Abstract:
We present a modeling framework for 1D fragmentation in brittle rods, in which the distribution of fragments is written explicitly in terms of the probability of breaks along the length of the rod. This work is motivated by the experimental observation of several preferred lengths in the fragment distribution of shattered brittle rods after dynamic buckling [1]. Our approach allows for non-constant spatial breaking probabilities, which can lead to preferred fragment sizes, derived equivalently from either combinatorics or a nonhomogeneous Poisson process. The resulting relation qualitatively matches the experimentally observed fragment distribution, as well as some other common distributions such as a power law with a cutoff.

Keywords: fragmentation; Poisson processes; fracture; combinatorics.

1 Introduction
The fragmentation of a brittle solid is an everyday example of probability and apparent randomness in a physical system: a glass or plate dropped on the floor usually shatters “into a million pieces” - very few large ones and many more small ones - and experience shows that no two plates break in the same way. While the phenomenon of a single crack or fracture has now advanced in many aspects to the stage of a general agreement between a well-developed mathematical theory and many careful experimental studies (see e.g. [2, 3]), the mathematical study of fragmentation, which can be thought of as the multiple fracture limit, is still at a developmental stage, with no universally accepted theoretical approach to the wealth of empirical information [4, 5, 6, 7]. The process of fragmentation has probably been known since prehistoric times, and observations of the phenomenon can be traced back to the ancients, as illustrated by this passage from the Iliad [8]:

But fierce Atrides wav’d his sword and strook
Full on his casque: the crested helmet shook;
The brittle steel, unfaithful to his hand,
Broke short: the fragments glitter’d on the sand.
The fundamental question of fragmentation is this: how does a single solid object break into many pieces? Certainly a number of different but interrelated physical mechanisms are involved in this breaking, and there may be different kinds of fragmentation [4, 7], for instance ductile vs brittle [9, 10], kinetic energy-dominated vs static stress-dominated [11], or even the fragmentation of a thin brittle coating attached to an easily deformed unbroken substrate [12]. Mathematical approaches to modeling this process, in order to obtain observed quantities such as the mean fragment size or the fragment distribution, fall into two broadly defined categories: those starting from a mechanical perspective of the stressed material, and those aiming to derive a specific functional form for the distribution (typically power law) in a post hoc manner. These latter approaches are usually motivated by the many experimental studies which have reported power laws in measured fragment distributions, such as those observed in the impact fragmentation of brittle glass rods [5, 13], disks, or spheres [14]. A power law dependence, with an exponential cutoff at larger sizes, is also seen in explosively fragmented distributions [15]. Some of the other physical situations in which power law particle distributions have been reported include ice floe size distribution in the Arctic [16], meteor shower mass distributions [17], and the size distribution of mercury drops which break into many pieces upon impact [18]. Post hoc models used to produce power laws in fragmentation include many suggesting the iterative breaking of a body [19, 18, 20]. The models range from simple to complex, but do not necessarily include a physical motivation. In contrast, Astrom described iterative breaking in a model motivated by the branching and merging of cracks along a fracture surface [6, 21]; some similar results were given earlier by Gilvarry, but based on different considerations [22].

The interest in power laws is that they are indicative of a self-similar process, which can suggest a universal theory; however, there is no single exponent which is observed. Moreover, there are also several reports of two scaling regimes (two different power laws); this has been related in some models to a transition from 1D to 3D effects [13], and indeed a recent model treats dynamic fragmentation for $D$ (arbitrary) dimensions [6]. The value of the exponent is usually a free parameter in these models. While the fragmentation process shares several similarities with turbulence in fluids [23], the fact that there is no universal power law exponent is a major difference, and so far this analogy has not been useful.

One early approach to modeling fragmentation was the work of Kolmogorov [24], inspired by the measurement of a log-normal distribution of fragment sizes produced by grinding. Kolmogorov used a few mathematical parameters to describe the continual grinding of larger particles into smaller particles. The primary requirement in his approach is that the fragmentation process reaches a condition where it is independent of particle size, independent of the fragmentation of other particles, and independent of the starting time - this last point implies that the probabilities are independent of the history of the particle in question. Under these assumptions, and two others involving the size and integrability of the expected number of particles resulting from a single particle per unit time, Kolmogorov deduced that the long time limit of the fragment distribution was log-normal.
Figure 1: Measured number density of fragments $n(\ell)$ from one fragmentation data set for the dynamic buckling of a brittle rod (here spaghetti) [1]. The inset shows an experimental image of an event representing the observation that the rod tends to break either near the top or near the bottom [29].

