Piecewise Smooth Solutions to the Burgers-Hilbert Equation

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Abstract

The paper is concerned with the Burgers-Hilbert equation \( u_t + (u^2/2)_x = H[u] \), where the right hand side is a Hilbert transform. Unique entropy admissible solutions are constructed, locally in time, having a single shock. In a neighborhood of the shock curve, a detailed description of the solution is provided.

1 Introduction

Consider the balance law obtained from Burgers’ equation by adding the Hilbert transform as a source term:

\[
  u_t + \left( \frac{u^2}{2} \right)_x = H[u].
\]

(1.1)

Here

\[
  H[f](x) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(x-y)}{y} \, dy
\]

(1.2)

denotes the Hilbert transform of a function \( f \in L^2(\mathbb{R}) \). The above equation was derived in [1] as a model for nonlinear waves with constant frequency. For initial data

\[
  u(0, x) = \bar{u}(x),
\]

(1.3)
in \( H^2(\mathbb{R}) \), the local existence and uniqueness of the solution to (1.1) was proved in [7], together with a sharp estimate on the time interval where this solution remains regular. See also [8] for a shorter proof. For general initial data \( \bar{u} \in L^2(\mathbb{R}) \), the global existence of entropy weak solutions was recently proved in [4] together with a partial uniqueness result. We remark that, in this general setting, the well-posedness of the Cauchy problem remains a largely open question.

In the present paper we consider an intermediate situation. Namely, we construct solutions of (1.1) which are piecewise continuous, with a single shock. Our solutions have the form

\[
  u(t, x) = \varphi(x - y(t)) + w(t, x - y(t)),
\]

In...
where \( t \mapsto y(t) \) denotes the location of the shock. Here \( w \in H^2([-\infty, 0[ \cup]0, +\infty]) \), while \( \varphi(x) = \frac{2}{\pi} |x| \ln |x| \), for \( x \) near the origin.

In Section 2 we write (1.1) in an equivalent form, and state an existence-uniqueness theorem, locally in time. The key a priori estimates on approximate solutions, and a proof of the main theorem, are then worked out in Sections 3 to 5.

The present results can be easily extended to the case of solutions with finitely many, non-interacting shocks. An interesting open problem is to describe the local behavior of a solution in a neighborhood of a point \( (t_0, x_0) \) where either (i) a new shock is formed, or (ii) two shocks merge into a single one. Motivated by the analysis in [12] we conjecture that, for generic initial data

\[
\bar{u} \in H^2(\mathbb{R}) \cap C^3(\mathbb{R}),
\]

the corresponding solution of (1.1) remains piecewise smooth with finitely many shock curves on any domain of the form \([0, T] \times \mathbb{R}\). We thus regard the present results as a first step toward a description of all generic singularities. For other examples of hyperbolic equations where generic singularities have been studied we refer to [2, 3, 5, 6, 9]. The possible emergence of singularities, for more general dispersive perturbations of Burgers’ equation, has been recently studied in [10].

## 2 Statement of the main result

Consider a piecewise smooth solution of (1.1) with one single shock. Calling \( y(t) \) the location of the shock at time \( t \), by the Rankine-Hugoniot conditions we have

\[
\dot{y}(t) = \frac{u^-(t) + u^+(t)}{2}.
\]

(2.1)

where \( u^-, u^+ \) denote the left and right limits of \( u(t, x) \) as \( x \to y(t) \). Here and in the sequel, the upper dot denotes a derivative w.r.t. time. It is convenient to shift the space coordinate, replacing \( x \) with \( x - y(t) \), so that in the new coordinate system the shock is always located at the origin. In these new coordinates, the equation (1.1) takes the equivalent form

\[
u_t + \left( \frac{u^2}{2} \right)_x - \dot{y} u_x = H[u].
\]

(2.2)

We shall construct solutions to (2.2) in a special form, providing a cancellation between leading order terms in the transport equation and the Hilbert transform.

Consider a smooth function with compact support \( \eta \in C_\infty^\infty(\mathbb{R}) \), with \( \eta(x) = \eta(-x) \), and such that

\[
\begin{cases}
\eta(x) = 1 & \text{if } |x| \leq 1, \\
\eta(x) = 0 & \text{if } |x| \geq 2, \\
\eta'(x) \leq 0 & \text{if } x \in [1, 2].
\end{cases}
\]

(2.3)

Moreover, define

\[
\varphi(x) = \frac{2|x| \ln |x|}{\pi} \cdot \eta(x).
\]

(2.4)
Notice that \( \varphi \) has support contained in the interval \([-2, 2]\) and is smooth separately on the domains \( \{x < 0\} \) and \( \{x > 0\} \).

In addition, we consider the space of functions

\[
\mathcal{H} = H^2([-\infty, 0] \cup [0, +\infty]).
\]  

(2.5)

Every function \( w \in \mathcal{H} \) is continuously differentiable outside the origin. The distributional derivative of \( w \) is an \( L^2 \) function restricted to the half lines \([-\infty, 0) \) and \((0, +\infty) \). However, both \( w \) and \( w_x \) can have a jump at the origin. It is clear that the traces

\[
\begin{align*}
  u^- &\doteq w(0^-), \\
  u^+ &\doteq w(0^+), \\
  b^- &\doteq w_x(0^-), \\
  b^+ &\doteq w_x(0^+),
\end{align*}
\]  

(2.6)

are continuous linear functionals on \( \mathcal{H} \).

Figure 1: Decomposing a piecewise regular function \( u = \varphi + w \) as a sum of the function \( \varphi \) defined at (2.4) and a function \( w \in H^2(\mathbb{R} \setminus \{0\}) \), continuously differentiable outside the origin.

Solutions of (2.2) will be constructed in the form

\[
u(t, x) = \varphi(x) + w(t, x).
\]  

(2.7)

In order that the shock be entropy admissible, the function \( w \) should range in the open domain

\[
\mathcal{D} = \left\{ w \in H^2(\mathbb{R} \setminus \{0\}) ; \ w(0-) > w(0+) \right\}.
\]  

(2.8)

By (2.6)-(2.8), for \( x \approx 0 \) this solution has the asymptotic behavior

\[
u(t, x) = \begin{cases}
  u^-(t) + b^-(t) x + \frac{2|x| \ln |x|}{\pi} + \mathcal{O}(1) \cdot |x|^{3/2} & \text{if } x < 0, \\
  u^+(t) + b^+(t) x + \frac{2|x| \ln |x|}{\pi} + \mathcal{O}(1) \cdot |x|^{3/2} & \text{if } x > 0,
\end{cases}
\]  

(2.9)

for suitable functions \( u^\pm, b^\pm \). Here and throughout the sequel, the Landau symbol \( \mathcal{O}(1) \) denotes a uniformly bounded quantity.

Inserting (2.7) in the equation (2.2) and recalling (2.6), one obtains

\[
w_t + \left( \varphi + w - \frac{u^- + u^+}{2} \right) (\varphi_x + w_x) = H[\varphi] + H[w].
\]  

(2.10)

To derive estimates on the Hilbert transform, the following observation is useful. Consider a function \( f \) with compact support, continuously differentiable for \( x < 0 \) and for \( x > 0 \), with a
jump at the origin. Then, for any $x \neq 0$, an integration by parts yields\(^1\)
\[
H[f](x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f'(y) \ln|x-y| \, dy + \frac{1}{\pi} \left[ f(0+) - f(0-) \right] \ln|x|.
\]  
(2.11)

A similar computation shows that, to leading order, the Hilbert transform of $w$ near the origin is given by
\[
H[w](x) = \frac{u^+ - u^-}{\pi} \ln|x| + O(1),
\]  
(2.12)

with $u^-, u^+$ as in (2.6). On the other hand, for $x \approx 0$ one has
\[
\left( \varphi(x) + w(x) - \frac{w(0-) + w(0+)}{2} \right) \varphi_x(x) = \left( \text{sign}(x) \cdot \frac{u^+ - u^-}{2} + O(1) \cdot |x| \ln|x| \right) \cdot \frac{2 \text{sign}(x) \cdot (1 + \ln|x|)}{\pi} = \frac{u^+ - u^-}{\pi} \ln|x| + O(1).
\]  
(2.13)

