On Differential Systems with Vector-Valued Impulsive Controls.

ALBERTO BRESSAN - FRANCO RAMPazzo

Sunto. – Per un problema di Cauchy del tipo

\[ \dot{x}(t) = f(x) + \sum_{i=1}^{m} g_i(x) \dot{u}_i(t), \quad x(0) = \bar{x} \in \mathbb{R}^n, \]

si dimostra un teorema di dipendenza lipschitziana delle soluzioni dai controlli, nel caso in cui questi siano a loro volta lipschitziani. Come applicazione si dà una definizione di soluzione generalizzata per controlli (anche discominuti) a variazione limitata e si dimostra la dipendenza continua di tali soluzioni da certe estensioni del grafico del controllo. Lo studio del problema (\( \dot{+} \)) ha ultimamente ricevuto motivo di ulteriore interesse per le applicazioni alla meccanica degli impulsi (discominuità dei momenti) e degli iperimpulsi (discominuità delle configurazioni), in [3], [4], [5], [7].

1. – Introduction.

Let \( \Omega \) be an open subset of \( \mathbb{R}^n \) and let \( f, g_i \) \( (i = 1, ..., m) \) be \( C^1 \) functions from \( \Omega \) into \( \mathbb{R}^n \). Given a control \( u(t) = (u_1, ..., u_m)(t) \), we write \( x(u, \cdot) \) for the Carathéodory solution of the Cauchy Problem

\[
\begin{align*}
\dot{x}(t) &= f(x(t)) + \sum_{i=1}^{m} g_i(x(t)) \dot{u}_i(t) \\
x(0) &= \bar{x} \in \Omega
\end{align*}
\]

where, as usual, dots denote differentiation with respect to time. If \( u \) is Lipschitz continuous, then its derivative \( \dot{u} \) is a measurable bounded function, and a classical theorem of Carathéodory implies the existence and uniqueness of the solution of (1.1) on some positive interval \([0, \varepsilon)\).

In the present paper we develop a theory of solutions of (1.1) for the more general class of (possibly discontinuous) vector valued controls with bounded variation. Our results rely on a careful
study of the continuity of the input-output map \( \Phi: u(\cdot) \rightarrow x(u, \cdot) \)
with respect to suitable topologies on the spaces of controls \( u(\cdot) \)
and of trajectories \( x(\cdot) \).

In the case of scalar controls, i.e. \( m = 1 \), the Lipschitz continuity of \( \Phi \) w.r.t. the \( C^0 \) norms was proven in [8]. A similar theorem for the \( L^1 \) norms appears in [2]. It is well known that the above results cannot hold for vector-valued controls, unless all Lie brackets \( [y_i, y_j] \) vanish identically. Still, we prove that the input-output map \( \Phi \) is continuous w.r.t. the \( C^0 \) topologies when restricted to a subset of controls having a uniform Lipschitz constant. When the control \( u \) is discontinuous but has bounded variation, one can parametrize (a completion of) the graph of \( u \) in a Lipschitz continuous way, and apply the previous result to a suitable augmented system, with \((m + 1)\)-dimensional control. In § 5 we prove that the generalized solution thus obtained coincides almost everywhere with the limit of classical solutions of (1.1) corresponding to mollified controls.

The primary motivation for the present work came from the study of hyperimpulsive controls for Langrangean systems. More precisely, given a Langrangean system with \( d = r + m \) degrees of freedom, referred to the coordinates \( (q, \gamma) = (q_1, ..., q_r, \gamma_1, ..., \gamma_m) \), one considers a control determining the values of \( \gamma_1(t), ..., \gamma_m(t) \).
The implementation of such a control physically requires the addition of certain forces. It was shown in [5] that, if the coordinates \( (q, \gamma) \) satisfy suitable conditions and if the above additional forces have a character of ideal constraints with respect to the submanifolds \( \gamma = c \in \mathbb{R}^m \), then the coordinates \( (q', ..., q') \) and their corresponding momenta \( (p_1, ..., p_r) \) satisfy a set of equations of the form (1.1), with \( x = (q, p), u_i = \gamma_i \). In the theory of hyper-impulsive motions, when the coordinates \( \gamma \) vary rapidly over a very short interval of time, this behavior is modelled assuming that \( \gamma_1, ..., \gamma_m \) suffer a discontinuity at a certain instant \( t \). This leads to the problem of defining a class of generalized solutions for the Cauchy problem (1.1), in the case of discontinuous controls \( u \). Moreover, in order to retain the physical significance of this model, it is important to understand the relations between generalized and classical solutions of (1.1). This is the object of our theorems on continuous dependence and approximations given in §§ 4, 5.

2. – Graph completions.

Let \( u: [0, T] \rightarrow \mathbb{R}^m \) be continuously differentiable and let \( \varphi(s) = (\varphi_1, ..., \varphi_m)(s) = (t(s), u(t(s))) \), \( s \in [0, S] \), be a Lipschitz continuous parametrization of the graph of \( u \). Together with (1.1),
consider the \((n+1)\)-dimensional Cauchy problem

\[
\begin{align*}
\dot{\varphi}(s) &= \bar{f}(y(s))\dot{\varphi}_0(s) + \sum_{i=1}^{m} \bar{g}_i(y(s))\dot{\varphi}_i(s) \\
y(0) &= (0, \bar{x})
\end{align*}
\]  

with \(\bar{f}(x_0, x) = (1, \bar{f}(x)), \quad \bar{g}_i(x_0, x) = (0, g_i(x))\).

