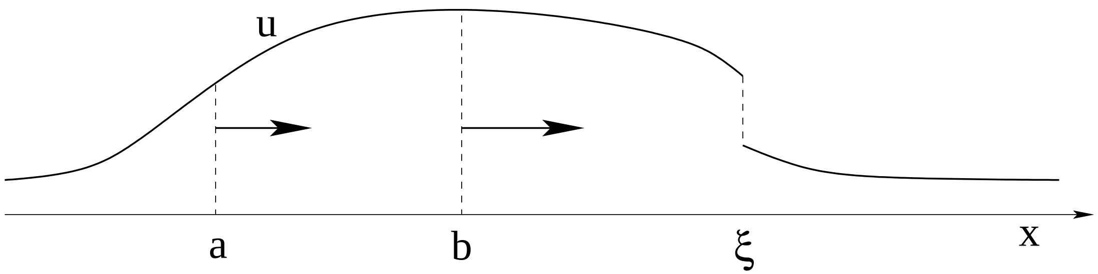
The Scalar Conservation Law

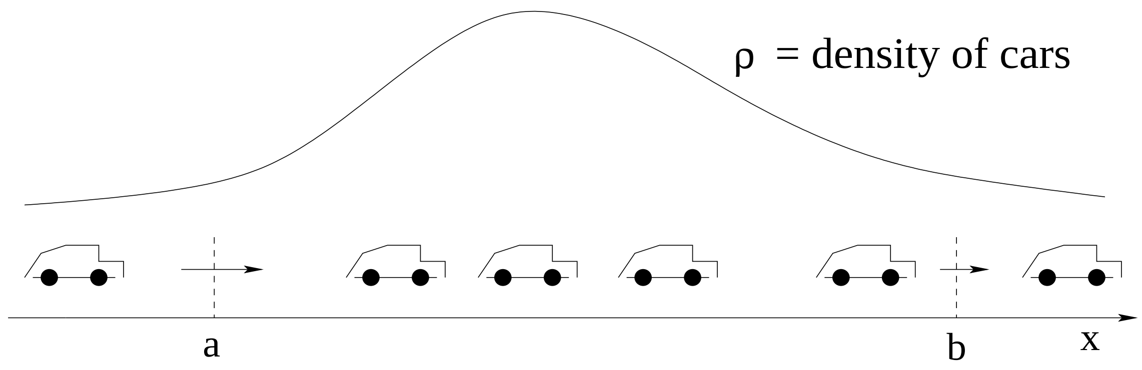
$u_t + f(u)_x = 0$ $\qquad$ u : conserved quantity $\qquad$ $f(u)$: flux

$$\frac{d}{dt}\int_a^b u(t,x)\ dx \;=\; \int_a^b u_t(t,x)\,dx \;=\; -\int_a^b f(u(t,x))_x\,dx$$

$$= f(u(t,a)) - f(u(t,b)) \;=\; [\text{inflow at } a] - [\text{outflow at } b]\,.$$



Example : Traffic Flow



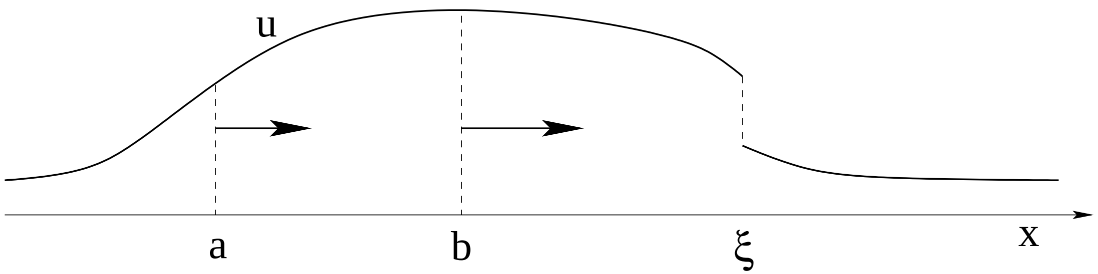
$$\frac{d}{dt}\int_a^b \rho(t,x)\,dx = \text{[flux of cars entering at } a] - \text{[flux of cars exiting at } b]$$

flux: $f(t,x) =$[number of cars crossing the point x per unit time]

$= \text{[density]}\times\text{[velocity]}$

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}[\rho\, v(\rho)] \;=\; 0$$

Weak solutions



conservation equation: $$u_t + f(u)_x = 0$$

quasilinear form: $$u_t + a(u)u_x = 0 \qquad a(u) = f'(u)$$

Conservation equation remains meaningful for $u = u(t,x)$ discontinuous, in distributional sense:

$$\iint \{u\phi_t + f(u)\phi_x\}\, dxdt = 0 \qquad \text{for all} \quad \phi \in \mathcal{C}_c^1$$

Need only : $u, f(u)$ locally integrable

Convergence

$$u_t + f(u)_x \;=\; 0$$

Assume: u_n is a solution for $n \geq 1$,

$$u_n \to u \qquad f(u_n) \to f(u) \qquad \text{in} \quad \mathbf{L}^1_{loc}$$

then

$$\iint \{u\phi_t + f(u)\phi_x\}\,dxdt \;=\; \lim_{n\to\infty} \iint \{u_n\phi_t + f(u_n)\phi_x\}\,dxdt \;=\; 0$$

for all $\phi \in \mathcal{C}^1_c$. Hence u is a weak solution as well.