Another historical strand goes back to the 1947 paper by Mott, motivated by military questions on the fragmentation of shell cases [25]. This approach originates from more physical constraints, treating local deformations and stress release after a break occurs. The literature in this area includes energy-based models in the dynamic regime (due to impact or stress-wave loading) [7, 26], as compared to more of a flaw-dominated approach [27]. Many developments have been made in the geophysical community, particularly regarding the fragmentation of rocks due to geological or blasting processes (for an overview see [7]).

More recently, another approach to fragmentation was taken by Audoly & Neukirch [28], in which the dynamics of curvature after an initial break in a 1D brittle solid (in this case spaghetti) is described by a dispersive equation. In their model, the breaking is governed by the intersection of reflected self-similar solutions to this equation, which drives a focusing of the curvature, leading to breaking events. The dynamic spreading of fragmentation probability was confirmed by their experiments, in which a first break leads to a second.

In contrast to the focus on scale-invariant distributions, a recent experiment on the dynamic buckling and fragmentation of thin brittle rods found fragment distributions with two peaks, indicating preferred fragment sizes [1]. These lengths apparently originated with the initial sinusoidal buckling of the rod, leading to local maxima in the fragment distribution near $1/2$ and $1/4$ of the buckling wavelength, see Fig. 1. An explanation was proposed based on the assumption that breaks in the rod were more likely to occur around the points of maximum curvature, although there were many observations of breaking when the spaghetti did not break at every maximum (Fig. 1, inset). The speculative conclusion was that the distribution of fragments was being determined primarily by the initial stress distribution rather than by a sequential, multiplicative process [1]. This indication that coherent patterns
in the deformation can play a role in determining the fragment distribution provided the impetus for the present work.

The challenge posed by these observations to mathematical modeling was the existence of multiple peaks in the fragment distribution - as opposed to a self-similar, scaling law or a single preferred fragment size. While it is clear that the coherent pattern comes in some way from the spatial distribution of stress, deformation, and perhaps other fields in the original object [1], none of the existing modeling approaches allowed for this possibility. Our goal in this paper is to provide a mathematical framework connecting the notion of a rod breaking in this fashion, and the distribution of fragment sizes that result from the event. This will provide a foundation for further study of the relation between stress dynamics and the sizes of the fragments produced. In this paper we confine ourselves to the 1D case, having in mind the breaking of spaghetti or glass rods under conditions of dynamic buckling. If the breaks in different intervals of the rod indeed occur independently of one another, but following some function \( \lambda(x) \) in such a way that the probability of having exactly one break in the interval \((x, x + \delta x)\) is \( \lambda(x)\delta x + o(x) \), and the probability of having more than one break is \( o(x) \), then the breaks follow a nonhomogeneous Poisson process [30]. That is, the probability of having exactly \( k \) breaks in the interval \((x_1, x_2)\) of the rod is

\[
P_k(x_1, x_2) = \frac{1}{k!} \left( \int_{x_1}^{x_2} \lambda(x) \, dx \right)^k \cdot \exp \left( -\int_{x_1}^{x_2} \lambda(x) \, dx \right),
\]

and we will usually be most interested in the probability of having no breaks

\[
P_0(x_1, x_2) = \exp \left( -\int_{x_1}^{x_2} \lambda(x) \, dx \right)
\]

or the probability of having one or more breaks, which is given by \( 1 - P_0(x_1, x_2) \). Note that \( \lambda(x) \) can be thought of as a probability density for breaking, or as the density of the expected number of breaks per unit length. We will show that for a unit rod breaking according to a nonhomogeneous Poisson process with density \( \lambda(x) \) [30], the associated number density of fragments is given by the formula for \( \ell \in (0, 1) \)

\[
n(\ell) = \int_0^{1-\ell} \lambda(s)\lambda(s + \ell) \exp \left( -\int_s^{s+\ell} \lambda(q) \, dq \right) \, ds
+ \lambda(\ell) \exp \left( -\int_0^\ell \lambda(q) \, dq \right) + \lambda(1 - \ell) \exp \left( -\int_{1-\ell}^1 \lambda(q) \, dq \right)
\]