If \(y(s) = (y_0, y_1, \ldots, y_n)(s)\) solves (2.1), then \(y_0(s) = t(s)\), moreover \(x(t) = (y_1, \ldots, y_n)(s(t))\) yields the solution of (1.1). In the case where \(u\) is not absolutely continuous, one may still be able to construct a Lipschitzian parametrization \(\varphi\) of the graph of \(u\), solve the corresponding Cauchy problem (2.1) and use this solution \(y(\cdot)\) to recover a generalized solution of (1.1). To implement this program, we introduce the following definition.

**Definition 1.** Let \(u : [0, T] \to \mathbb{R}^n\) have finite total variation. A graph-completion of \(u\) is a Lipschitz continuous map \(\varphi : [0, S] \to [0, T] \times \mathbb{R}^n, \varphi(s) = (t(s), v(s))\) such that

i) \(\varphi(0) = (0, u(0)), \varphi(S) = (T, u(T))\);

ii) \(0 < r < s < T \Rightarrow t(r) < t(s)\);

iii) \(\forall t \in [0, T], \exists s \in [0, S]\) such that \(\varphi(s) = (t, u(t))\).

Notice that by iii) the range of \(\varphi\) is a compact connected set containing the graph of \(u\). Among all possible graph-completions of a control \(u\), a natural one is obtained by «bridging» the discontinuities of \(u\), on the graph, by straight segments as shown in the following.

**Example 1.** Let \(u\) be right-continuous with bounded variation. For \(t \in [0, T]\), let \(V(t)\) be the total variation of \(u\) on \([0, t]\). Since the function \(t \mapsto t + V(t)\) is right-continuous and strictly increasing, for every \(s \in [0, T + V(T)]\) there exists a unique time \(t = t(s)\) such that

\[
s \leq t + V(t), \quad t' + V(t') < s \quad \forall t' < t.
\]

For \(t \in [0, T]\), define

\[
\begin{align*}
u^-(t) &= \lim_{t' \to t^-} u(t'), \quad V^-(t) = \lim_{t' \to t^-} V(t') \\
W(t) &= t + V(t), \quad W^-(t) = t + V^-(t).
\end{align*}
\]
If \( u \) has a jump at \( t \), we thus have

\[
\| u(t) - u^-(t) \| = V(t) - V^-(t) = W(t) - W^-(t).
\]

A map \( \varphi_u: [0, W(T)] \to [0, T] \times \mathbb{R}^n \) can now be defined by setting \( \varphi_u(s) = (t(s), v(s)) \), with \( t(s) \) defined by (2.2) and \( v(s) = u(t(s)) \) if \( u \) is continuous at \( t(s) \), while

\[
v(s) = \frac{s - W^-(t)}{W(t) - W^-(t)} u(t) + \frac{W(t) - s}{W(t) - W^-(t)} u^-(t)
\]

if \( u \) has a jump at time \( t = t(s) \).

Using (2.5) one checks that \( \varphi_u \) is continuous with Lipschitz constant 1. Moreover, conditions i)-iii) in Definition 1 hold. In particular, if \( t \in [0, T] \), \( (t, u(t)) = \varphi_u(s) \) for \( s = t + V(t) \). The function \( \varphi_u \), uniquely determined by the above construction, will be called the canonical graph-completion of \( u \).

**Definition 2.** – Let \( u: [0, T] \to \mathbb{R}^m \) be a control with bounded variation, let \( \varphi = (\varphi_0, ..., \varphi_m) : [0, S] \to [0, T] \times \mathbb{R}^m \) be a graph-completion of \( u \), and let \( y = (y_0, ..., y_n) \) be the corresponding solution of (2.1). The (possibly multivalued) map \( x(\varphi, \cdot) \) defined by

\[
x(\varphi, t) = \{(y_1, ..., y_n)(s); t = y_0(s)\}
\]

is called the generalized solution of (1.1) relative to \( \varphi \).

**Remark.** – By choosing \( s^+(t) = \max \{ s; t = y_0(s) \} \) and setting \( x^+(\varphi, t) = (y_1, ..., y_n)(s^+(t)) \) one obtains a right-continuous selection of the multivalued function \( x(\varphi, \cdot) \) defined at (2.7).

Since the graph of \( x(\varphi, \cdot) \) by definition coincides with the image \( y([0, S]) \), which is compact, it follows that the map \( t \to x(\varphi, t) \) is Hausdorff upper semicontinuous [1, p. 41]. In general it is clear that these generalized solutions are not unique, since they depend on the choice of a particular graph-completion \( \varphi \) of \( u \). Yet, one can obtain uniqueness by prescribing a canonical way to construct the map \( \varphi \), as in Example 1. In physical applications, it can also happen that the graph-completion is naturally suggested by the problem.

3. – Equivalent paths.

Our goal is to investigate the dependence of the generalized solution \( x(\varphi, \cdot) \) on the particular graph-completion \( \varphi \), and the
relation between classical and generalized solutions of (1.1). As
a preliminary, observe that every Lipschitz continuous map
φ: [0, S] → R^s can be reparametrized by means of its total variation.
More precisely, for s ∈ [0, S], let V(s) be the total variation of φ
on [0, S]. V is thus a non-decreasing, Lipschitz continuous map
from [0, S] onto [0, V(S)]. For τ ∈ [0, V(S)], set φ(τ) = φ(s) if
τ = V(s). It is readily checked that φ is a well defined map with
Lipschitz constant 1.