The identity between the leading terms in (2.12) and (2.13) achieves a crucial cancellation between the two sides of (2.10). It is thus convenient to write this equation in the equivalent form
\[
w_t + \left( \varphi + w - \frac{u^- + u^+}{2} \right) w_x = H[\varphi] - \varphi \varphi_x + \left( H[w] - \left( w - \frac{u^- + u^+}{2} \right) \varphi_x \right).
\]  
(2.14)

**Definition.** By an entropic solution to the Cauchy problem (2.10) with initial data
\[
w(0, \cdot) = \bar{w} \in \mathcal{D},
\]  
(2.15)

we mean a function $w : [0, T] \times \mathcal{I} \mapsto \mathcal{I}$ such that

(i) For every $t \in [0, T]$, the norm $\|w(t, \cdot)\|_{H^2(\mathcal{I}; \mathcal{D})}$ remains uniformly bounded. As $x \to 0$, the limits satisfy
\[
u^-(t) \doteq u(t, 0-) > u(t, 0+) \doteq u^+(t).
\]  
(2.16)

\(^1\)Indeed, if $f \in C^\infty_c(\mathcal{I})$, then for a suitably large constant $M$ we have
\[
\pi \cdot H[f](x) = \lim_{\varepsilon \to 0^+} \int_{|y-x| > \varepsilon} \frac{f(x-y)}{y} \, dy = -\lim_{\varepsilon \to 0^+} \int_{|y-x| > \varepsilon} \frac{f(x+y)}{y} \, dy
\]
\[
= -\lim_{\varepsilon \to 0^+} \left( \int_{-M}^{-\varepsilon} + \int_{\varepsilon}^{M} \right) \frac{f(x+y) - f(x)}{y} \, dy
\]
\[
= \lim_{\varepsilon \to 0^+} \left( \int_{-M}^{-\varepsilon} + \int_{\varepsilon}^{M} \right) f'(x+y) \ln|y| \, dy - \lim_{\varepsilon \to 0^+} [f(x-\varepsilon) - f(x)] \ln \varepsilon
\]
\[
+ \lim_{\varepsilon \to 0^+} [f(x+\varepsilon) - f(x)] \ln \varepsilon + [f(x-M) - f(x)] \ln M - [f(x+M) - f(x)] \ln M
\]
\[
= \int_{-\infty}^{\infty} f'(x+y) \ln|y| \, dy = \int_{-\infty}^{\infty} f'(y) \ln|x-y| \, dy.
\]

By approximating $f$ with a sequence of smooth functions with compact support we obtain (2.11).
(ii) The equation (2.14) is satisfied in integral sense. Namely, for every \( t_0 \geq 0 \) and \( x_0 \neq 0 \), calling \( t \mapsto x(t; t_0, x_0) \) the solution to the Cauchy problem

\[
\dot{x} = \varphi(x) + w(t, x) - \frac{u^-(t) + u^+(t)}{2}, \quad x(t_0) = x_0,
\]

one has

\[
w(t_0, x_0) = \bar{w}(x(0; t_0, x_0)) + \int_{t_0}^t F(t, x(t; t_0, x_0)) \, dt,
\]

with

\[
F = H[\varphi] - \varphi \varphi_x + \left( H[w] - \left( w - \frac{u^- + u^+}{2} \right) \varphi_x \right).
\]

A few remarks are in order:

(i) The bound on the norm \( \|w(t, \cdot)\|_{H^2} \) implies that the limits in (2.16) are well defined. By requiring that the inequality in (2.16) holds we make sure that the shock is entropy admissible.

(ii) Since \( w(t, \cdot) \in H^2(\mathbb{R} \setminus \{0\}) \), the right hand side of the ODE in (2.17) is continuously differentiable w.r.t. \( x \). Combined with the inequalities in (2.16), this implies that the backward characteristic \( t \mapsto x(t; t_0, x_0) \) is well defined for all \( t \in [0, t_0] \).

(iii) In [11], a function satisfying the integral equations (2.18) was called a broad solution. The regularity assumption on \( w(t, \cdot) \) and the fact that the source term \( F \) in (2.19) is continuous outside the origin imply that \( w = w(t, x) \) is continuously differentiable w.r.t. both variables \( t, x \), for \( x \neq 0 \). Therefore, the identity in (2.14) is satisfied at every point \( (t, x) \), with \( x \neq 0 \).

The main result of this paper provides the existence and uniqueness of an entropic solution, locally in time.

**Theorem 1.** For every \( \bar{w} \in \mathcal{D} \) there exists \( T > 0 \) such that the Cauchy problem (2.2), (2.15) admits a unique entropic solution, defined for \( t \in [0, T] \).

In turn, Theorem 1 yields the existence of a piecewise regular solution to the Burgers-Hilbert equation (1.1), locally in time, for initial data of the form

\[ u(0, x) = \varphi(x) + \bar{w}(x), \]

with \( \bar{w} \in \mathcal{D} \).

The solution \( w = w(t, x) \) of (2.14) will be obtained as a limit of a sequence of approximations. More precisely, for \( n = 1 \), we define

\[
w_1(t, \cdot) = \bar{w} \quad \text{for all } t \geq 0.
\]
Next, let the \( n \)-th approximation \( w_n(t, x) \) be constructed. By induction, we then define \( w_{n+1}(t, x) \) to be the solution of the linear, non-homogeneous Cauchy problem

\[
wt + \left( \varphi + w_n - \frac{u^+ - u^-}{2} \right) wx = H[\varphi] - \varphi \varphi_x + \left( H[w] - \left( w - \frac{u^+ - u^-}{2} \right) \varphi_x \right). \tag{2.21}
\]

with initial data (2.15).

The induction argument requires three steps:

(i) Existence and uniqueness of solutions to the linear problem (2.21) with initial data (2.15).

(ii) A priori bounds on the strong norm \( \|w_n(t)\|_{H^2(\mathbb{R}\setminus\{0\})} \), uniformly valid for \( t \in [0, T] \) and all \( n \geq 1 \).

(iii) Convergence in a weak norm. This will follow from the bound

\[
\sum_{n \geq 1} \|w_{n+1}(t) - w_n(t)\|_{H^1(\mathbb{R}\setminus\{0\})} < \infty.
\]

In the following sections we shall provide estimates on each term on the right hand side of (2.21), and complete the above steps (i)–(iii).

### 3 Estimates on the source terms

To estimate the right hand side of (2.21), we consider again the cutoff function \( \eta \) in (2.3) and split an arbitrary function \( w \in H^2(\mathbb{R}\setminus\{0\}) \) as a sum:

\[
w = v_1 + v_2 + v_3, \tag{3.1}
\]

where

\[
v_1(x) \doteq \begin{cases} w(0-) \cdot \eta(x) & \text{if } x < 0, \\ w(0+) \cdot \eta(x) & \text{if } x > 0, \end{cases} \quad v_2(x) \doteq \begin{cases} w_x(0-) \cdot x \eta(x) & \text{if } x < 0, \\ w_x(0+) \cdot x \eta(x) & \text{if } x > 0, \end{cases} \quad v_3 = w - v_1 - v_2. \tag{3.2}
\]

The right hand side of (2.21) can be expressed as the sum of the following terms:

\[
A \doteq H[\varphi], \quad B \doteq \varphi \varphi_x, \quad C \doteq H[v_2 + v_3], \quad D \doteq H[v_1] - \left( w - \frac{u^+ - u^-}{2} \right) \varphi_x. \tag{3.4}
\]

The goal of this section is to provide a priori bounds of the size of these source terms and on their first and second derivatives.