(1)

## 2 The single fracture Poisson process

One of the basic modeling assumptions in the study of fragmentation involves the obvious question of what order the pieces are created in - all at once, or sequentially; few experiments actually measure the temporal sequence of the breaks [28]. Starting with Kolmogorov, many
models are built on the conception of multiplicative breaking [31]. On the other hand there is a tradition of simplifying the treatment in terms of a “single fragmentation event”, based on the idea of multiple nucleation sites for the initial cracks that produce the fragments, in the limit of fast dynamic fragmentation [22, 32, 26]. Here we will take the latter approach.

In this section we review the single fracture Poisson process [7] as derived in the current literature in the context of 1D fragmentation [7, 20, 33]. In the simplest case, the expected number of breaks per unit length is constant, i.e. \( \lambda(x) = C \), and Eq. 1 reduces to

\[
n(\ell) = C(2 + C - C\ell)e^{-C\ell}.
\]  

This would correspond to breaks following a homogeneous Poisson process, where the probability of having \( k \) breaks in a section of length \( x \) is given by the Poisson distribution

\[
P(k; C, x) = \frac{(Cx)^k e^{-Cx}}{k!}.
\]

Interestingly, existing literature on this subject uses Eq. 3 to derive a fragment number density \( n(\ell) = C^2 e^{-C\ell} \) which is different than Eq. 2. The derivation starts from a probability density \( p(\ell) \) for the fragment size calculated by considering a fragment of length between \( \ell \) and \( \ell + \Delta \ell \). For this to exist, there must be an unbroken section of length \( \ell \) starting from the left tip of the fragment, which occurs with probability \( P(0; C\ell) = e^{-C\ell} \), followed by one or more breaks within \( \Delta \ell \) of the right tip, which occurs with probability \( 1 - P(0; C\Delta \ell) = 1 - e^{-C\Delta \ell} \). This implies that

\[
\int_{\ell}^{\ell+\Delta \ell} p(s) \, ds = e^{-C\ell} \left( 1 - e^{-C\Delta \ell} \right).
\]

Dividing both sides of this equation by \( \Delta \ell \) and letting \( \Delta \ell \to 0 \) leads to \( p(\ell) = Ce^{-C\ell} \). For this distribution, the expected number of breaks is \( C \), yielding \( C + 1 \) expected fragments. However the unit rod is assumed to be a section of an infinite rod, so the end fragments are counted only as half fragments. In this way the expected number of fragments is also \( C \).

The density of the expected number of fragments of length \( \ell \) is obtained in this approach by multiplying the expected number of fragments \( C \) by the probability \( p(\ell) \) for fragments of length \( \ell \)

\[
n(\ell) = C^2 e^{-C\ell}.
\]  

This formula would be correct if the sizes of fragments were independent of the number of fragments produced in an event, but in fact it is not at all independent. For example, consider an event with only two fragments; it is certain that exactly one of them is of length less than \( \frac{1}{2} \) (since the probability of a break in the exact center is zero). This would mean
that \( \int_0^{1/2} p(\ell) \, d\ell = 1/2 \). However the actual distribution gives

\[
\int_0^{1/2} p(\ell) \, d\ell = 1 - e^{-C/2},
\]

which depends on the value of \( C \). The contradiction indicates that \( p(\ell) \) should depend on the number of fragments; so while Eq. 4 appears to be commonly accepted, it is incorrect. Concerns over this method of derivation have been noted [6, 27], but left unaddressed. Our derivation of the correct version of Eq. 4 proceeds along different lines, and avoids this issue altogether.

3 Derivation for a Segmented Rod

To highlight the combinatorial source of the fragment distributions (Eqs.1 and 2) and their dependence on the underlying probability of breaking, we will first consider a simple discrete model. We begin with a rod of unit length, comprising \( J + 1 \) indestructible segments joined by \( J \) joints. For convenience we will discuss fragment lengths in units of segments, so that for instance the length of the shortest fragment is 1, instead of \((J+1)^{-1}\). We consider an event in which every joint breaks with probability \( p \) (independently of the others). Our goal is to find the expected number of fragments of exactly \( s \) segments long, \( E_{J+1}(s) \), which will depend among other things on the expected number of breaks in the rod, denoted \( C = Jp \). \( E_{J+1}(s) \) is given by the sum of the number of fragments of length \( s \) in each possible configuration of the broken rod weighted by the probability of achieving that particular configuration. Configurations in which there are no fragments of length \( s \) can thus be ignored.