Systems of Conservation Laws

$$\begin{cases} \dfrac{\partial}{\partial t} u_1 + \dfrac{\partial}{\partial x} f_1(u_1, \ldots, u_n) & = & 0, \\ \qquad \cdot \quad \cdot \quad \cdot & & \\ \dfrac{\partial}{\partial t} u_n + \dfrac{\partial}{\partial x} f_n(u_1, \ldots, u_n) & = & 0. \end{cases}$$

$$u_t + f(u)_x = 0$$

$u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ conserved quantities

$f = (f_1, \ldots, f_n) : \mathbb{R}^n \mapsto \mathbb{R}^n$ fluxes

Euler equations of gas dynamics (1755)

$$\begin{cases} \rho_t + (\rho v)_x &= 0 \qquad \text{(conservation of mass)} \\ (\rho v)_t + (\rho v^2 + p)_x &= 0 \qquad \text{(conservation of momentum)} \\ (\rho E)_t + (\rho E v + p v)_x &= 0 \qquad \text{(conservation of energy)} \end{cases}$$

ρ = mass density $\qquad v$ = velocity

$E = e + v^2/2$ = energy density per unit mass (internal + kinetic)

$p = p(\rho, e)$ constitutive relation

Hyperbolic Systems

$$u_t + f(u)_x = 0 \qquad\qquad u = u(t,x) \in \mathbb{R}^n$$

$$u_t + A(u)u_x = 0 \qquad\qquad A(u) = Df(u)$$

The system is **strictly hyperbolic** if each $n \times n$ matrix $A(u)$ has real distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u)$$

right eigenvectors $r_1(u), \ldots, r_n(u)$ (column vectors)
left eigenvectors $l_1(u), \ldots, l_n(u)$ (row vectors)

$$Ar_i = \lambda_i r_i \qquad\qquad l_i A = \lambda_i l_i$$

Choose bases so that $l_i \cdot r_j = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$