DEFINITION 3. - The map φ = [0, V(S)] → R^s constructed above
is the canonical parametrization of φ. We say that two continuous
maps φ_1: [0, S_1] → R^s (i = 1, 2) are equivalent, and write φ_1 ∼ φ_2,
if their canonical parametrizations coincide.

PROPOSITION 1. - If φ is a graph completion of a control u, such
is also its canonical parametrization φ. Moreover, the generalized
solutions x(φ, ·) and x(φ, ·) of (1.1) coincide.

PROOF. - For the map φ, conditions i) and ii) in Def. 1 are
obvious. Moreover, iii) holds because graph (u) ⊆ range (φ) =
range (φ). This yields the first assertion. If y, y are the
solutions of (2.1) corresponding to φ, φ respectively, then
y(s) =
= y(φ(V(s)) for all s ∈ [0, S], hence

x(u, t) = \{y_1, ..., y_n\}(s); t = y_0(s) =
= \{y_1, ..., y_n\}(V(s)); t = y_0(V(s)) = x(φ, t).

In view of the above result, equivalent graph-completions of u
yield the same generalized solution of (1.1). In the study of the
dependence of x(φ, ·) on φ, it is therefore natural to look for esti-
mates which do not depend on the particular parametrization of
φ. This is achieved by using a metric which is parameter-free.

DEFINITION 4. - Given two continuous paths φ_i: [0, S_i] → R^s,
\ i = 1, 2, their distance δ(φ_1, φ_2) is defined as

(3.1) δ(φ_1, φ_2) = \inf \max_{\gamma_1, \gamma_2 \in \{0, 1\}} \|φ_1(\gamma_1(s)) - φ_2(\gamma_2(s))\|,

the inf being taken over all couples of continuous, non-decreasing,
surjective maps γ_i: [0, 1] → [0, S_i].

It is interesting to observe that the inf in (3.1) is actually a
minimum. To prove this, we need

LEMMA 1. - Given two maps γ_i: [0, 1] → [0, S_i], \ i = 1, 2,
as in Definition 4, there exist nondecreasing surjective maps
\( \gamma_i : [0, 1] \to [0, S_i] \), continuous with Lipschitz constant \( L = 1 + S_1 + S_2 \), such that

\[
\max_{s \in [0, 1]} \| \varphi_1(\gamma_1(s)) - \varphi_2(\gamma_2(s)) \| = \max_{s \in [0, 1]} \| \varphi_1(\gamma_1'(s)) - \varphi_2(\gamma_2'(s)) \|.
\]

**Proof.** For \( t \in [0, 1] \), set \( \beta(t) = \frac{t + \gamma_1(t) + \gamma_2(t)}{[1 + S_1 + S_2]} \). \( \beta \) is then a strictly increasing, continuous surjective map from \([0, 1]\) onto \([0, 1]\). Since \( d\beta(t)/dt \geq [1 + S_1 + S_2]^{-1} \) a.e., the inverse map \( \beta^{-1} \) is also increasing with Lipschitz constant \( L = 1 + S_1 + S_2 \). Define \( \gamma_i', i = 1, 2 \), by setting \( \gamma_i(s) = \gamma_i(\beta^{-1}(s)) \). It is now clear that (3.2) holds. Concerning the Lipschitz condition, if \( r < s \) we have

\[
| \gamma_i'(r) - \gamma_i'(s) | =
= | \gamma_i(\beta^{-1}(r)) - \gamma_i(\beta^{-1}(s)) |
\leq [\beta^{-1}(r) + \gamma_1(\beta^{-1}(s)) + \gamma_2(\beta^{-1}(s))] -
- [\beta^{-1}(r) + \gamma_1(\beta^{-1}(r)) + \gamma_2(\beta^{-1}(r))] =
= \gamma_1(\beta^{-1}(s)) - \beta^{-1}(r)]
= [1 + S_1 + S_2] \cdot [s - r] \quad (i = 1, 2).
\]

**Proposition 2.** Given two continuous paths \( \varphi_i : [0, S_i] \to \mathbf{R}^n \), \( i = 1, 2 \), there exist two continuous nondecreasing surjective maps \( \gamma_i : [0, 1] \to [0, S_i] \) such that

\[
\delta(\varphi_1, \varphi_2) = \max_{s \in [0, 1]} \| \varphi_1(\gamma_1(s)) - \varphi_2(\gamma_2(s)) \|.
\]

**Proof.** For every \( n \geq 1 \), choose \( \gamma_1^n, \gamma_2^n \) such that

\[
\max \| \varphi_1(\gamma_1^n(s)) - \varphi_2(\gamma_2^n(s)) \| < \delta(\varphi_1, \varphi_2) + \frac{1}{n}.
\]

By Lemma 1, it is not restrictive to assume that \( \gamma_1^n, \gamma_2^n \) are both Lipschitz continuous with constant \( 1 + S_1 + S_2 \). Ascoli’s theorem now implies the existence of two maps \( \gamma_1, \gamma_2 \) satisfying the conditions in Definition 4 and

\[
\gamma_i(s) = \lim_{n \to \infty} \gamma_i^n(s)
\]

for some subsequence \( n' \to \infty \), uniformly on \([0, 1]\). The continuity of \( \varphi_1, \varphi_2 \) together with (3.4) now imply (3.3).