The case where \( s = J + 1 \) is easily dealt with, since there is only one such fragment in all configurations of the broken rod, and the configuration occurs with probability \((1 - p)^J\). Then \( E_{J+1}(J + 1) = (1 - p)^J \).

For the general case where \( 1 \leq s \leq J \), we first consider the set \( S_k \) of all configurations with exactly \( k \) breaks. Because we treat all breaks as independent (occurring simultaneously), each configuration with \( k \) breaks has the same probability of occurring, and will produce \( k + 1 \) fragments. For one of the fragments to have a length \( s \) requires at least \( s + k \) total segments in the rod, thus we will stipulate that \( k \leq J + 1 - s \). We now consider the subset of \( S_k \) for which the first fragment on the left is \( s \) segments long. This leaves \( J - s \) joints among which to distribute the remaining \( k - 1 \) breaks; there are exactly \((J-s)_{k-1}\) unique ways in which this can occur.

Consider next the subset of \( S_k \) for which the second fragment on the left is \( s \) segments long. Each element of this subset can be obtained by permuting the order of the fragments from the previous subset in a one-to-one map, thus there are also \((J-s)_{k-1}\) configurations. In fact there are the same number of configurations for each of the \( k+1 \) positions of the \( s \)-length
segment, each being a simple rearrangement of the others. Thus

\[ E_{J+1}(s) = \sum_{k=1}^{J+1-s} p^k (1-p)^{J-k} \binom{J-s}{k-1}. \quad (6) \]

Note that this method of enumerating by the number of breaks allows us to sum neatly over \( k \), sidestepping the need to treat explicitly the configurations which have more than one piece of length \( s \), as these cases were naturally included in the enumeration.

To evaluate this sum, we define the function

\[ h(\mu, \eta) = \mu^2 (1 + \mu)^n = \mu^2 \sum_{m=0}^{n} \mu^m \binom{\eta}{m} \]

so that

\[ \frac{\partial h}{\partial \mu} = 2\mu(1 + \mu)^n + \eta \mu^2(1 + \mu)^{n-1} = \sum_{m=0}^{n} (m + 2) \mu^{m+1} \binom{\eta}{m}. \]

Using this we set \( m = k - 1 \) and rewrite Eq. 6 as

\[
E_{J+1}(s) = (1 - p)^J \sum_{m=0}^{J-s} \left( \frac{p}{1-p} \right)^{m+1} (m + 2) \binom{J-s}{m} = \left(1 - p\right)^J \left( \frac{\partial h(\mu, J-s)}{\partial \mu} \right)_{\mu = (\frac{p}{1-p})} \\
= (1 - p)^J \left[ 2 \left( \frac{p}{1-p} \right) \left( 1 + \frac{p}{1-p} \right)^{(J-s)} + (J - s) \left( \frac{p}{1-p} \right)^2 \left( 1 + \frac{p}{1-p} \right)^{(J-s-1)} \right] \\
= p(1 - p)^{(s-1)} \cdot [2 + (J - s)p].
\]

4 Towards a Continuum Rod

To produce a model which gives a fragment distribution \( n(\ell) \) for a 1D continuum from the discrete \( E_{J+1}(s) \), we need to specify how the probability of breaking at a joint relates to the probability density of breaks in the continuum rod. We will do this by fixing the expected number of breaks \( C \) and calculating \( p = C/J \) for the discrete rod.