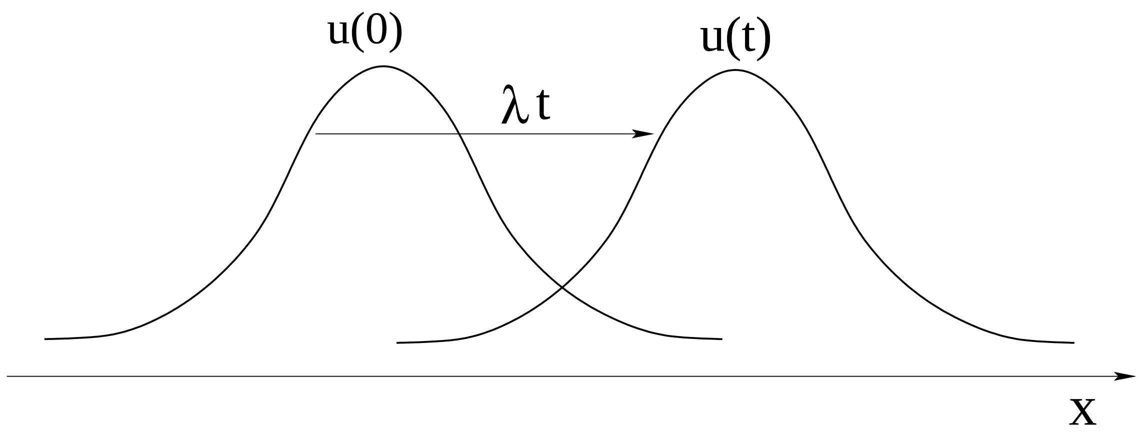
Scalar Equation with Linear Flux

$$u_t + f(u)_x \;=\; 0 \qquad\qquad f(u) = \lambda u + c$$

$$u_t + \lambda u_x \;=\; 0 \qquad\qquad u(0, x) = \phi(x)$$

Explicit solution: $u(t, x) = \phi(x - \lambda t)$

traveling wave with speed $f'(u) \;=\; \lambda$



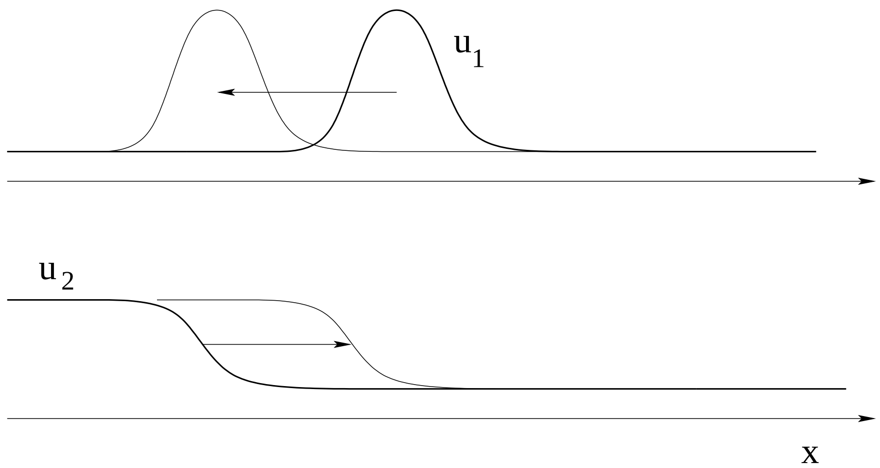
A Linear Hyperbolic System

$$u_t + Au_x = 0 \qquad\qquad u(0,x) = \phi(x)$$

$$\lambda_1 < \cdots < \lambda_n \quad \text{eigenvalues} \qquad \begin{cases} r_1, \ldots, r_n & \text{right eigenvectors} \\ l_1, \ldots, r_n & \text{left eigenvectors} \end{cases}$$

Explicit solution: **linear superposition of travelling waves**

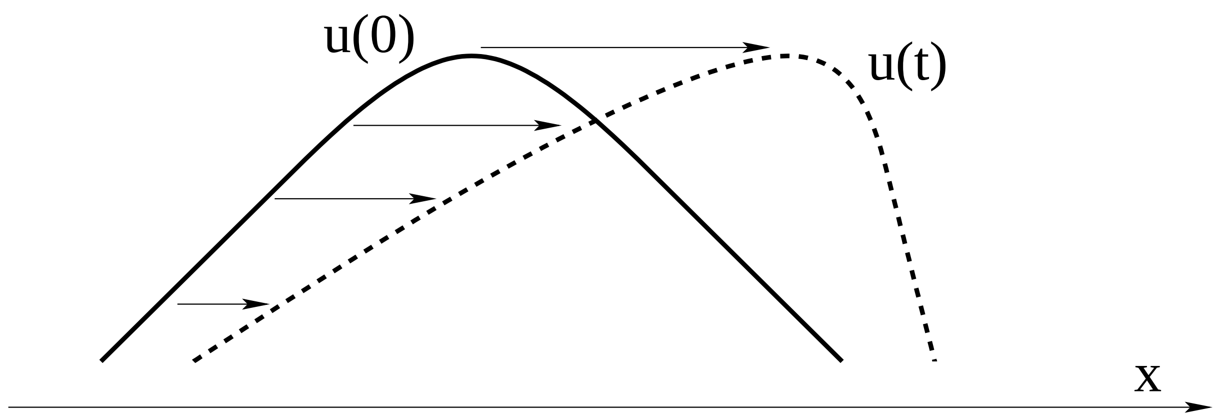
$$u(t,x) = \sum_i \phi_i(x - \lambda_i t) r_i \qquad\qquad \phi_i(s) = l_i \cdot \phi(s)$$



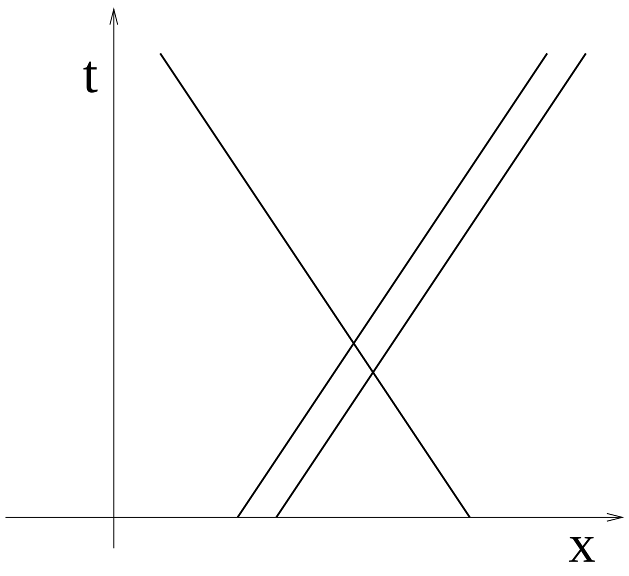
Nonlinear Effects

$$u_t + A(u)u_x = 0$$

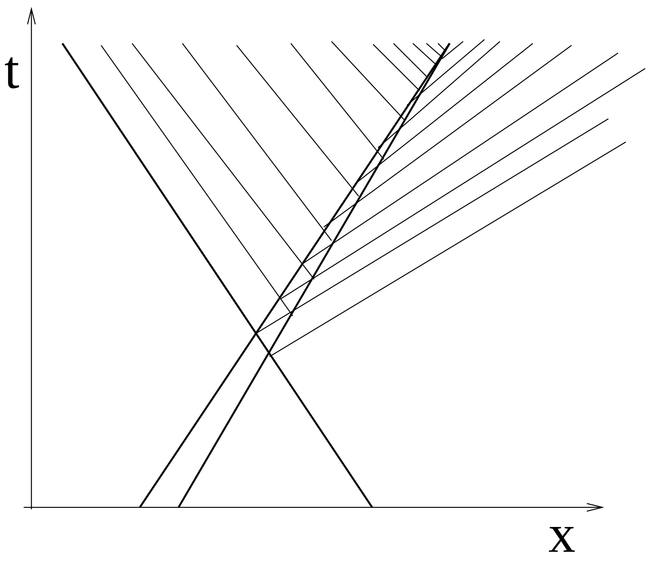
eigenvalues depend on u $\Longrightarrow$ waves change shape