**Lemma 1.** There exists constants \( K_0, K_1 \) such that the following holds. For any \( \delta \in ]0, 1/2] \) and any \( w \in H^2(\mathbb{R}\setminus\{0\}) \), the source terms in (3.4) satisfy

\[
\|A\|_{H^2(\mathbb{R}\setminus[-\delta, \delta])} + \|B\|_{H^2(\mathbb{R}\setminus[-\delta, \delta])} \leq K_0 \cdot \delta^{-2/3}, \tag{3.5}
\]

\[
\|C\|_{H^2(\mathbb{R}\setminus[-\delta, \delta])} \leq K_1 \cdot \delta^{1/3},
\]

\[
\|D\|_{H^2(\mathbb{R}\setminus[-\delta, \delta])} \leq K_2 \cdot \delta^{-1/3}.
\]
\[ \| C \|_{H^2(\mathbb{R}[\delta,\delta])} + \| D \|_{H^2(\mathbb{R}[\delta,\delta])} \leq K_1 \delta^{-2/3} \cdot \| w \|_{H^2(\mathbb{R}\setminus \{0\})}. \]  

(3.6)

**Proof.** 1. We begin by observing that the function \( \varphi \) is continuous with compact support, smooth outside the origin. Therefore, the Hilbert transform \( A = \mathcal{H}[\varphi] \) is smooth outside the origin. As \( |x| \to \infty \) one clearly has
\[ A(x) = \mathcal{O}(1) \cdot x^{-1}, \quad A_x(x) = \mathcal{O}(1) \cdot x^{-2}, \quad A_{xx}(x) = \mathcal{O}(1) \cdot x^{-3}. \]  

(3.7)

In addition, as \( x \to 0 \), we claim that
\[ A(x) = \mathcal{O}(1) \cdot x \ln^2 |x|, \quad A_x(x) = \mathcal{O}(1) \cdot \ln^2 |x|, \quad A_{xx}(x) = \mathcal{O}(1) \cdot \frac{\ln |x|}{x}. \]  

(3.8)

Indeed, to fix the ideas, let \( 0 < x < 1/2 \). By (2.11) we have
\[ \pi \cdot \mathcal{H}[\varphi](x) = \int_{-2}^{2} \varphi'(y) \ln |x-y| \, dy = I_1 + I_2 + I_3, \]  

(3.9)

where:
\[ I_1 = \left( \int_{-2}^{-1} + \int_{1}^{2} \right) \varphi'(y) \ln |x-y| \, dy = \mathcal{O}(1) \cdot x, \]  

(3.10)

\[ \frac{\pi}{2} I_2 = \int_{-1}^{0} - \ln |x-y| \, dy + \int_{0}^{1} \ln |x-y| \, dy = \left( \int_{-x}^{x} - \int_{1-x}^{0} \right) \ln |y| \, dy = \mathcal{O}(1) \cdot x \ln x, \]  

(3.11)

and moreover,
\[ \frac{\pi}{2} I_3 = \int_{0}^{1} \ln |y| \ln |x-y| \, dy + \int_{-1}^{0} - \ln |y| \ln |x-y| \, dy \]
\[ = \left( \int_{0}^{x/2} + \int_{x/2}^{x} + \int_{x-1}^{1} - \int_{1}^{0} \right) \ln |y| \ln |x-y| \, dy \]
\[ = \left( \int_{0}^{x/2} + \int_{x/2}^{x} \right) \ln |y| \ln |x-y| \, dy - \int_{0}^{x} \ln |y - 1| \ln |x-y + 1| \, dy \]
\[ = I_{31} + I_{32} + I_{33}. \]  

(3.12)

We now have
\[ |I_{31}| \leq \ln \left| \frac{x}{2} \right| \cdot \int_{0}^{x/2} \ln |y| \, dy = \mathcal{O}(1) \cdot x \ln^2 |x|, \]  

(3.13)

\[ |I_{32}| \leq \ln \left| \frac{x}{2} \right| \cdot \int_{x/2}^{x} \ln |x-y| \, dy = \mathcal{O}(1) \cdot x \ln^2 |x|, \]  

\[ |I_{33}| \leq \int_{0}^{x} \ln |1-x| \ln |1+x| \, dy = \mathcal{O}(1) \cdot x^3. \]

Hence \( \mathcal{H}[\varphi] = \mathcal{O}(1) \cdot x \ln^2 |x| \). This yields the first estimate in (3.8).
Next, we estimate the derivative $\pi \partial_x H[\varphi] = \partial_x I_1 + \partial_x I_2 + \partial_x I_3$. The term $|\partial_x I_1|$ is uniformly bounded, while
\[
\frac{\pi}{2} \partial_x I_2 = \int_0^{2x} \frac{1}{x-y} \, dy + \int_{2x}^{1} \frac{1}{x-y} \, dy - \int_{-1}^{0} \frac{1}{x-y} \, dy = O(1) \cdot \ln |x|.
\] (3.14)

Differentiating $I_3$ w.r.t. $x$ we obtain
\[
\frac{\pi}{2} \partial_x I_3 = \left( \int_{-x/2}^{-x/2} + \int_{-x/2}^{0} \right) - \ln |y| \, dy + \left( \int_0^{x/2} + \int_0^{1} \right) \frac{\ln |y|}{x-y} \, dy
\]
\[
+ \lim_{\epsilon \to 0} \left( \int_{x/2}^{x/2} + \int_{x+\epsilon}^{3x/2} \right) \frac{\ln y}{x-y} \, dy.
\] (3.15)

Assuming $0 < x < 1/2$, we obtain
\[
\int_{-x/2}^{-x/2} \frac{-\ln |y|}{x-y} \, dy \leq \int_{-x/2}^{x/2} \frac{-\ln |y|}{|y|} \, dy = O(1) \cdot \ln^2 |x|,
\]
\[
\int_{-x/2}^{0} \frac{-\ln |y|}{x-y} \, dy \leq \int_{-x/2}^{0} \frac{-\ln |y|}{x} \, dy = O(1) \cdot \ln |x|,
\]
\[
\int_{0}^{x/2} \frac{\ln |y|}{x-y} \, dy \leq \int_{0}^{x/2} \frac{\ln |y|}{x/2} \, dy = O(1) \cdot \ln |x|,
\]
\[
\int_{3x/2}^{1} \frac{\ln |y|}{x-y} \, dy \leq \ln \left| \frac{3x}{2} \right| \int_{3x/2}^{1} \frac{1}{x-y} \, dy = O(1) \cdot \ln^2 |x|.
\]

The remaining term is estimated as
\[
\left( \int_{x/2}^{x-\epsilon} + \int_{x-\epsilon}^{3x/2} \right) \frac{\ln y}{x-y} \, dy = \left( \int_{x/2}^{x-\epsilon} + \int_{x-\epsilon}^{3x/2} \right) \frac{\ln y - \ln x}{x-y} \, dy \leq \frac{2}{x} (x-2\epsilon) \leq 2.
\]

Combining the previous estimates we obtain $\partial_x H[\varphi](x) = O(1) \cdot \ln^2 |x|$. This gives the second estimate in (3.8).

Finally, we estimate the second derivative of the Hilbert transform $\partial_{xx} H[\varphi] = \sum_{i=1}^{3} \partial_{xx} (I_i)$.

By (3.10) and (3.14) we obtain
\[
\frac{\pi}{2} \partial_{xx} I_2 = - \int_{2x}^{1} \frac{1}{(x-y)^2} \, dy + \int_{-1}^{0} \frac{1}{(x-y)^2} \, dy = O(1) \cdot \frac{\ln |x|}{x}.
\] (3.16)

\[
\frac{\pi}{2} \partial_{xx} I_3 = \left( \int_{-x/2}^{-x/2} + \int_{-x/2}^{0} \right) \frac{\ln |y|}{(x-y)^2} \, dy - \left( \int_{0}^{x/2} + \int_{0}^{1} \right) \frac{\ln |y|}{(x-y)^2} \, dy
\]
\[
+ \frac{\ln |x/2|}{x} + \frac{3 \ln |3x/2|}{x} + \partial_x \left( \int_{x/2}^{3x/2} \frac{\ln |y|}{x-y} \, dy \right).
\] (3.17)