Because \( \ell \in (0, 1) \) and \( s \) is an integer number of segments, we also need to provide an appropriate map between \((0, 1)\) and \( \mathbb{Z} \). We define \( s(\ell, J) \) to be the integer part of \( \ell \), such that \( s(\ell, J) \leq \ell \leq s(\ell, J) + 1 \). Note that

\[
\frac{s(\ell, J)}{J} = \frac{s(\ell, J)}{J + 1} \cdot \frac{J + 1}{J} \to \ell
\]

as \( J \to \infty \). Also, the difference in the lengths of fragments contemplated by \( n(\ell) \) and \( E_{J+1}(s(\ell, J)) \) is less than \((J + 1)^{-1}\). Thus it is natural to interpret fragments of \( s(\ell, J)\)
segments to represent all fragments of lengths \( \left[ \frac{s(\ell, J)}{J+1}, \frac{s(\ell, J)+1}{J+1} \right] \), and

\[
E_{J+1}(s(\ell, J)) = \int_{s(\ell, J)/J+1}^{s(\ell, J)+1/J+1} n(t) \, dt.
\]  

(7)

In this way we can calculate \( n(\ell) \) as \( J \to \infty \) to be

\[
(J+1)E_{J+1}(s(\ell, J)) = (J+1) \int_{s(\ell, J)/J+1}^{s(\ell, J)+1/J+1} n(t) \, dt \to n(\ell).
\]  

(8)

We now calculate

\[
(J+1)E_{J+1}(s(\ell, J)) = p(1-p)^{(s(\ell, J)-1)} \cdot [2 + (J - s(\ell, J))p]
\]

\[
= \left( (1-p)^{-\frac{1}{p}} \right)^{-p s(\ell, J)} (1-p)^{-1} (J+1)p \cdot [2 + Jp - np]
\]

\[
= \left( (1-p)^{-\frac{1}{p}} \right)^{-C s(\ell, J)} \frac{C + p}{1-p} \cdot [2 + C - \frac{C s(\ell, J)}{J}],
\]

so that

\[
n(\ell) = \lim_{J \to \infty} (J+1)E_{J+1}(s(\ell, J)) = C(2 + C - C\ell)e^{-C\ell}
\]  

(9)

for \( \ell \in (0, 1) \).

There is a small discrepancy in our derived formula for \( n \): the expected number of fragments should be \( C + 1 \). If we integrate by parts we find

\[
\int_0^1 n(\ell) \, d\ell = -(2 + C - C\ell)e^{-C\ell}\bigg|_0^1 - C \int_0^1 e^{-C\ell} \, dx = C + 1 - e^{-C}.
\]

The discrepancy is resolved by including the pieces which remain unbroken, for which in the discrete case \( E_{J+1}(J+1) = (1-p)^J \). Since \((1-p)^J = \left( (1-p)^{-\frac{1}{p}} \right)^{-C} \to e^{-C} \) as \( J \to \infty \), the expected number of pieces of length 1 is \( e^{-C} \). The correct distribution is

\[
n(\ell) = C(2 + C - C\ell)e^{-C\ell} + \delta(\ell, 1)e^{-C}, \quad \ell \in (0, 1]
\]  

(10)

where \( \delta \) is the Kronecker delta for the contribution of the unbroken rods. Note that only in the limit where \( C \to \infty \) and \( \ell \) is small does one obtain \( n(\ell) \simeq C^2e^{-C\ell} \) (Eq. 4), which is the one often quoted in the literature [7, 20, 33].
5 The Nonhomogeneous Case

For a brittle rod which does not have a constant probability of breaking along its length, one can write the expected number of breaks \( C \) as a function of position \( x \). In this case one can derive Eq. 1 from Eq. 10 by considering the limiting case of a discrete but local combinatoric calculation \[34\]. However it is more direct, and possibly more fundamental, to derive Eq. 1 from the nonhomogeneous Poisson process \[30\]. Specifically, we assume that a unit rod breaks in such a manner that the probability of having one or more breaks in the interval \((x_0, x_0 + s)\) is

\[
1 - \exp \left( - \int_{x_0}^{x_0 + s} \lambda(q) \, dq \right)
\]
i.e. with the probability density

\[
\lambda(x_0 + s) \exp \left( - \int_{x_0}^{x_0 + s} \lambda(q) \, dq \right).
\]

Here \( \lambda \) is the Poisson process parameter \[30\]. Our approach to define \( n(\ell) \) will be to calculate \( N(\ell + \delta) - N(\ell) \), the expected number of fragments of length ranging from \( \ell \) to \( \ell + \delta \) (we will assume that \( \delta << \ell, 1 - \ell \)), and then take the limit as \( \delta \to 0 \)

\[
n(\ell) = \lim_{\delta \to 0} \frac{N(\ell + \delta) - N(\ell)}{\delta}
\]