eigenvectors depend on u $\Longrightarrow$ nontrivial wave interactions



linear



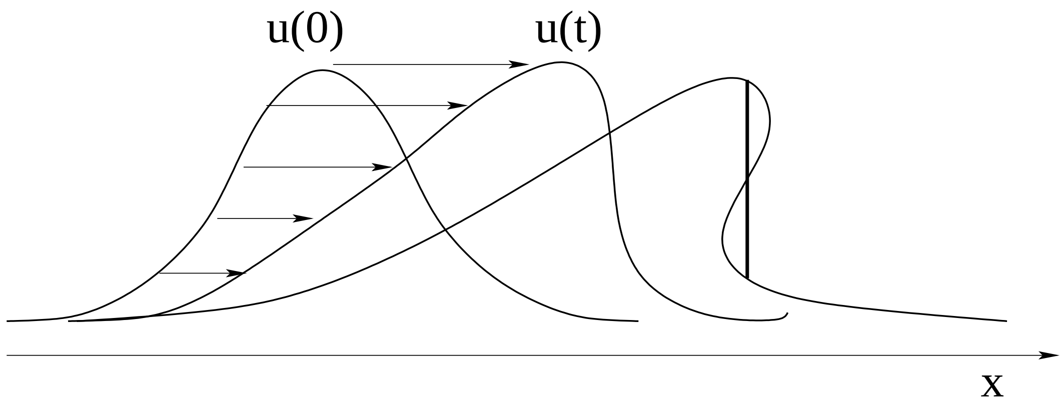
nonlinear

Loss of Regularity

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \phi(x)$$

Method of characteristics yields: $\quad u\big(t,\ x_0 + t\, f'(\phi(x_0))\big) \ = \ \phi(x_0)$

characteristic speed $=$ $f'(u)$



Global solutions only in a space of discontinuous functions

$$u(t,\cdot) \in BV$$

Wave Interactions

$$u_t \;=\; -A(u)u_x$$

$\lambda_i(u) = i$-th eigenvalue $\qquad$ $l_i(u)$, $r_i(u) = i$-th eigenvectors

$u_x^i \;\doteq\; l_i \cdot u_x \;=\; [i\text{-th component of } u_x] \;=\; [\text{density of } i\text{-waves in } u]$

$$u_x \;=\; \sum_{i=1}^{n} u_x^i r_i(u) \qquad\qquad u_t \;=\; -\sum_{i=1}^{n} \lambda_i(u) u_x^i r_i(u)$$

differentiate first equation w.r.t. t, second one w.r.t. x

$\Longrightarrow$ evolution equation for scalar components u_x^i

$$(u_x^i)_t + (\lambda_i u_x^i)_x \;=\; \sum_{j>k} (\lambda_j - \lambda_k)\big(l_i \cdot [r_j, r_k]\big) u_x^j u_x^k$$

source terms: $(\lambda_j - \lambda_k)\big(l_i \cdot [r_j, r_k]\big) u_x^j u_x^k$

$=$ amount of i-waves produced by the interaction of j-waves with k-waves

$$\begin{aligned} \lambda_j - \lambda_k \;&=\; \text{[difference in speed]} \\ &=\; \text{[rate at which } j\text{-waves and } k\text{-waves cross each other]} \end{aligned}$$

$$u_x^j u_x^k \;=\; \text{[density of } j\text{-waves]} \times \text{[density of } k\text{-waves]}$$

$$[r_j, r_k] \;=\; (Dr_k)r_j - (Dr_j)r_k \qquad \text{(Lie bracket)}$$

$$\begin{aligned} &=\; \text{[directional derivative of } r_k \text{ in the direction of } r_j] \\ &\qquad -\text{[directional derivative of } r_j \text{ in the direction of } r_k] \end{aligned}$$

$l_i \cdot [r_j, r_k] \;=\; i$-th component of the Lie bracket $[r_j, r_k]$ along the basis of eigenvectors $\{r_1, \ldots, r_n\}$

Shock solutions

$$u_t + f(u)_x \ = \ 0$$

$$u(t,x) = \begin{cases} u^- & \text{if} \quad x < \lambda t \\ u^+ & \text{if} \quad x > \lambda t \end{cases}$$