We conclude this section by showing that the distance \( \delta \) is in fact a pseudometric.
PROPOSITION 3. --

i) \( \delta(q_1, q_2) = \delta(q_2, q_1) > 0 \),

ii) \( \delta(q_1, q_2) = 0 \) if and only if \( q_1 \sim q_2 \),

iii) \( \delta(q_1, q_3) + \delta(q_2, q_3) \geq \delta(q_1, q_3) \).

The proofs of i) and ii) are straightforward. To prove iii), let \( q_i: [0, S_i] \rightarrow \mathbb{R}^n \) \((i = 1, 2, 3)\) be continuous, and use Proposition 2 to construct functions \( \alpha_1, \alpha_2, \beta_2, \beta_3 \) such that

\[
\delta(q_1, q_3) = \max_{s \in [0,1]} \| q_1(\alpha_1(s)) - q_3(\alpha_2(s)) \|,
\]

\[
\delta(q_2, q_3) = \max_{s \in [0,1]} \| q_2(\beta_2(s)) - q_3(\beta_3(s)) \|.
\]

For every \( \varepsilon > 0 \), the map \( \alpha_{22}: [0, 1] \rightarrow [0, S_2], \alpha_{22}(s) = (1 - \varepsilon) \cdot \alpha_2(s) + \varepsilon S_2 \) is strictly increasing, hence it has a continuous inverse \( \alpha_{22}^{-1} \). Define \( \alpha_{23} = \alpha_1 \circ \alpha_{22}^{-1} \circ \beta_3: [0, 1] \rightarrow [0, S_1] \). Then, for every \( \varepsilon > 0 \),

\[
\delta(q_1, q_3) < \max_{s \in [0,1]} \| q_1 \circ \alpha_{13}(s) - q_3 \circ \beta_3(s) \| <
\]

\[
\max_{s \in [0,1]} \| q_1 \circ \alpha_1 \circ \alpha_{22}^{-1} \circ \beta_3(s) - q_2 \circ \alpha_2 \circ \alpha_{22}^{-1} \circ \beta_2(s) \| +
\]

\[
\max_{s \in [0,1]} \| q_2 \circ \alpha_2 \circ \alpha_{22}^{-1} \circ \beta_2(s) - q_2 \circ \beta_2(s) \| + \max_{s \in [0,1]} \| q_2 \circ \beta_2(s) - q_3 \circ \beta_2(s) \| =
\]

\[
\delta(q_1, q_3) + \sigma(\varepsilon) + \delta(q_2, q_3),
\]

with

\[
\sigma(\varepsilon) = \max_{s \in [0,1]} \| q_2 \circ \alpha_2 \circ \alpha_{22}^{-1} \circ \beta_2(s) - q_2 \circ \beta_2(s) \|.
\]

Since \( \sigma(\varepsilon) \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \) because of Lemma 2 below, this yields iii).

LEMMA 2. -- Let \( \alpha: [a, b] \rightarrow [0, S] \) be continuous, surjective and non-decreasing. For \( 0 < \varepsilon \ll 1 \) the map \( \alpha_\varepsilon \) defined by

\[
(3.5) \quad \alpha_\varepsilon(r) = [1 - \varepsilon] \alpha(r) + \varepsilon S[r - a]/[b - a]
\]

is strictly increasing, and its inverse \( \alpha_\varepsilon: [0, S] \rightarrow [a, b] \) satisfies

\[
(3.6) \quad |\alpha(\alpha_\varepsilon^{-1}(t)) - t| \leq \varepsilon S \quad \forall t \in [0, S].
\]

Indeed

\[
|\alpha \circ \alpha_\varepsilon^{-1}(t) - t| = |\alpha \circ \alpha_\varepsilon^{-1}(t) - \alpha_\varepsilon \circ \alpha_\varepsilon^{-1}(t)| =
\]

\[
= |\varepsilon \circ \alpha_\varepsilon^{-1}(t) - \varepsilon S[\alpha_\varepsilon^{-1}(t) - a]/[b - a]| \leq \varepsilon S.
\]

Consider again a control system of the form

\begin{equation}
\dot{y}(t) = \sum_{i=1}^{m} g_i(y(t)) \dot{u}_i(t), \quad y(0) = \bar{y} \in \mathbb{R}^n \quad t \in [0, T]
\end{equation}

where the vector fields $g_i$ are defined and continuously differentiable in some open set $\Omega \subseteq \mathbb{R}^n$. We denote by $y(u, \cdot)$ the solution of (4.1), if it exists, corresponding to the control $u$. The next results establish the continuity of the input-output map, when restricted to suitable sets of controls.

**Theorem 1.** Fix $L > 0$, let $\mathcal{U} \subseteq C^0([0, T]; \mathbb{R}^n)$ be the set of all controls with Lipschitz constant $L$, and call $\mathcal{U}^*$ the set of all controls $u \in \mathcal{U}$ for which the corresponding solution $y(u, \cdot)$ of (4.1) exists in $\Omega$.

Then

i) $\mathcal{U}^*$ is a relatively open subset of $\mathcal{U}$.

ii) The restriction to $\mathcal{U}^*$ of the map $\Phi: u(\cdot) \to y(u, \cdot)$ is locally Lipschitz continuous with respect to the $C^0$ norms on the spaces of controls and trajectories.