8
Assuming $0 < x < 1/2$, we obtain

$$\left| \int_{-x/2}^{x/2} \frac{\ln |y|}{(x - y)^2} \, dy \right| \leq \frac{\ln x}{2} \int_{-1}^{1} \frac{1}{(x - y)^2} \, dy = O(1) \cdot \ln \frac{x}{x},$$

$$\int_{-x/2}^{x/2} \frac{\ln |y|}{(x - y)^2} \, dy \leq \int_{-x/2}^{x/2} \frac{-\ln |y|}{x^2} \, dy = O(1) \cdot \ln \frac{x}{x},$$

$$\int_{x/2}^{x/2} \frac{\ln |y|}{(x - y)^2} \, dy \leq \int_{x/2}^{x/2} \frac{\ln |y|}{(x/2)^2} \, dy = O(1) \cdot \ln \frac{x}{x},$$

$$\int_{3x/2}^{x/2} \frac{\ln |y|}{(x - y)^2} \, dy \leq \ln \left| \frac{3x}{2} \right| \int_{3x/2}^{x} \frac{1}{(x - y)^2} \, dy = O(1) \cdot \ln \frac{x}{x}.$$  

The remaining term is estimated by

$$\partial_x \left( \int_{x/2}^{x/2} \frac{\ln |y|}{x - y} \, dy \right) = \partial_x \left( \int_{-x/2}^{x/2} \frac{\ln |x - y|}{y} \, dy \right) = \int_{-x/2}^{x/2} \frac{1}{y(x - y)} \, dy + \frac{\ln |x/2|}{x} - \frac{\ln |3x/2|}{x},$$

$$\left| \int_{-x/2}^{x/2} \frac{1}{y(x - y)} \, dy \right| = \left| \int_{-x/2}^{x/2} \frac{1}{y(x - y)} \, dy \right| = \left| \int_{-x/2}^{x/2} \frac{1}{y(x - y)} \, dy \right| \leq \frac{2}{x}. \quad (3.20)$$

Therefore, by (3.16) and (3.18) – (3.20), we have $\partial_{xx} H[\varphi](x) = O(1) \cdot \frac{\ln |x|}{x}$.

2. The function $B = \varphi \varphi_x$ is smooth outside the origin and vanishes for $|x| \geq 2$. As $x \to 0$, the following estimates are straightforward:

$$B(x) = O(1) \cdot |x| \ln^2 |x|, \quad B_x(x) = O(1) \cdot \ln |x|, \quad B_{xx}(x) = O(1) \cdot \frac{\ln |x|}{|x|}. \quad (3.21)$$

3. Next, we observe that $v_3 \in H^2(\mathbb{R})$. Moreover, there exists a constant $C_\eta$ such that

$$\|v_3\|_{H^2(\mathbb{R})} \leq C_\eta \cdot \|w\|_{H^2(\mathbb{R}\setminus\{0\})}.$$ 

Clearly, the Hilbert transform $H[v_3]$ satisfies the same bounds. Hence

$$\|H[v_3]\|_{H^2(\mathbb{R})} = O(1) \cdot \|w\|_{H^2(\mathbb{R}\setminus\{0\})}. \quad (3.22)$$

We observe that $v_2$ is Lipschitz continuous, has compact support and is continuously differentiable outside the origin. Since $v_2$ has better regularity properties than $\varphi$, the same arguments used to estimate the Hilbert transform of $\varphi$ also apply to $H[v_2]$. More precisely, as in (3.7) for $|x| \to \infty$ we have

$$H[v_2](x) = O(1) \cdot x^{-1}, \quad H[v_2]_x(x) = O(1) \cdot x^{-2}, \quad H[v_2]_{xx}(x) = O(1) \cdot x^{-3}. \quad (3.23)$$
As in (3.8), for \( x \to 0 \) we have
\[
H[v_2](x) = \mathcal{O}(1) \cdot x \ln^2 |x|,
\]
\[
H[v_2]'(x) = \mathcal{O}(1) \cdot \ln^2 |x|,
\]
\[
H[v_2]''(x) = \mathcal{O}(1) \cdot \frac{\ln |x|}{x}.
\]
(3.24)
The only difference is that in (3.23)-(3.24) by \( \mathcal{O}(1) \) we now denote a quantity such that
\[
|\mathcal{O}(1)| \leq C \cdot \|w\|_{H^2(\mathbb{R} \setminus \{0\})},
\]
(3.25)
for some constant \( C \) independent of \( w \).

4. Finally, observing that the the function \( v_1 \) in (3.2) has compact support, for \( |x| \to \infty \) we have the bounds
\[
D(x) = H[v_1](x) = \mathcal{O}(1) \cdot x^{-1} \quad D_x(x) = \mathcal{O}(1) \cdot x^{-2}, \quad D_{xx}(x) = \mathcal{O}(1) \cdot x^{-3}.
\]
(3.26)
On the other hand, for \( x \to 0 \) we claim that
\[
D(x) = \mathcal{O}(1), \quad D_x(x) = \mathcal{O}(1) \cdot \ln |x|, \quad D_{xx}(x) = \mathcal{O}(1) \cdot |x|^{-1},
\]
(3.27)
where \( \mathcal{O}(1) \) is a quantity satisfying (3.25). Indeed, without loss of generality we can assume \( 0 < x < 1/2 \). Recalling the construction of \( w \) and \( \varphi \), we have
\[
\left( w - \frac{u^- + u^+}{2} \right) \varphi_x = \frac{(u^+ - u^-) \ln |x|}{\pi} + \mathcal{O}(1).
\]
(3.28)
The Hilbert transform of \( v_1 \) is computed by
\[
\pi H[v_1] = \int_{-\infty}^{+\infty} \frac{v_1(y)}{x-y} \, dy
\]
\[
= \left( \int_{-2}^{-1} + \int_{1}^{2} \right) \frac{v_1(y)}{x-y} \, dy + \int_{-1}^{0} \frac{u^-}{x-y} \, dy + \left( \int_{0}^{x/2} + \int_{1}^{3x/2} \right) \frac{u^+}{x-y} \, dy + \int_{x/2}^{3x/2} \frac{u^+}{x-y} \, dy
\]
The first term on the right hand side is bounded and the last term vanishes, in the principal value sense. The second term is computed by
\[
\int_{-1}^{0} \frac{u^-}{x-y} \, dy = u^- (- \ln |x| + \ln |x + 1|) = -u^- \ln |x| + \mathcal{O}(1) \cdot |x|,
\]
while the remaining integrals are estimated by
\[
\left( \int_{0}^{x/2} + \int_{3x/2}^{1} \right) \frac{u^+}{x-y} \, dy = u^+ (\ln |x| - \ln |x - 1|) = u^+ \ln |x| + \mathcal{O}(1) \cdot |x|.
\]
Combining the previous estimates we obtain
\[
H[v_1] = \frac{(u^+ - u^-) \ln |x|}{\pi} + \mathcal{O}(1).
\]
(3.29)
Next, we estimate the derivative \( D_x(x) \). We have
\[
\partial_x \left( w - \frac{u^+ + u^-}{2} \right) \varphi_x = \mathcal{O}(1) \cdot \ln |x|, \quad \left( w - \frac{u^+ + u^-}{2} \right) \varphi_{xx} = \frac{u^+ - u^-}{\pi x} + \mathcal{O}(1).
\]
(3.30)
To estimate the derivative of \( H[v_1] \) we write
\[
\pi \cdot \partial_x H[v_1] = \left( \int_{-2}^{-1} + \int_{1}^{2} \right) \frac{-v_1(y)}{(x-y)^2} \, dy - \int_{-1}^{0} \frac{u^-}{(x-y)^2} \, dy \\
+ \partial_x \left( \int_{0}^{x/2} + \int_{3x/2}^{1} \right) \frac{v_1(y)}{x-y} \, dy + \partial_x \int_{x/2}^{3x/2} \frac{v_1(y)}{x-y} \, dy.
\]
(3.31)

The first term on the right hand side of (3.31) is uniformly bounded. The second term is estimated by
\[
- \int_{-1}^{0} \frac{u^-}{(x-y)^2} \, dy = - \frac{u^-}{x} + O(1).
\]