First note that the left end of a fragment of length \((\ell, \ell + \delta)\) must have its left edge in the interval \((0, 1 - \ell)\). We will partition this into sections of length \( \delta \). Because we do not require that \((1 - \ell)\) is an integer multiple of \( \delta \), there will be \( M \) such sections, where \( M \) is the integer part of \((1 - \ell)/\delta\), and possibly an extra section \((M\delta, 1 - \ell)\) remaining on the right. With probability one, the left end of any interior fragment lies in one of these \( M \) (or \( M + 1 \)) sections, and there can only be one such fragment originating in any of these sections. Then for the \( j \)th segment, if \( P(x_0, \ell; \delta) \) is the probability of having a fragment of length \((\ell, \ell + \delta)\) with its left edge in \((x_0, x_0 + \delta)\), then

\[
N(\ell + \delta) - N(\ell) = P_L + P_R + P_E + \sum_{j=0}^{M-1} P(j\delta, \ell; \delta)
\]

where \( P_L \) and \( P_R \) are the probabilities that the fragments from the left and right ends of the rod are of length \((\ell, \ell + \delta)\) and \( P_E = P(M\delta, \ell; 1 - \ell - M\delta) \) is the probability of having such a fragment begin in the extra section.

From (11) we can calculate

\[
P_L = \exp \left( - \int_{0}^{\ell} \lambda(t) \, dt \right) \left( 1 - \exp \left( - \int_{\ell}^{\ell + \delta} \lambda(t) \, dt \right) \right)
\]

(13)
and

\[ P_R = \exp \left( - \int_{1-\ell}^{1} \lambda(t) \, dt \right) \left( 1 - \exp \left( - \int_{1-\ell}^{1-\ell-\delta} \lambda(t) \, dt \right) \right). \quad (14) \]

We will make use of the limits

\[ \lim_{\delta \to 0} \frac{1}{\delta} P_L = \lambda(\ell) \exp \left( - \int_{0}^{\ell} \lambda(t) \, dt \right) \]

and

\[ \lim_{\delta \to 0} \frac{1}{\delta} P_R = \lambda(1-\ell) \exp \left( - \int_{1-\ell}^{1} \lambda(t) \, dt \right). \quad (15) \]

To calculate \( P(x_0, \ell; \delta) \) we write \( P(x_0, \ell; \delta) = P_1(x_0, \ell; \delta) + P_2(x_0, \ell; \delta) \). \( P_1 \) and \( P_2 \) represent two mutually exclusive cases: in the first case there are one or more breaks in the section of the rod between \( x_0 + \delta \) and \( x_0 + \delta + \ell \). We will deal with three parts of the fragment: the center unbroken section of length \( \ell \), the left tip, and the right tip. We will not know where these tips are located precisely, only to within a small region. Let the left tip lie within an interval of length \( s \), and the right tip in an interval of length \( w \). This gives us the constraints \( s + w < \delta \) with \( s, w > 0 \).

In this case if the left tip of the rod lies in the interval \( (x_0, x_0 + s) \) then right tip of the rod lies in \( (x_0 + s + \ell, x_0 + s + \ell + w) \). By twice invoking (11) together with the probability \( \exp \left( - \int_{x_0+s+\ell}^{x_0+s+\ell+w} \lambda(t) \, dt \right) \) that the center section remains unbroken we have

\[ P_1(x_0, \ell; \delta) = \int_{0}^{\delta} \int_{0}^{\delta-s} \lambda(x_0 + s) \lambda(x_0 + s + \ell + w) \exp \left( - \int_{x_0}^{x_0+s+\ell+w} \lambda(t) \, dt \right) \, dw \, ds. \quad (17) \]

By noting that the domain of integration is a triangle of area \( \frac{1}{2} \delta^2 \) we can calculate

\[ \lim_{\delta \to 0} \frac{P_1}{\delta^2} = \frac{1}{2} \lambda(x_0) \lambda(x_0 + \ell) \exp \left( - \int_{x_0}^{x_0+\ell} \lambda(t) \, dt \right). \quad (18) \]