The proof will rely on the following corollary of the Contraction Mapping Theorem [6]. A quite similar result appears in [2].

**Lemma 3.** Let $A, B$ be closed subsets of the Banach spaces $E, F$ respectively. Let $\Psi: A \times B \to B$ be a map such that $\forall u, v \in A, \forall x, y \in B$ one has

\begin{align}
\|\Psi(u, x) - \Psi(u, y)\|_F &\leq \frac{1}{2}\|x - y\|_F, \\
\|\Psi(u, x) - \Psi(v, x)\|_F &\leq C\|u - v\|_E,
\end{align}

for some constant $C$. Then for each $u \in A$ there exists a unique $x = x(u) \in B$ such that $x(u) = \Psi(u, x(u))$. Moreover

\begin{equation}
\|x(u) - x(v)\|_F \leq 2C\|u - v\|_E.
\end{equation}

**Proof of Theorem 1.** Let $u^* \in \mathcal{U}^*$ and let $y^* = y(u^*, \cdot)$ be the corresponding solution of (4.1). Since the image $y^*([0, T])$ is a compact subset of $\Omega$, there exists $\delta > 0$ so small that

\begin{equation}
\Omega' = \bigcup_{t \in [0, T]} B(y(u^*, t), \delta) \subseteq \Omega.
\end{equation}
Here and in the sequel $B(y, \delta)$ denotes the open ball centered at $y$ with radius $\delta$. For $i = 1, \ldots, m$, construct a $C^1$ vector field $h_i$ with compact support in $\mathbb{R}^n$, which coincides with $g_i$ on $\Omega'$. Let $N$ be an upper bound for both $h_i$ and the operator norms of the Jacobian matrices of $h_i$, i.e.

\[
(4.6) \quad \|h_i(x)\| < N, \quad \left\| \frac{\partial h_i}{\partial x}(x) \right\| < N \quad \forall x \in \mathbb{R}^n, \quad i = 1, \ldots, m.
\]

Consider the spaces $E = C^0([0, T]; \mathbb{R}_n)$ with the usual norm and $F = C^0([0, T]; \mathbb{R}_n)$ with the equivalent norm \[\|x\|_F = \sup \{e^{-\lambda t}\|x(t)\| : t \in [0, T]\}, \quad \lambda = 2NmL.\]

The new Cauchy problem

\[
(4.8) \quad \dot{z}(t) = \sum_{i=1}^{m} h_i(z(t)) \dot{u_i}(t) \quad z(0) = \bar{y} \quad t \in [0, T]
\]

now has a unique solution $z(u, \cdot)$ for every $u \in \mathcal{U}$.

Observe that $z(u, \cdot)$ is the solution of the implicit equation

\[
(4.9) \quad z = \Psi(u, z)
\]

with

\[
(4.10) \quad \Psi(u, z)(t) = \bar{y} + \int_0^t \sum_{i=1}^{m} h_i(z(s)) \dot{u}_i(s) \, ds
\]

We now apply Lemma 3, taking $A = \mathcal{U}$ and letting $B \subset F$ be the set of all trajectories with Lipschitz constant $mNL$. This choice guarantees that $\Psi$ in (4.10) maps $A \times B$ into $B$. To check (4.2), let $\|x - y\|_F = \mu$. Since $\|x(t) - y(t)\| < \mu e^{\lambda t}$ we have

\[
(4.11) \quad e^{-\lambda t}\|\Psi(u, x)(t) - \Psi(u, y)(t)\| < \mu MNL\lambda^{-1} = \frac{1}{2}\|x - y\|_F.
\]
To obtain (4.3), an integration by parts yields

$$
(4.12) \quad \|P(u, x)(t) - P(v, x)(t)\| \leq \int_0^t \sum_{i=1}^m \left\| \frac{d}{ds} g_i(x(s)) \right\| \|u_i(s) - v_i(s)\| ds + \\
\quad \quad \quad + \sum_{i=1}^m \|g_i(x(0))\| \cdot \|u_i(0) - v_i(0)\| + \sum_{i=1}^m \|g_i(x(t))\| \cdot \|u_i(t) - v_i(t)\| \\
\quad \quad \quad \leq Tm^2 N^2 L \|u - v\|_x + 2mN \|u - v\|_x,
$$

because

$$
\left\| \frac{d}{ds} g_i(x(s)) \right\| \leq \|g_i(x(s))\| \cdot \|\dot{x}(s)\| \leq N \cdot mNL.
$$

This yields (4.3) with $C = Tm^2 N^2 L + 2mN$. By Lemma 3, the input-output map $u \to z(u, \cdot)$ is thus Lipschitz continuous on $\mathcal{U}$. Statements i) and ii) are now clear, because whenever the trajectory $z(u, \cdot)$ of (4.8) is entirely contained inside $\Omega'$, it coincides with the solution $y(u, \cdot)$ of (4.1).

The next result is an analog of Theorem 1 for the pseudometric $\delta$ defined at (3.1).

**Theorem 2.** Fix $K > 0$, let $\mathcal{U}$ be the set of all Lipschitz continuous maps $v : [0, S_u] \to \mathbb{R}^m$ with total variation $\leq K$, and call $\mathcal{U}^*$ the set of maps $v \in \mathcal{U}$ for which the corresponding solution $y(v, \cdot)$ of (4.1) exists. Then

i) $\mathcal{U}^*$ is relatively open in $\mathcal{U}$,

ii) the restriction of the input-output map $v(\cdot) \to y(v, \cdot)$ to $\mathcal{U}^*$ is continuous, provided that the pseudometric defined at (3.1) is used on the spaces of controls and trajectories.