Furthermore, we have
\[
\partial_x \left( \int_{0}^{x/2} + \int_{3x/2}^{1} \right) \frac{v_1(y)}{x-y} \, dy = \left( \int_{0}^{x/2} + \int_{3x/2}^{1} \right) \frac{-v_1(y)}{(x-y)^2} \, dy + \frac{4u^+}{x} \\
= -\frac{3u^+}{x} + O(1) + \frac{4u^+}{x} = \frac{u^+}{x} + O(1).
\]
(3.32)

Lastly, since \( v_1(x) = u^+ \) for \( x \in [0,1] \), we have
\[
\partial_x \int_{x/2}^{3x/2} \frac{v_1(y)}{x-y} \, dy = \partial_x \int_{-x/2}^{x/2} \frac{u^+}{y} \, dy = 0.
\]
(3.33)

Combining the previous estimates we thus obtain
\[
\partial_x H[v_1](x) = \frac{u^+ - u^-}{\pi x} + O(1).
\]

Together with (3.30), as \( x \to 0 \) this yields the asymptotic estimate
\[
D_x(x) = H[v_1]_x - \left[ \left( w - \frac{u^- + u^+}{2} \right) \varphi_x \right]_x = O(1) \cdot \ln|x|.
\]
(3.34)

The second derivative \( D_{xx} \) is estimated in a similar way. Indeed, by (3.1)–(3.3) and (3.30), we have
\[
\partial_{xx} \left( w - \frac{u^- + u^+}{2} \varphi_x \right) = \partial_{xx} \left( w - \frac{u^+ + u^-}{2} \right) \varphi_x + \partial_x \left( w - \frac{u^+ + u^-}{2} \right) \varphi_{xx} \\
+ \partial_x \left( w - \frac{u^- + u^+}{2} \varphi_x \right) \varphi_{xx} + \left( w - \frac{u^- + u^+}{2} \right) \varphi_{xxx}
\]
(3.35)

On the other hand, differentiating (3.31) and recalling (3.32) and (3.33) we have
\[
\pi \cdot \partial_{xx} H[v_1] = \left( \int_{-2}^{-1} + \int_{1}^{2} \right) \frac{2v_1(y)}{(x-y)^3} \, dy + \int_{-1}^{0} \frac{2u^-}{(x-y)^3} \, dy \\
+ \partial_x \left( \int_{0}^{x/2} + \int_{3x/2}^{1} \right) \frac{-v_1(y)}{(x-y)^2} \, dy - \frac{4u^+}{x^2} + \partial_x \int_{x/2}^{3x/2} \frac{u^+}{y} \, dy.
\]
(3.36)
As before, the first term is uniformly bounded while the last term is zero. The second term is computed by
\[
\int_{-1}^{0} \frac{2u^-}{(x-y)^3} dy = \frac{u^-}{x^2} + \mathcal{O}(1).
\] (3.37)
The third term is estimated by
\[
\partial_x \left( \int_0^{x/2} + \int_{3x/2}^1 \frac{-v_1(y)}{(x-y)^2} dy \right) = \left( \int_0^{x/2} + \int_{3x/2}^1 \frac{2v_1(y)}{(x-y)^3} dy - \frac{2u^+}{x^2} + \frac{6u^+}{x^2} \right)
\]
\[
= \frac{3u^+}{x^2} + \mathcal{O}(1).
\] (3.38)
Combining above estimates (3.35)–(3.38) we obtain
\[
D_{xx} = H[v_1]_{xx} - \left[ \left( w - \frac{u^- + u^+}{2} \right) \varphi_x \right]_{xx}
\]
\[
= \frac{1}{\pi} \left( \frac{u^-}{x^2} + \frac{3u^+}{x^2} - \frac{4u^+}{x^2} \right) + \frac{u^+ - u^-}{\pi x^2} + \mathcal{O}(1) \cdot \frac{1}{x} = \mathcal{O}(1) \cdot \frac{1}{x}.
\] (3.39)

5. By the estimates (3.8), (3.21) it follows
\[
\|A + B\|_{H^2(\mathbb{R} \setminus [-\delta, \delta])} = \mathcal{O}(1) \cdot \left( \int_{-\delta}^{\delta} \frac{1}{x^2} \ln^2 |x| dx \right)^{1/2} = \mathcal{O}(1) \cdot \left( \int_{-\delta}^{\delta} \frac{1}{x^{7/3}} dx \right)^{1/2}
\]
\[
= \mathcal{O}(1) \cdot (\delta^{-4/3})^{1/2} = \mathcal{O}(1) \cdot \delta^{-2/3}.
\] (3.40)
Similarly, the estimates (3.6) follow from (3.22), and (3.26)-(3.27).

4 Construction of approximate solutions

In this section, given an initial datum \( \bar{w} \in \mathcal{D} \), we prove that all the approximate solutions \( w_n \) at (2.20)-(2.21) are well defined, on a suitably small time interval \([0,T]\).

As in (2.6), we define
\[
\begin{align*}
\bar{u}^- & \doteq \bar{w}(0^-), \\
\bar{u}^+ & \doteq \bar{w}(0^+), \\
u_n^-(t) & \doteq w_n(t,0^-), \\
u_n^+(t) & \doteq w_n(t,0^+).
\end{align*}
\]

To fix the ideas, assume that the initial data \( \bar{w} \in H^2(\mathbb{R} \setminus \{0\}) \) satisfies
\[
\bar{u}^- - \bar{u}^+ = 6\delta_0, \quad \|\bar{w}\|_{H^2(\mathbb{R} \setminus \{0\})} = \frac{M_0}{2},
\] (4.1)
for some (possibly large) constants \( \delta_0, M_0 > 0 \).

Choosing a time interval \([0,T]\) sufficiently small, we claim that for each \( n \geq 1 \) the approximate solution \( w_n \) satisfies the a priori bounds
\[
\begin{align*}
|u_n^-(t) - \bar{u}^-| & \leq \delta_0, \quad ||w_n(t)||_{H^2(\mathbb{R} \setminus \{0\})} \leq M_0, \quad \text{for all } t \in [0,T].
\end{align*}
\] (4.2)
This will be proved by induction. For \( n = 1 \) these bounds are a trivial consequence of the definition (2.20). In the following, we assume that the function \( w_n = w_n(t, x) \) satisfies (4.2), and show that the same bounds are satisfied by \( w_{n+1} \). We recall that \( w_{n+1} \) is defined as the solution to the linear equation (2.21), with initial data (2.15).

A sequence of approximate solutions \( w^{(k)} \) to the linear equation (2.21) will be constructed by induction on \( k = 1, 2, \ldots \). For notational convenience we introduce the function

\[
a(t, x) \doteq \varphi(x) + \frac{u_n^-(t) + u_n^+(t)}{2}.
\]

(4.3)

As in (2.17), call \( t \mapsto x(t; t_0, x_0) \) the solution to the Cauchy problem

\[
\dot{x} = a(t, x(t)), \quad x(t_0) = x_0.
\]

(4.4)

We begin by defining

\[
w^{(1)}(t, x) \doteq \bar{w}(x).
\]

(4.5)

By induction, if \( w^{(k)} \) has been constructed, we then set

\[
w^{(k+1)}(t_0, x_0) = \bar{w}(x(0; t_0, x_0)) + \int_{t_0}^{t_0} F^{(k)}(t, x(t; t_0, x_0)) dt,
\]

(4.6)

where \( F^{(k)} \) is defined as in (2.19), with \( w \) replaced by \( w^{(k)} \) and \( u^\pm(t) = w(t, 0 \pm) \) replaced by \( w^{(k)}(t, 0 \pm) \), respectively.

Assuming that \( w_n \) satisfies (4.2), we will show that every approximation \( w^{(k)} \) to the linear Cauchy problem (2.21), (2.15) satisfies the same bounds, on a sufficiently small time interval \([0, T]\). Our first result deals with solution to the linear transport equation (4.7). We show that, within a sufficiently short time interval, the \( H^2 \) norm of the solution can be amplified at most by a factor of \( 3/2 \).