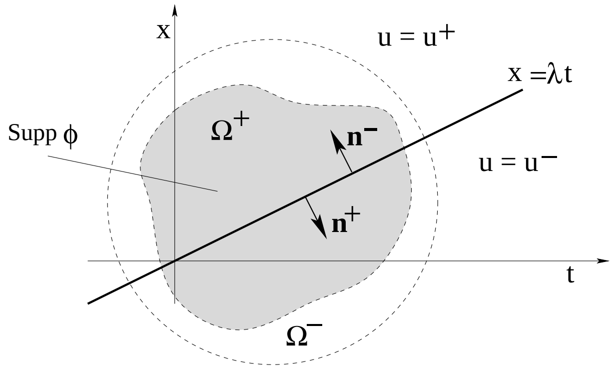
is a weak solution if and only if

$$\lambda \cdot [u^+ - u^-] \ = \ f(u^+) - f(u^-)$$ **Rankine - Hugoniot equations**

[speed of the shock] $\times$ [jump in the state] $=$ [jump in the flux]

Derivation of the Rankine - Hugoniot Equations

$$\iint \left\{ u\phi_t + f(u)\phi_x \right\} \, dxdt \;=\; 0 \qquad \text{for all} \quad \phi \in \mathcal{C}^1_c$$



$$\mathbf{v} \;\doteq\; \big(u\phi\,,\;\; f(u)\phi\big)$$

$$0 \;=\; \iint_{\Omega^+\cup\Omega^-} \operatorname{div} \mathbf{v}\, dxdt \;=\; \int_{\partial\Omega^+} \mathbf{n}^+ \cdot \mathbf{v}\, ds + \int_{\partial\Omega^-} \mathbf{n}^- \cdot \mathbf{v}\, ds$$

$$= \int [\lambda u^+ - f(u^+)]\,\phi(t,\lambda t)\,dt + \int [\,-\,\lambda u^- + f(u^-)]\,\phi(t,\lambda t)\,dt$$

$$= \int \Big[\lambda(u^+ - u^-) - (f(u^+) - f(u^-))\Big]\,\phi(t,\lambda t)\,dt\,.$$

Alternative formulation:

$$\lambda\,(u^+ - u^-) \;=\; f(u^+) - f(u^-) \;=\; \int_0^1 Df(\theta u^+ + (1-\theta)u^-)\cdot(u^+ - u^-)\;d\theta$$

$$=\; A(u^+, u^-)\cdot(u^+ - u^-)$$

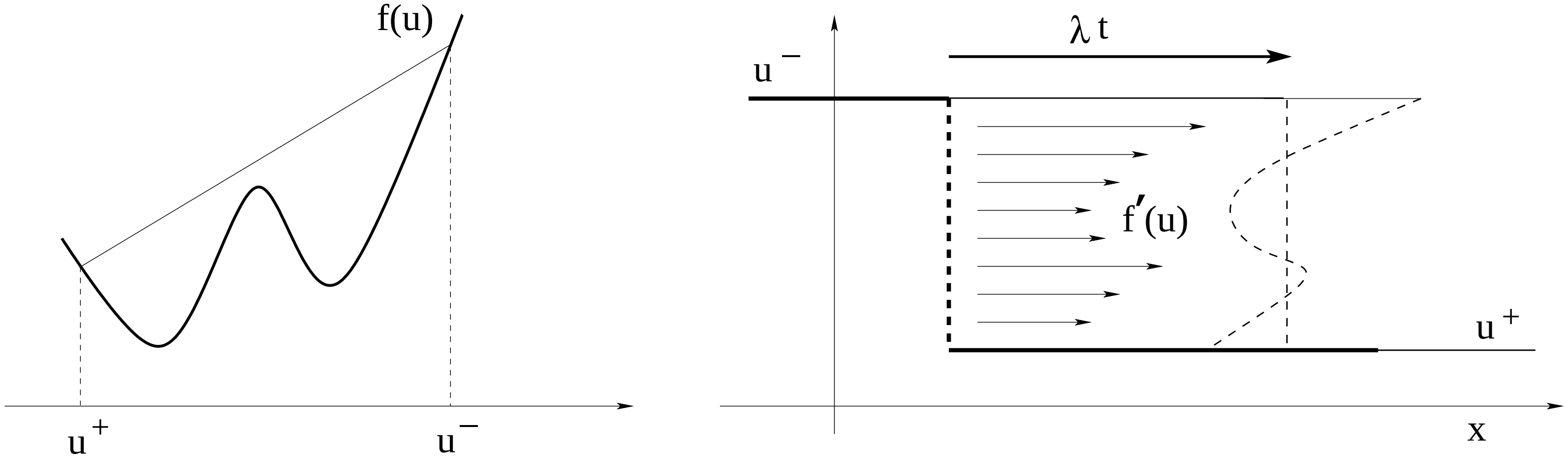
$$A(u,v) \doteq \int_0^1 Df(\theta u + (1-\theta)v)\;d\theta \;=\; \text{[averaged Jacobian matrix]}$$

The Rankine-Hugoniot conditions hold if and only if

$$\lambda(u^+ - u^-) \;=\; A(u^+, u^-)\,(u^+ - u^-)$$

- The jump $u^+ - u^-$ is an eigenvector of the averaged matrix $A(u^+, u^-)$
- The speed λ coincides with the corresponding eigenvalue

scalar conservation law: $\qquad u_t + f(u)_x = 0$



$$\lambda \;=\; \frac{f(u^+) - f(u^-)}{u^+ - u^-} \;=\; \frac{1}{u^+ - u^-}\int_{u^-}^{u^+} f'(s)\,ds$$