The theorem will be proved by showing that, given any sequence $(v_n)_{n \geq 0}$ of elements of $\mathcal{U}$ such that $v_0 \in \mathcal{U}^*$ and

$$
(4.13) \quad \lim_{n \to \infty} \delta(v_n, v_0) = 0,
$$

there exists a subsequence $v_{n'}$ such that $v_{n'} \in \mathcal{U}^*$ for all $n'$ suitably large and

$$
(4.14) \quad \lim_{n' \to \infty} \delta(y(v_{n'}, \cdot), y(v_0, \cdot)) = 0.
$$
By the results in § 3, it is not restrictive to assume that each map \( v_n \) coincides with its canonical parametrization. In particular, we can assume that every \( v_n \) has Lipschitz constant 1 and is defined on some interval \([0, S_n] \subset [0, K]\). If (4.13) holds, by Definition 4 there exist maps \( \beta_n, \gamma_n \) \( (n \geq 1) \) such that

\[
(4.15) \quad \lim_{n \to \infty} \max_{t \in [0, 1]} \| v_n(\beta_n(t)) - v_0(\gamma_n(t)) \| = 0.
\]

By Lemma 1 we can also assume that \( \beta_n \) and \( \gamma_n \) are all Lipschitz continuous with constant \( 2K + 1 \). Hence, for a suitable subsequence \( n' \to \infty \), there exists a map \( \gamma \) such that \( \gamma_n(t) \to \gamma(t) \) uniformly on \([0, 1]\). Moreover, (4.15) implies that the sequence of controls \( w_n = v_n \circ \beta_n \) converges to \( w_0 = v_0 \circ \gamma \) uniformly on \([0, 1]\). Since the solution of (4.1) corresponding to \( u_0 \) exists by assumption, Theorem 1 implies that the trajectories \( y(w_n, \cdot) \) also exist for all \( n' \) suitably large, and tend to \( y(w_0, \cdot) \) uniformly on \([0, 1]\). This proves (4.14) because

\[
\lim_{n' \to \infty} \delta(y(v_{n'}, \cdot), y(v_0, \cdot)) \leq \lim_{n' \to \infty} \max_{t \in [0, 1]} \| y(v_{n'}, \beta_{n'}(t)) - y(v_0, \gamma(t)) \| = \lim_{n' \to \infty} \max_{t \in [0, 1]} \| y(v_{n'} \circ \beta_{n'}, t) - y(v_0 \circ \gamma, t) \| = 0.
\]

We now specialize the above result in the case where the \( v_n \) represent the graph-completions of a sequence of controls \( u_n \).

**Corollary 1.** For every \( n > 0 \), let \( u_n: [0, T] \to \mathbb{R}^m \) be a control with bounded variation, let \( \varphi_n \) be a graph-completion of \( u_n \) and let \( x(\varphi_n, \cdot) \) be the corresponding generalized solution of (1.1). If \( \delta(\varphi_n, \varphi_0) \to 0 \) as \( n \to \infty \) and if the total variation of the maps \( \varphi_n \) are uniformly bounded, then the graphs of \( x(\varphi_n, \cdot) \) tend to the graph of \( x(\varphi_0, \cdot) \) in the Hausdorff metric [1, p. 65].

**Corollary 2.** Let \( \varphi_0: [0, S] \to \mathbb{R}^{m+1} \) be a graph-completion of a control \( u: [0, T] \to \mathbb{R}^m \), and let \( (u_n)_{n \geq 1} \) be a sequence of Lipschitz continuous controls with uniformly bounded variation which approximate \( \varphi_0 \) in the sense that, setting \( \varphi_n(t) = (t, u_n(t)) \), one has

\[
\lim_{n \to \infty} \delta(\varphi_0, \varphi_n) = 0.
\]

Then the generalized solution \( x(\varphi_0, \cdot) \) of (1.1) relative to \( \varphi_0 \) satisfies

\[
x(\varphi_0, t) = \lim_{n \to \infty} x(u_n, t)
\]
at every \( t \in [0, T] \) where \( x(q_0, t) \) is single-valued, hence almost everywhere.

**Proof.** - Set \( x_0 = x(q_0, \cdot) \), \( x_n = x(u_n, \cdot) \) for \( n \geq 1 \). Fix \( \varepsilon > 0 \) and \( t \in [0, T] \) such that \( x(q_0, t) \) is single-valued. Since \( x_0 \) is upper semicontinuous at \( t \), there exists \( \delta \in (0, \varepsilon/2) \) such that \( x_0(s) \subseteq B(x_0(t), \varepsilon/2) \) whenever \( |s - t| < \delta \). Choose \( N \) so large that the Hausdorff distance between the graphs of \( x_n \) and \( x_0 \) is smaller than \( \delta \), for all \( n \geq N \). This implies the existence of \( s_n \in [t - \delta, t + \delta] \) such that

\[
x_n(t) \in B(x_0(s_n), \delta) \subseteq B(x_0(t), \varepsilon), \quad \forall n \geq N.
\]

Since \( \varepsilon \) was arbitrary, Corollary 2 is proved.