**Lemma 2.** Let \( w_n = w_n(t, x) \) be a function that satisfies the bounds (4.2) for all \( t > 0 \), and define \( a = a(t, x) \) as in (4.3). Then there exists \( T > 0 \) small enough, depending only on \( \delta_0, M_0 \), so that the following holds. For any \( \tau \in [0, T] \) and any solution \( w \) of the linear equation

\[
w_t + a(t, x)w_x = 0
\]

(4.7)

with initial datum

\[
w(0) = \bar{w} \in H^2(\mathbb{R} \setminus [-\delta_0 \tau, \delta_0 \tau]),
\]

one has

\[
\|w(\tau)\|_{H^2(\mathbb{R} \setminus \{0\})} \leq \frac{3}{2} \|\bar{w}\|_{H^2(\mathbb{R} \setminus [-\delta_0 \tau, \delta_0 \tau])}.
\]

(4.8)

**Proof.** 1. The equation (4.7) can be solved by the method of characteristics, separately on the regions where \( x < 0 \) and \( x > 0 \). We observe that characteristics move toward the origin from both sides. In this first step we prove that all characteristics starting at time \( t = 0 \) inside the interval \([−\delta_0 \tau, \delta_0 \tau]\) hit the origin before time \( \tau \) (see Fig. 2). Hence the profile \( w(\tau, \cdot) \) does not depend on the values of \( \bar{w} \) on this interval.
We claim that there exists $\delta_1 > 0$ such that
\[
\begin{cases}
a(t, x) \leq -\delta_0 & \text{for all } x \in [0, \delta_1], \\
a(t, x) \geq \delta_0 & \text{for all } x \in [-\delta_1, 0].
\end{cases}
\tag{4.9}
\]
Indeed, (4.1) and (4.2) imply
\[
a(t, 0+) = u_n^+(t) - u_n^-(t) \leq -2\delta_0. \tag{4.10}
\]
Moreover, for $x > 0$ we have
\[
|a(t, x) - a(t, 0+)| \leq \frac{2}{\pi} |x \ln x| + \int_0^x |w_{n,x}(t, y)| dy \leq C_0 |x|^{1/2}, \tag{4.11}
\]
for some constant $C_0$ depending only on the norm $\|w_n(t, \cdot)\|_{H^2}$, hence only on $M_0$ in (4.2). Choosing $\delta_1 > 0$ small enough so that $C_0\delta_1^{1/2} < \delta_0$, from (4.10)-(4.11) we obtain the first inequality in (4.9). The second inequality is proved in the same way. In addition, by choosing the time interval $[0, T]$ small enough, we can also assume
\[
\delta_0 T \leq \delta_1. \tag{4.12}
\]

2. Multiplying (4.7) by $2w$ one finds
\[
(w^2)_t + (aw^2)_x = a_x w^2. \tag{4.13}
\]
Integrating (4.13) over the domain
\[
\Omega \equiv \{(t, x); \ |x| > \delta_0(\tau - t), \ t \in [0, \tau]\} \tag{4.14}
\]
shown in Fig. 2, we obtain
\[
\int_{-\infty}^{\infty} w^2(\tau, x) dx \leq \int_{|x| > \delta_0\tau} \hat{w}^2 dx + \int_0^{\tau} \int_{|x| > \delta_0(\tau - t)} a_x w^2 dx dt. \tag{4.15}
\]
Indeed, by (4.9) and (4.12), for every $\tau \in [0, T]$ the flow points outward along the boundary of the domain $\Omega$. By (4.3) the derivative $a_x$ satisfies a bound of the form
\[
|a_x(t, x)| \leq C_a (1 + |\ln |x||), \tag{4.16}
\]
where $C_a$ is a constant depending only on the norm $\|w_n\|_{H^2}$ in (4.2). Taking the supremum of $|a_x(t, x)|$ over the set
\[
\Omega_t \equiv \{x; \ |x| > \delta_0(\tau - t)\}, \tag{4.17}
\]
from (4.15) we thus obtain
\[
\|w(\tau)\|_{L^2(\mathcal{R}^2)}^2 \leq \|\tilde{w}\|_{L^2(\Omega_0)}^2 + \int_0^\tau C_a \left(1 + |\ln(\delta_0(\tau - t))|\right) \|w(t)\|_{L^2(\Omega_t)}^2 dt. \tag{4.18}
\]
By Gronwall’s lemma, this yields a bound on $\|w(\tau)\|_{L^2}^2$. 

14
Figure 2: The norm \( \|w(\tau)\|_{H^2(I \setminus \{0\})} \) is estimated by using the balance laws for \( w^2, w_x^2, w_{xx} \) on the shaded domain \( \Omega \). By (4.9), along the boundary where \( |x| = \delta_0(\tau - t) \) all characteristics move outward. Hence no inward flux is present.

3. Next, differentiating (4.7) w.r.t. \( x \) and multiplying by \( 2w_x \) we obtain

\[
\begin{align*}
    w_x + a w_{xx} &= -a_x w_x, & w_x(0, \cdot) &= \bar{w}_x. \\
    (w^2_x)_t + (aw^2_x)_x &= -a_x w_x^2.
\end{align*}
\]  

(4.19)

Integrating (4.20) over the domain \( \Omega \) in (4.14) and using the bound (4.16), by similar computations as before we now obtain

\[
\|w_x(\tau)\|_{L^2(I \setminus \{0\})}^2 \leq \|\bar{w}_x\|_{L^2(\Omega_0)}^2 + \int_0^\tau C_a \left( 1 + |\ln(\delta_0(\tau - t))| \right) \|w_x(t)\|_{L^2(\Omega_t)}^2 \, dt.
\]  

(4.21)

By Gronwall’s lemma, this yields a bound on \( \|w_x(\tau)\|_{L^2}^2 \).

4. Differentiating (4.19) once again and multiplying all terms by \( 2w_{xx} \) we find

\[
\begin{align*}
    w_{xx} + a w_{xxx} &= -2a_x w_{xx} - a_{xx} w_x, & w_{xx}(0, \cdot) &= \bar{w}_{xx}, \\
    (w^2_{xx})_t + (aw^2_{xx})_x &= -3a_x w_{xx}^2 - 2a_{xx} w_x w_{xx}.
\end{align*}
\]  

(4.22)

Integrating (4.23) over the domain \( \Omega \) in (4.14), we obtain

\[
\int_{-\infty}^\infty w_{xx}^2(\tau, x) \, dx \leq \int_{|x|>\delta_0} w_{xx}^2(y) \, dy + \int_0^\tau \int_{|x|>\delta_0(\tau - t)} \left( -3a_x w_{xx}^2 - 2a_{xx} w_x w_{xx} \right) \, dx \, dt.
\]  

(4.24)

To estimate the right hand side of (4.24) we observe that, for \( |x| \) small,

\[
\begin{align*}
    |a_x| &= |\varphi_x + w_{n,xx}| = \mathcal{O}(1) \left( |\ln |x|| + \|w_n\|_{H^2} \right), & |a_{xx}| &= |\varphi_{xx} + w_{n,xx}| = \mathcal{O}(1) \frac{1}{|x|} + |w_{n,xx}|.
\end{align*}
\]  

(4.25)

Recalling that \( \varphi(x) = 0 \) for \( |x| \geq 2 \), we have the bounds

\[
\begin{align*}
    E &\leq |3a_x w_{xx}^2 + 2a_{xx} w_x w_{xx}| \\
    &\leq \mathcal{O}(1) \cdot (1 + |\ln |x||) w_{xx}^2 + \mathcal{O}(1) \cdot \left( \frac{1}{|x|} + |w_{n,xx}| \right) \|w\|_{H^2} w_{xx},
\end{align*}
\]  

(4.26)
\[ \int_{\delta_0(t-s)}^{\delta_0(t)} \left( \frac{1}{x^2} \right) \frac{1}{2} \| w_{xx} \|_{L^2(\Omega_t)} \leq \left( \frac{1}{\delta_0(t-s)} \right)^{1/2} \| w_{xx} \|_{L^2(\Omega_t)} , \]

\[ \int_{0}^{T} \int_{|x|>\delta_0(t)} E(t,x) \, dx \, dt \leq \mathcal{O}(1) \cdot \int_{0}^{T} (1 + |\ln \delta_0(t)|) \cdot \| w(t) \|_{H^2(\Omega_t)} \, dt \]

\[ + \mathcal{O}(1) \cdot \int_{0}^{T} \left[ \delta_0(t) \right]^{-1/2} \cdot \| w(t) \|_{H^2(\Omega_t)} \, dt + \mathcal{O}(1) \cdot \int_{0}^{T} \| w_n(t) \|_{H^2} \cdot \| w(t) \|_{H^2(\Omega_t)} \, dt. \]  