[speed of the shock] $=$ [slope of secant line through u^-, u^+ on the graph of f]

$=$ [average of the characteristic speeds between u^- and u^+]

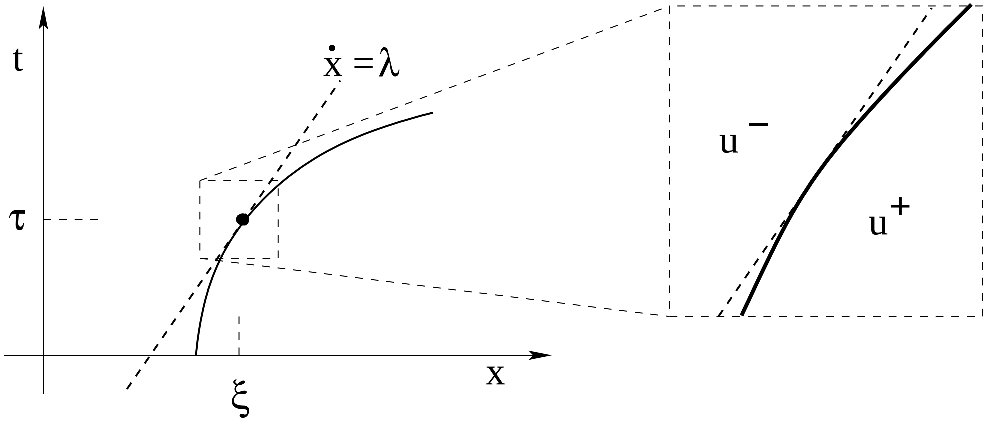
Points of Approximate Jump

The function $u = u(t,x)$ has an **approximate jump** at a point (τ, ξ) if there exists states $u^- \neq u^+$ and a speed λ such that, calling

$$U(t,x) \doteq \begin{cases} u^- & \text{if} \quad x < \lambda t, \\ u^+ & \text{if} \quad x > \lambda t, \end{cases}$$

there holds

$$\lim_{\rho \to 0+} \frac{1}{\rho^2} \int_{\tau-\rho}^{\tau+\rho} \int_{\xi-\rho}^{\xi+\rho} \Big| u(t,x) - U(t-\tau,\ x-\xi) \Big| \, dxdt = 0 \qquad (2)$$



Theorem. *If u is a weak solution to the system of conservation laws $u_t + f(u)_x = 0$ then the Rankine-Hugoniot equations hold at each point of approximate jump.*

Construction of Shock Curves

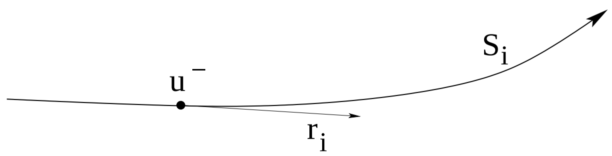
Problem: Given $u^- \in \mathbb{R}^n$, find the states $u^+ \in \mathbb{R}^n$ which, for some speed λ, satisfy the Rankine - Hugoniot equations

$$\lambda(u^+ - u^-) \;=\; f(u^+) - f(u^-) \;=\; A(u^-, u^+)(u^+ - u^-)$$

Alternative formulation: Fix $i \in \{1, \ldots, n\}$. The jump $u^+ - u^-$ is a (right) i-eigenvector of the averaged matrix $A(u^-, u^+)$ if and only if it is orthogonal to all (left) eigenvectors $l_j(u^-, u^+)$ of $A(u^-, u^+)$, for $j \neq i$

$$l_j(u^-, u^+) \cdot (u^+ - u^-) \;=\; 0 \qquad \text{for all } j \neq i \qquad (RH_i)$$

Implicit function theorem $\Longrightarrow$ for each i there exists a curve



$s \mapsto S_i(s)(u^-)$ of points that satisfy (RH$_i$)

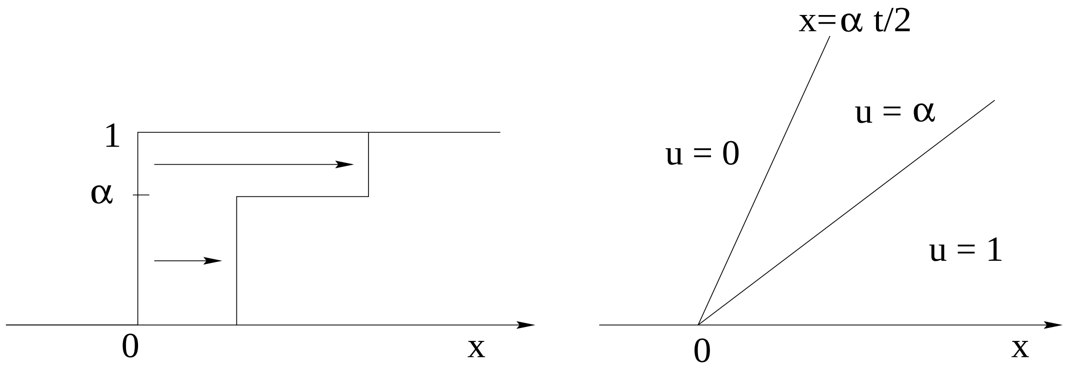
Weak solutions can be non-unique

Example: a Cauchy problem for Burgers' equation

$$u_t + (u^2/2)_x = 0 \qquad\qquad u(0,x) = \begin{cases} 1 & \text{if} \quad x \geq 0, \\ 0 & \text{if} \quad x < 0. \end{cases}$$