**Corollary 3.** - In addition to the assumption of Corollary 2, suppose that \( u_n \) converges to \( u \) uniformly on some interval \([\tau, \tau + \sigma]\). Then

\[
\lim_{n \to \infty} x(u_n, \tau) = x^*(q_0, \tau)
\]

where \( x^*(q_0, \cdot) \) was defined in the Remark following Definition 2.

**Proof.** - Assume that there exists a subsequence \( u_n', \) and a point \( x^* \) such that

\[
\lim_{n' \to \infty} x(u_{n'}, \tau) = x^* \neq x^*(q_0, \tau).
\]

Then on the interval \([\tau, \tau + \sigma]\) the sequence of solutions \( x(u_{n'}, \cdot) \) converges uniformly to the solution \( x(t) \) of the Cauchy Problem

\[
\dot{x}(t) = f(x) + \sum_{i=1}^m g_i(x) \dot{u}_i(t) \quad x(\tau) = x^*. 
\]

Since \( x^*(q_0, \cdot) \) is a right continuous function of \( t \) and \( x(\cdot) \) is continuous on \([\tau, \tau + \sigma]\), the distance \( |x^*(q_0, t) - x(t)| \) remains strictly positive on some interval of the form \([\tau, \tau + \varepsilon]\). This is impossible because Corollary 2 implies \( x^*(q_0, t) = x(t) \) almost everywhere.

5. - Mollifications.

Given a (possibly discontinuous) control \( u: [0, T] \to \mathbb{R}^m \), approximate solutions of (1.1) can be constructed by means of a mollification of the control.
lification. More precisely, let $\psi$ be a $C^1$ function with compact support for which

$$
\int_{-\infty}^{\infty} \psi(t) \, dt = 1, \quad \psi(t) \geq 0 \quad \forall t \in \mathbb{R}.
$$

For $\eta > 0$, set $\varphi_\eta(t) = \eta^{-1} \psi(\eta^{-1} t)$, and define the convolution $u_\eta = u * \varphi_\eta$:

$$
u_\eta(t) = \int_{-\infty}^{\infty} u(s) \varphi_\eta(t-s) \, ds
$$

with the convention that in (5.2) the function $u$ has been extended outside $[0, T]$ by setting $u(t) = u(0)$ if $t < 0$, $u(t) = u(T)$ if $t > T$. The mollified control $u_\eta$ is then $C^1$, and one can now look for a classical solution of (1.1) corresponding to $u_\eta$. It is interesting to determine the limit of this solution as $\eta \to 0$.

**Theorem 3.** Assume that $u:[0, T] \to \mathbb{R}^n$ is right continuous with bounded variation, and that $\lim_{t \to T^-} u(t) = u(T)$. Fix a $C^1$ map $\psi$ with compact support, satisfying (5.1) and define $u_\eta = u * \varphi_\eta$ as in (5.2). Then, as $\eta \to 0$, the graph of the corresponding solution $x(u_\eta, \cdot)$ of (1.1) tends to the graph of $x(q_u, \cdot)$ in the Hausdorff metric, $q_u$ being the canonical graph-completion of $u$. In particular,

$$
\lim_{\eta \to 0} x(u_\eta, t) = x(q_u, t)
$$

at every time $t$ where $u$ is continuous.

**Proof.** Define $q_\eta: [0, T] \to \mathbb{R}^{n+1}$ by setting $q_\eta(t) = (t, u_\eta(t))$ and let $q_u: [0, T + V(T)] \to \mathbb{R}^{n+1}$ be the canonical graph-completion of $u$. All notations introduced in Example 1, § 2 will be again used here. Since $q_u$ and $q_\eta$ ($\eta > 0$) have uniformly bounded variation, by Corollaries 1 and 2 it suffices to prove that $\delta(q_u, q_\eta) \to 0$ as $\eta \to 0$. For simplicity, we assume that the support of $\psi$ is contained inside $[-1, 1]$, which is not restrictive. Fix $\varepsilon > 0$. Since $u$ has bounded variation, there exist at most finitely many times $t_i$, $0 < t_1 < \ldots < t_k < T$ where the jump $\|u(t_i) - u(t_i^-)\|$ is larger than $\varepsilon/4$. Choose $\varepsilon' \in (0, \varepsilon/4)$ such that

i) the intervals $(t_i - 2\varepsilon', t_i + 2\varepsilon')$ are all disjoint and contained inside $[0, T]$.

ii) $V^+(t_i + 2\varepsilon') - V(t_i) \leq \varepsilon/4$, $V^-(t_i) - V(t_i - 2\varepsilon') \leq \varepsilon/4$ for all $i = 1, \ldots, k$. 
Now choose $\sigma \in (0, \varepsilon')$ such that, for every $t$ in the compact set $J = [0, T] \setminus \bigcup_i B(t_i, \varepsilon')$,

\begin{equation}
V(t + \sigma) - V(t - \sigma) \leq \varepsilon / 2.
\end{equation}

The theorem will be proved by showing that

\begin{equation}
\delta(\varphi_s, \varphi_\eta) \leq \varepsilon \quad \forall \eta \in (0, \sigma).
\end{equation}

Indeed, for any $\eta \in (0, \sigma)$, we will construct a non-decreasing, continuous, surjective map $\beta: [0, W(T)] \to [0, T]$ such that

\begin{equation}
\|\varphi_s - \varphi_\eta(\beta(s))\| \leq \varepsilon \quad \forall s \in [0, W(T)].
\end{equation}

Let the canonical graph-completion $\varphi_s(t(s), \varphi(s))$ be defined as in Example 1. In the construction of $\beta(s)$ we consider three cases.