(4.27)

\[ \int_{\delta_0(t-s)}^{\delta_0(t)} \left( \frac{1}{x^2} \right) \frac{1}{2} \| w_{xx} \|_{L^2(\Omega_t)} \leq \left( \frac{1}{\delta_0(t-s)} \right)^{1/2} \| w_{xx} \|_{L^2(\Omega_t)} , \]

(4.28)

5. Calling \( Z(t) \doteq \| w(t) \|_{H^2(\Omega_t)} \), by the estimates (4.18), (4.21), and (4.28) we obtain an integral inequality of the form

\[ Z^2(\tau) \leq Z^2(0) + C_1 \cdot \int_{0}^{\tau} \left( 1 + |\ln \delta_0(t)| + \left[ \delta_0(t) \right]^{-1/2} + M_0 \right) Z^2(t) \, dt . \]  

(4.29)

By Gronwall's lemma, if \( \tau > 0 \) is sufficiently small this yields \( Z(\tau) \leq \frac{3}{2} Z(0) \), proving (4.8).

The above estimate can be easily extended to the linear, non-homogeneous problem

\[ w_t + a(t,x)w_x = F(t,x), \quad w(0,x) = \bar{w}(x). \]  

(4.30)

Indeed, in the same setting as Lemma 2, using (4.8) and Duhamel's formula, for \( \tau \in [0,T] \) we obtain

\[ \| w(\tau,\cdot) \|_{H^2(\mathbb{R}\setminus\{0\})} \leq \frac{3}{2} \| \bar{w} \|_{H^2(\mathbb{R}\setminus[-\delta_0(\tau),\delta_0(\tau)])} + \frac{3}{2} \int_{0}^{\tau} \| F(t,\cdot) \|_{H^2(\mathbb{R}\setminus[-\delta_0(\tau-t),\delta_0(\tau-t)])} \, dt . \]  

(4.31)

Relying on Lemma 1 we now prove uniform \( H^2 \) bounds on all approximations \( w^{(k)} \), on a suitably small time interval \([0,T]\).

**Lemma 3.** Let \( w_n = w_n(t,x) \) be a function that satisfies the bounds (4.2) for all \( t > 0 \), and define \( a = a(t,x) \) as in (4.3). Then there exists \( T > 0 \) small enough, depending only on \( \delta_0, M_0 \) in (4.1), so that the following holds. For every \( k \geq 1 \) and every \( \tau \in [0,T] \), one has

\[ \| w^{(k)}(\tau) \|_{H^2(\mathbb{R}\setminus\{0\})} \leq M_0 , \]  

(4.32)

\[ |w^{(k)}(\tau,0-) - \bar{u}_-| \leq \delta_0 , \quad |w^{(k)}(\tau,0+) - \bar{u}_+| \leq \delta_0 . \]  

(4.33)

**Proof.** 1. Recalling the constants \( K_0, K_1 \) in Lemma 1, choose \( T > 0 \) small enough so that

\[ \int_{0}^{T} (\delta_0 s)^{-2/3} \, ds < \frac{M_0}{6(K_0 + K_1 M_0)}. \]  

(4.34)
2. The estimate (4.32) trivially holds for $w^{(1)}(\tau) \equiv \bar{w}$. Assuming that it holds for $w^{(k)}(t)$, $t \in [0,T]$, by (4.31) for any $\tau \in [0,T]$ we have the estimate

$$\|w^{(k+1)}(\tau)\|_{H^2(\mathbb{R} \setminus \{0\})} \leq \frac{3}{2} \|\bar{w}\|_{H^2(\mathbb{R} \setminus \{0\})} + \frac{3}{2} \int_0^T \|A + B + C + D\|_{H^2(\mathbb{R} \setminus [-\delta_0(\tau-t), \delta_0(\tau-t)])} \, ds$$

$$\begin{align*}
&\leq \frac{3}{4} M_0 + \frac{3}{2} \int_0^T K_0[\delta_0(\tau-t)]^{-2/3} \, dt + \frac{3}{2} \int_0^T K_1[\delta_0(\tau-t)]^{-2/3} \|w^{(k)}(t)\|_{H^2(\mathbb{R} \setminus \{0\})} \, dt \\
&\leq \frac{3}{4} M_0 + \frac{3}{2} (K_0 + K_1 M_0) \int_0^T (\delta_0 s)^{-2/3} \, ds \\
&< \frac{3}{4} M_0 + \frac{3}{2} (K_0 + K_1 M_0) \cdot \frac{M_0}{6(K_0 + K_1 M_0)} = M_0.
\end{align*}$$

(4.35)

By induction, this proves the bound (4.32).

3. To prove the two estimates in (4.33), we write

$$|w^{(k+1)}(\tau, 0+) - \bar{u}^+| \leq |\bar{w}(x(0; \tau, 0+)) - \bar{u}^+| + \tau \cdot \sup_{t \in [0,\tau]} \|A + B + C + D\|_{L^\infty} \cdot (4.36)$$

The a priori bound on $\|w^{(k)}(t, \cdot)\|_{H^2(\mathbb{R} \setminus \{0\})}$ implies that the $L^\infty$ norm in (4.36) is uniformly bounded. By possibly choosing a smaller $T > 0$, both terms on the right hand side of (4.36) will be $< \delta_0/2$. This yields the second inequality in (4.33). The first inequality is proved in the same way.

The next lemma shows that the sequence of approximations $w^{(k)}$ defined at (4.5)–(4.6) converges to a solution to (2.21).

**Lemma 4.** For some $T > 0$ sufficiently small, the sequence of approximations $w^{(k)}(t, \cdot)$ converges in $H^2(\mathbb{R} \setminus \{0\})$ to a function $w = w(t, \cdot)$. The convergence is uniform for $t \in [0,T]$. This limit function provides a solution to the initial value problem (2.21) with initial data (2.15).

**Proof.** 1. By the previous bounds, the difference between two approximations can be estimated by

$$\|w^{(k+1)}(\tau) - w^{(k)}(\tau)\|_{H^2(\mathbb{R} \setminus \{0\})}$$

$$\begin{align*}
&\leq \frac{3}{2} \int_0^T [\delta_0(\tau-t)]^{-2/3} K_1 \|w^{(k)}(t) - w^{(k-1)}(t)\|_{H^2(\mathbb{R} \setminus [-\delta_0(\tau-t), \delta_0(\tau-t)])} \, dt.
\end{align*}$$

(4.37)

If $T > 0$ is small enough, so that

$$\frac{3}{2} \int_0^T (\delta_0 s)^{-2/3} K_1 \, ds \leq \frac{1}{2},$$

then for every $\tau \in [0,T]$ the sequence $w^{(k)}(\tau, \cdot)$ is Cauchy in $H^2(\mathbb{R} \setminus \{0\})$, hence it converges to a unique limit function $w(\tau, \cdot)$.

2. It remains to prove that that $w$ provides a solution to (2.21) with initial data (2.15), in the sense that the integral identities (2.18) are satisfied for all $t_0 \in [0,T]$ and $x_0 \neq 0$.
This is clear, because for every $\epsilon > 0$ as $k \to \infty$ the source terms on the right hand side of (2.21) converge uniformly on the set $\{(t,x) ; t \in [0,T], x \geq \epsilon\}$.

5 Convergence of the approximate solutions

By the analysis in the previous section, the sequence of approximate solutions $w_n$ of (2.21), (2.15) is well defined, on a suitably small time interval $[0,T]$. Moreover, the uniform bounds (4.2) hold.