Each $\alpha \in [0,1]$ yields a weak solution

$$u_\alpha(t,x) = \begin{cases} 0 & \text{if} \quad x < \alpha t/2, \\ \alpha & \text{if} \quad \alpha t/2 \leq x < (1+\alpha)t/2, \\ 1 & \text{if} \quad x \geq (1+\alpha)t/2. \end{cases}$$



Admissibility Conditions on Shocks

For physical systems:

a concave entropy should not decrease

For general hyperbolic systems:

require stability w.r.t. small perturbations

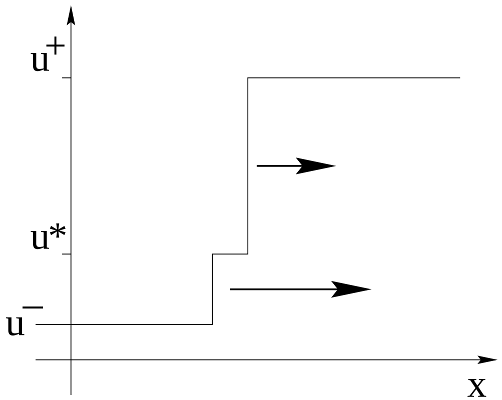
Stability Conditions: the Scalar Case

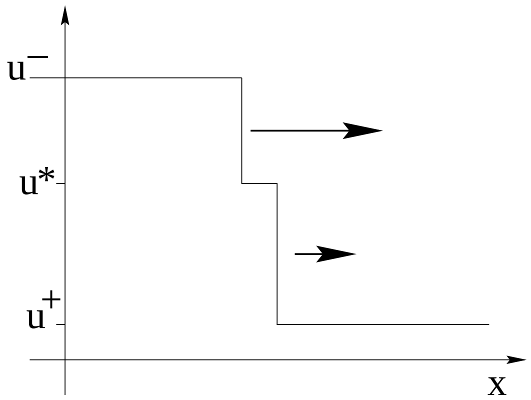
Perturb the shock with left and right states u^-, u^+ by inserting an intermediate state $u^* \in [u^-, u^+]$

Initial shock is stable $\Longleftrightarrow$

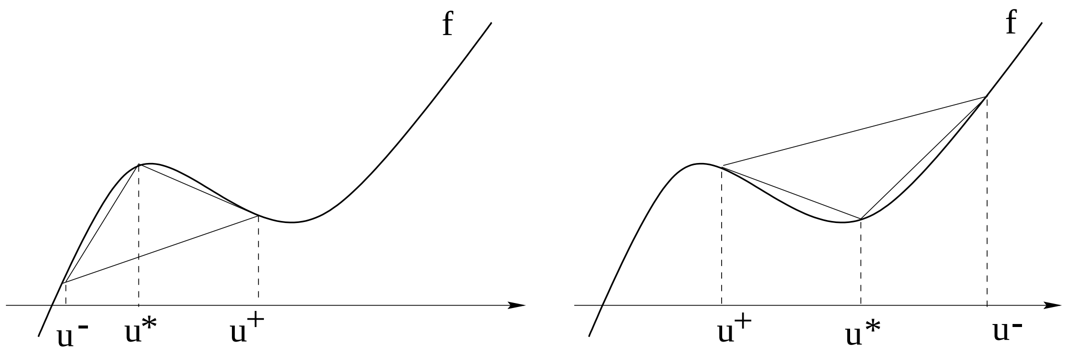
[speed of jump behind] $\geq$ [speed of jump ahead]

$$\frac{f(u^*) - f(u^-)}{u^* - u^-} \geq \frac{f(u^+) - f(u^*)}{u^+ - u^*}$$





speed of a shock = slope of a secant line to the graph of f



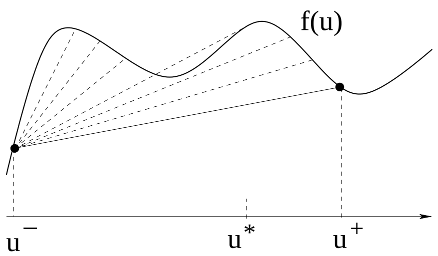
Stability conditions:

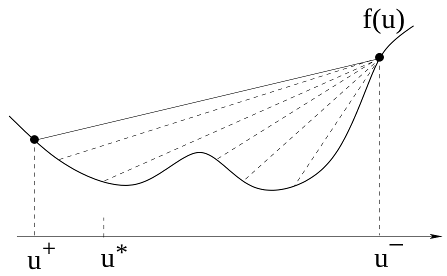
- when $u^- < u^+$ the graph of f should remain above the secant line
- when $u^- > u^+$, the graph of f should remain below the secant line

General Stability Conditions

Scalar case: stability holds if and only if

$$\frac{f(u^*) - f(u^-)}{u^* - u^-} \geq \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$

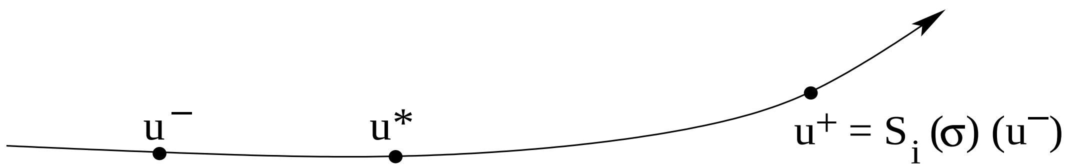


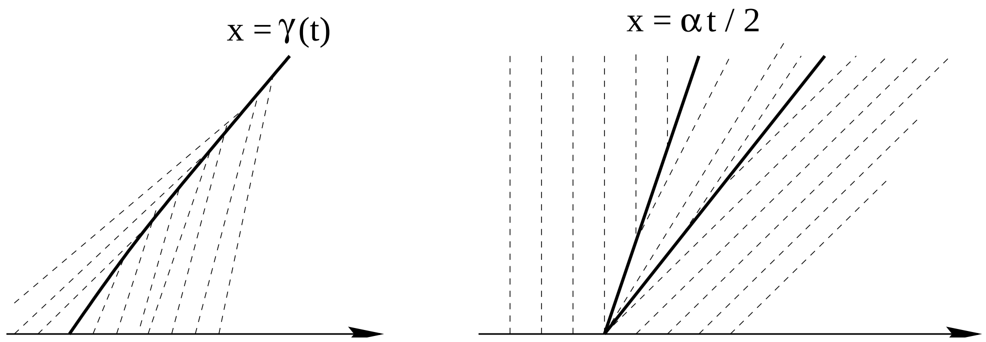


Vector valued case: $u^+ = S_i(\sigma)(u^-)$ for some $\sigma \in I\!R$.

Admissibility Condition (T.P.Liu, 1976). The speed $\lambda(\sigma)$ of the shock joining u^- with u^+ must be less or equal to the speed of every smaller shock, joining u^- with an intermediate state $u^* = S_i(s)(u^-)$, $s \in [0, \sigma]$.

$$\lambda(u^-, u^+) \;\leq\; \lambda(u^-, u^*)$$





Admissibility Condition (P. Lax, 1957) A shock connecting the states u^-, u^+, travelling with speed $\lambda = \lambda_i(u^-, u^+)$ is *admissible* if

$$\lambda_i(u^-) \geq \lambda_i(u^-, u^+) \geq \lambda_i(u^+)$$

[Liu condition] $\Longrightarrow$ [Lax condition]

Entropy - Entropy Flux

$$u_t + f(u)_x = 0$$

Definition. A function $\eta : \mathbb{R}^n \mapsto \mathbb{R}$ is called an **entropy**, with **entropy flux** $q : \mathbb{R}^n \mapsto \mathbb{R}$ if

$$D\eta(u) \cdot Df(u) = Dq(u)$$

For **smooth** solutions $u = u(t,x)$, this implies

$$\eta(u)_t + q(u)_x \;=\; D\eta \cdot u_t + Dq \cdot u_x \;=\; -D\eta \cdot Df \cdot u_x + Dq \cdot u_x \;=\; 0$$

$\Longrightarrow$ $\eta(u)$ is an additional conserved quantity, with flux $q(u)$

Entropy Admissibility Condition

A weak solution u of the hyperbolic system $\quad u_t + f(u)_x = 0$

is **entropy admissible** if

$$[\eta(u)]_t + [q(u)]_x \leq 0$$

in the sense of distributions, for every pair (η, q), where η is a **convex entropy** and q is the corresponding entropy flux.

$$\iint \{\eta(u)\varphi_t + q(u)\varphi_x\}\ dxdt \geq 0 \qquad\qquad \varphi \in \mathcal{C}_c^1, \quad \varphi \geq 0$$

- smooth solutions conserve all entropies

- solutions with shocks are admissible if they dissipate all convex entropies

Existence of entropy - entropy flux pairs

$$D\eta(u) \cdot Df(u) = Dq(u)$$

$$\begin{pmatrix} \frac{\partial \eta}{\partial u_1} & \cdots & \frac{\partial \eta}{\partial u_n} \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_n} \\ & \cdots & \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial q}{\partial u_1} & \cdots & \frac{\partial q}{\partial u_n} \end{pmatrix}$$

- a system of n equations for 2 unknown functions: $\eta(u), \ q(u)$

- over-determined if $n > 2$

- however, some physical systems (described by several conservation laws) are endowed with natural entropies