I) If $t(s) \in J = [0, T] \setminus \bigcup_i B(t_i, \varepsilon')$, set $\beta(s) = t(s)$.

II) If $s \in (W(t_i - \varepsilon'), W(t_i))$, set $\beta(s) = t_i - \varepsilon'$.

III) If $s \in (W(t_i), W(t_i + \varepsilon'))$, set $\beta(s) = t_i + \varepsilon'$.

For every $\mu \in (0, 1)$, $\alpha_\mu$ is strictly increasing, continuous and surjective. Call $\alpha_\mu^{-1}$ its inverse. Define

\begin{equation}
\beta(s) = t_i + \alpha_\mu^{-1}(\lambda(s))
\end{equation}

with

\begin{equation}
\lambda(s) = \frac{s - W(t_i)}{W(t_i) - W(t_i)}, \quad \mu = \min\left\{\frac{\varepsilon / 2}{W(t_i) - W(t_i)}, 1\right\}.
\end{equation}

Relying on the fact that $\text{supp} (\varphi_\eta) \subseteq [-\eta, \eta]$, with $\eta < \sigma < \varepsilon'$, we will prove that, in all three cases, (5.5) holds.

I) If $t(s) \in J$, then

\begin{equation}
\|\varphi_s - \varphi_\eta(\beta(s))\| = \|u(t(s)) - u_\eta(t(s))\| \leq \varepsilon / 2
\end{equation}
because both \( u(t(s)) \) and \( u_\eta(t(s)) \) lie in the convex closure of the values of \( u \) on \((t(s) - \sigma, t(s) + \sigma)\) and, by (5.3), the oscillation of \( u \) on this interval is bounded by \( \varepsilon/2 \).

II) If \( s \in (W(t_i - \varepsilon'), W^- (t_i)) \), then

\[
\| \varphi(u(s)) - \varphi(u_\eta(\beta(s))) \| \leq \| u(t(s)) - u(t(s) - \varepsilon') \| + \| u_\eta(t_i - \varepsilon) \| \\
\leq \varepsilon' + \varepsilon/4 \leq \varepsilon/2.
\]

Indeed, \( t(s) \in (t_i - \varepsilon', t_i) \), both \( u(t(s)) \) and \( u_\eta(t_i - \varepsilon') \) lie in the convex closure of the values of \( u \) on \((t_i - 2\varepsilon', t_i)\), and the oscillation of \( u \) on such interval is bounded by \( \varepsilon/4 \). The case \( s \in (W(t_i), W^- (t_i + \varepsilon')) \) is entirely similar.

III) If \( t(s) = t_i \), then

\[
\| \varphi(u(s)) - \varphi(u_\eta(\beta(s))) \| \leq \| t_i - \beta(s) \| + \| u(s) - u_\eta(\beta(s)) \|. 
\]

By construction, \( \| t_i - (\beta(s)) \| \leq \varepsilon' \leq \varepsilon/4 \). To estimate the second term, we use Lemma 2 in § 3 and obtain

\[
|\alpha_0(\alpha_\mu^{-1}(\lambda)) - \lambda| \leq \mu \leq \frac{\varepsilon/2}{W(t_i) - W^-(t_i)} \quad \forall \lambda \in [0, 1]. 
\]

Moreover, (5.6) and (5.7) imply

\[
\int_{t_i}^{\beta(s) - t_i} \Psi_{\eta}(\beta(s) - \xi) \, dt = \int_{-\infty}^{\Psi_{\eta}(\xi')} d\xi' = \alpha_0(\beta(s) - t_i) = \alpha_0(\alpha_\mu^{-1}(\lambda(s))). 
\]

Recalling that the oscillation of \( u \) on \((t_i - 2\varepsilon', t_i)\) and on \((t_i, t_i + 2\varepsilon')\) is bounded by \( \varepsilon/4 \) and using (5.9), (5.10) with \( \lambda = \lambda(s) \) we now obtain

\[
\| u(s) - u_\eta(\beta(s)) \| \leq \| [\lambda u(t_i) + (1 - \lambda) u^-(t_i)] - [\alpha_0 \circ \alpha_\mu^{-1}(\lambda) u(t_i) + \\
+ (1 - \alpha_0 \circ \alpha_\mu^{-1}(\lambda)) u^-(t_i)] \| + \| \int_{t_i}^{t_i + 2\varepsilon'} u^-(t) \Psi_{\eta}(\beta(s) - \xi) \, d\xi + \\
+ \int_{t_i}^{t_i + 2\varepsilon'} u(t) \Psi_{\eta}(\beta(s) - \xi) \, d\xi - \int_{t_i}^{t_i + 2\varepsilon'} u(t) \Psi_{\eta}(\beta(s) - \xi) \, d\xi \| \leq \| u(t_i) - u^-(t_i)\| \mu + \varepsilon/4 \leq 3\varepsilon/4.
\]

This completes the proof of (5.5), which in turn yields (5.4). ■
REFERENCES


A. Bressan: Dept. of Mathematics, University of Colorado
Boulder, Co 80309 (U.S.A.)
F. Rampazzo: SISSA, Trieste

Permessula in Redazione
il 22 settembre 1987