To complete the proof of Theorem 1, it remains to show that the $w_n$ converge to a limit function $w$, providing an entropic solution to the Cauchy problem (2.10), (2.15). Toward this goal we prove that on a suitably small time interval $[0,T]$ the sequence $(w_n)_{n \geq 1}$ (2.21) is Cauchy w.r.t. the norm of $H^1(\mathbb{R} \setminus \{0\})$, hence it converges to a unique limit. This will be achieved in several steps.

1. For a fixed $n$, consider the differences

$$
\begin{align*}
W & \doteq w_{n+1} - w_n, \\
W_n & \doteq w_n - w_{n-1}, \\
U^- & \doteq u_{n+1}^- - u_n^-, \\
U^-_n & \doteq u_n^+ - u_{n-1}^-,
\end{align*}
$$

From (2.21) we deduce

$$
W_t + \left( \varphi + w_n - \frac{u_n^- - u_n^+}{2} \right) W_x + \left( W_n - \frac{U^-_n + U^+_n}{2} \right) w_{n,x} = H[W] - \left( W - \frac{U^- + U^+}{2} \right) \varphi_x.
$$

Multiplying both sides by $2W$ we obtain the balance law

$$
(W^2)_t + \left[ \left( \varphi + w_n - \frac{u_n^- - u_n^+}{2} \right) W^2 \right]_x = (\varphi + w_n)_x W^2 - \left( W_n - \frac{U^-_n + U^+_n}{2} \right) 2W w_{n,x} + 2H[W] \cdot W - \left( W - \frac{U^- + U^+}{2} \right) 2W \varphi_x.
$$

Integrating over the domain $\Omega$ in (4.14) and observing that $\varphi_x(x) = O(1)(1 + |\ln |x||)$, we obtain

$$
\frac{1}{2} \int_0^T W^2(\tau,x) dx \leq - \int_0^\tau \int_{|x| > \delta_0(\tau - t)} \left\{ (\varphi + w_n)_x : W^2 \right. \\
- \left. \left( W_n - \frac{U^-_n + U^+_n}{2} \right) 2W w_{n,x} + 2H[W] \cdot W - \left( W - \frac{U^- + U^+}{2} \right) 2W \varphi_x \right\} dx dt
\right.
$$

$$
= O(1) \cdot \int_0^\tau \left\{ |\ln(\tau - t)| \cdot \|W(s)\|_3^3 + \|W_n(t)\|_{H^1} \|W(t)\|_3^3 + \|W(t)\|_3^3 \right.
$$

$$
+ \left. |\ln(\tau - t)| \cdot \|W(t)\|_{H^1} \right\} dt
\right.
$$

$$
\leq C_3 \cdot \int_0^\tau \|W(t)\|_3^3 \cdot \left( \|W_n(t)\|_{H^1} + |\ln(\tau - t)| \|W(t)\|_{H^1} \right) dt,
$$

(5.3)
for some constant $C_3$.

2. Next, differentiating (5.1) w.r.t. $x$ we obtain

$$
W_{xt} + \left( \varphi + w_n - \frac{u_n^- - u_n^+}{2} \right) W_{xx} + (\varphi_x + w_{n,x}) W_x + \left( W_n - \frac{U_n^- + U_n^+}{2} \right) w_{n,xx} + W_{n,x} w_{n,x}
$$

$$
= H[W_x] - \left( W - \frac{U^- + U^+}{2} \right) \varphi_{xx} - \varphi_x W_x.
$$

Multiplying both sides by $2W_x$ we obtain the balance law

$$
(W_x^2)_t + \left[ \left( \varphi + w_n - \frac{u_n^- - u_n^+}{2} \right) W_x^2 \right]_x = - (\varphi_x + w_{n,x}) W_x^2 - \left( W_n - \frac{U_n^- + U_n^+}{2} \right) 2W_x w_{n,xx}

- 2w_{n,x}W_{n,x} W_x + 2H[W_x] W_x - \left( W - \frac{U^- + U^+}{2} \right) 2W_x \varphi_{xx} - 2\varphi_x W_x^2.
$$

By the definition (2.4) one has

$$
\| \varphi_x \|_{L^2(\mathbb{R}\setminus[-\delta_0(\tau-t),\delta_0(\tau-t)])} = \mathcal{O}(1) \cdot (\tau - t)^{-1/2}.
$$

Integrating (5.5) over the domain $\Omega$ in (4.14) we obtain

$$
\int_0^\infty W_x^2(t, x) \, dx = \mathcal{O}(1) \cdot \int_0^\tau \left\{ \| \ln(\tau - t) \| \| W_x(t) \|_{L^2}^2 + \| W_n(t) \|_{H^1} \| W_x(t) \|_{L^2} \right.

\left. + \| W(t) \|_{H^1} \| W_x(t) \|_{L^2} \cdot (\tau - t)^{-1/2} \right\} \, dt.
$$

3. Calling $Z(t) \doteq \| W(t) \|_{H^1(\mathbb{R}\setminus\{0\})}$, from (5.3) and (5.7) we obtain an integral inequality of the form

$$
Z^2(\tau) \leq C_4 \int_0^\tau Z(t) \cdot \left( \| W_n(t) \|_{H^1} + Z(t) \right) \cdot (\tau - t)^{-1/2} \, dt,
$$

for some constant $C_4$.

We now set

$$
\varepsilon_0 \doteq \sup_{t \in [0,T]} \| W_n(t) \|_{H^1(\mathbb{R}\setminus\{0\})}.
$$

Since $Z(0) = 0$, calling $\tau^*$ the first time where $Z \geq \varepsilon_0/2$ one has

$$
\frac{\varepsilon_0}{2} \leq C_4 \int_0^{\tau^*} \frac{\varepsilon_0}{2} \cdot \left( \varepsilon_0 + \frac{\varepsilon_0}{2} \right) (\tau^* - t)^{-1/2} \, dt = \frac{3}{2} C_4 \varepsilon_0^2 \tau^*.
$$

Hence $\tau^* \geq (3C_4)^{-1}$. Choosing $0 < T < (3C_4)^{-1}$, we thus obtain

$$
Z(t) \leq \frac{\varepsilon_0}{2} \quad \text{for all} \quad t \in [0, T].
$$

This establishes the desired contraction property:

$$
\sup_{t \in [0,T]} \| w_{n+1}(t) - w_n(t) \|_{H^1(\mathbb{R}\setminus\{0\})} \leq \frac{1}{2} \cdot \sup_{t \in [0,T]} \| w_n(t) - w_{n-1}(t) \|_{H^1(\mathbb{R}\setminus\{0\})}.
$$
4. By (5.9), for every \( t \in [0, T] \) the sequence of approximations \( w_n(t, \cdot) \) is Cauchy in the space \( H^1(\mathbb{R} \setminus \{0\}) \), hence it converges to a unique limit \( w(t, \cdot) \).

It remains to check that this limit function \( w \) is an entropic solution, i.e. it satisfies the integral equation (2.18). But this is clear, because for every \( \epsilon > 0 \) the sequence of functions

\[
F_n \doteq H[\varphi] - \varphi \varphi_x + \left( H[w_n] - \left( w_n - \frac{u_n^- + u_n^+}{2} \right) \varphi_x \right)
\]

converges to the corresponding function \( F \) in (2.19), uniformly for \( t \in [0, T] \) and \( |x| \geq \epsilon \).

5. Finally, to prove uniqueness, assume that \( w, \tilde{w} \) are two entropic solutions. Consider the differences

\[
W \doteq w - \tilde{w}, \quad \begin{cases} U^- \doteq u^- - \tilde{u}^-, \\ U^+ \doteq u^+ - \tilde{u}^+ \end{cases}
\]

and call \( Z(t) \doteq \|W(t)\|_{H^1(\mathbb{R} \setminus \{0\})} \). Since \( Z(0) = 0 \), the same arguments used to prove (5.8) now yield

\[
Z^2(\tau) \leq C_4 \int_0^\tau Z(t) \cdot [Z(t) + Z(t)] \cdot (\tau - t)^{-1/2} \, dt.
\]

For \( \tau \in [0, T] \) sufficiently small, we thus obtain \( Z(\tau) = 0 \). This completes the proof of Theorem 1.

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References