In the second and mutually exclusive case, there are no breaks in the section of the rod between \( x_0 + \delta \) and \( x_0 + \delta + \ell \). Now if the left tip of the fragment lies in \( (x_0 + \delta - s, x_0 + \delta) \), the right tip lies in \( (x_0 + \delta + \ell, x_0 + \delta + \ell + w) \). This gives us

\[ P_2(x_0, \ell; \delta) = \int_{0}^{\delta} \int_{0}^{\delta-s} \lambda(x_0 + \delta - s) \lambda(x_0 + \delta + \ell + w) \exp \left( - \int_{x_0+\delta-s}^{x_0+\delta+\ell+w} \lambda(t) \, dt \right) \, dw \, ds \quad (19) \]

with

\[ \lim_{\delta \to 0} \frac{P_2}{\delta^2} = \frac{1}{2} \lambda(x_0) \lambda(x_0 + \ell) \exp \left( - \int_{x_0}^{x_0+\ell} \lambda(t) \, dt \right). \quad (20) \]

We said that \( P = P_1 + P_2 \), and this is usually true. The exception lies with interior fragments whose left tip is in the interval \( (1-\ell-M\delta, 1-\ell) \). In this case, \( P_2 \) would describe
a fragment formed by a break past the end of the rod. For these fragments then, we simply have
\( P = P_1 \). This applies either to \( P_E \) or \( P((M - 1)\delta, \ell; \delta) \) (the latter if \( \delta \) divides \( 1 - \ell \)). In either case, Eq. 12 becomes

\[
N(\ell + \delta) - N(\ell) = P_L + P_R + \sum_{i=0}^{M-2} P(i\delta, \ell; \delta) + o(\delta).
\]  \hspace{1cm} (21)

We can now derive the fragment distribution in Eq. 1:

\[
n(\ell) = \lim_{\delta \to 0} \frac{N(\ell + \delta) - N(\ell)}{\delta} = \lim_{\delta \to 0} \frac{1}{\delta} \sum_{i=0}^{M-2} P(i\delta, \ell; \delta) + \frac{P_L}{\delta} + \frac{P_R}{\delta}
\]

\[
= \lim_{\delta \to 0} \frac{1}{\delta} \sum_{i=0}^{M-2} \frac{P(i\delta, \ell; \delta)}{\delta^2} + \frac{P_L}{\delta} + \frac{P_R}{\delta}
\]

\[
= \int_0^{1-\ell} \lambda(s)\lambda(s + \ell) \exp \left( - \int_s^{s+\ell} \lambda(t) \, dt \right) \, ds
\]

\[
+ \lambda(\ell) \exp \left( - \int_0^\ell \lambda(t) \, dt \right) + \lambda(1-\ell) \exp \left( - \int_{1-\ell}^1 \lambda(t) \, dt \right)
\]

6 Fragment distributions for several basic \( \lambda(x) \)

We will discuss here a few examples of how patterns in \( \lambda(x) \) produce different distributions \( n(\ell) \). Let \( \lambda(x) = Af(x) \) and consider \( n(\ell) \) for very small values of \( A \). In this case, the first integral in Eq. 1 is \( O(A^2) \) compared to \( O(A) \) for the last two terms. The dominant terms

\[
Af(\ell) \exp \left( \int_0^{1-\ell} Af(x) \, dx \right) + Af(1-\ell) \exp \left( \int_{1-\ell}^1 Af(x) \, dx \right)
\]  \hspace{1cm} (22)

are symmetric about \( \ell = \frac{1}{2} \) whether or not \( f \) has any such symmetry, as shown in Fig. 2. In the special case where \( f(x) \) is itself symmetric about \( x = \frac{1}{2} \), Eq. 22 reduces to \( 2Af(\ell) \), see Fig. 2(a-b). It is interesting to note in Fig. 2(b) that \( n(0.5) \approx 0 \). This is because the remaining terms are end fragments, and breaks near the center are unlikely. Since this is the only way to produce end fragments of length \( 1/2 \), \( n(\ell) \) is close to zero. Contrast this to Fig. 2(d), where fragments of any length can be produced from at least one of the ends.

These arguments made for small \( A \) may not be observed in practice. As \( A \) increases, symmetry is quickly lost as the two dominant terms discussed above lose their significance. The overall mass of the graph shifts to the left as the increased likelihood of breaks creates a greater number of shorter pieces, see Figures 3 and 4.

Of particular relevance to the fragmentation data obtained from dynamic buckling [1] is the effect illustrated in Fig. 4. The fragment distributions shown are obtained from a
Figure 2: Examples of fragment distributions $n(\ell)$ produced by simple parabolic functions: a)-b) $\lambda(x) = \frac{1}{2}(x - \frac{1}{2})^2$; c)-d) $\lambda(x) = \frac{1}{2}(x - \frac{3}{4})^2$

Gaussian breaking probability with a sharp maximum at the center of the rod:

$$\lambda(x) = A \exp \left( -100 \left( x - \frac{1}{2} \right)^2 \right)$$

Not surprisingly, this leads to a significant number of fragments of length $\ell \approx 1/2$. However as the overall probability of breaking increases (increasing $A$), it becomes more likely to have multiple breaks near the center. Multiple breaks near the regions of high curvature were clearly evident in the images shown in Fig. 1 of [1]. The resulting $n(\ell)$ shown in Fig. 4 shows that this results in a downward shift in the fragment peak at 1/2. The inset to the figure traces this downward shift in the location of the maximum as a function of $A$; this shift was also seen experimentally [1].

Another aspect of our formalism is the delineation of contributions to the fragment distribution $n(\ell)$ from the end pieces and the central pieces. This is illustrated in Fig. 5 for $\lambda(x) = 1 + \sin^2(4\pi x)$, which shows the contributions to the total distribution from end fragments (dashed line) and center fragments (dashed and dotted line).

Note that for this figure $\lambda(x)$ would have four peaks along the length of the rod, which produces an $n(\ell)$ with four peaks (Fig. 5). However, in Fig. 4 there are two peaks visible in the fragment distribution $n(\ell)$, generated by a $\lambda(x)$ consisting of a single peak. As our original concern was with fragment distribution data exhibiting two distinct peaks (see Fig. 1), it
Figure 3: Fragment distributions $n(\ell)$ produced by $\lambda(x) = A\sin^2(5\pi x)$ for $A = 1, 5, 10, 15$ (lowest to highest). Inset in the upper right hand corner is a log-log plot for $A = 1, 10, 10^2, 10^3$ (lowest to highest).

Figure 4: The distributions $n(\ell)$ produced by $\lambda(x) = A\exp(-100(x - \frac{1}{2})^2)$, shown for $A = 1, 5, 10, 15$ (with increasing height). The inset shows the shift in the peak location with increasing $A$ (linear-log scale).
is natural to inquire in general how peaks in $\lambda(x)$ are expressed in $n(\ell)$. We answer this question when $\lambda(x)$ takes the form of $N$ individual pulses in $(0, 1)$ and is zero elsewhere.

The ends of a fragment produced in such a case must either be located within one of the pulses or coincide with one of the ends of the rods. Fragments produced by breaks at different pulses (or that go to the end of the rod) will produce a peak in $n(\ell)$ at about the distance between the two. By counting the pairs of possible fragment-end locations, we find that these produce at most $\binom{N+2}{2} - 1$ possible peaks in $n(\ell)$; we exclude the case of the unbroken rod, which is not shown in plots of $n(\ell)$. Fragments produced by multiple breaks within one pulse (assuming it is relatively narrow) will show up as a peak in $n$ near $\ell = 0$. Including this peak at $\ell = 0$ gives a total of $\binom{N+2}{2}$ possible peaks in $n(\ell)$.

Not all of these peaks will necessarily be seen: some may be relatively small. Alternately, when pairs of pulses in $\lambda(x)$ have similar spacing, the peaks in $n(\ell)$ may occur at about the same value of $\ell$ and produce a single wider or higher peak. In Fig. 6, a $\lambda(x)$ with $N = 3$ staggered pulses (at incommensurate distances) produces the full compliment of $\binom{5}{2} = 10$ peaks in $n(\ell)$.

7 Distributions with Power Law Scaling

As mentioned in the introduction, many experimental studies have reported a power law or scaling region of $n(\ell)$ over several decades, especially in the regime where $\ell$ is small and the total number of expected fragments is large [5, 13]. In terms of our approach, we ask if there
Figure 6: Three distinct but incommensurate pulses in $\lambda(x)$ can produce 10 peaks in $n(\ell)$, as calculated by Eq. 1.

is a distribution $\lambda(x)$ which would lead to a power law when $\ell$ is small in Eq. 1.

Although we developed Eq. 1 for integrable functions, we here adapt it for functions where $\int_0^1 \lambda(x) = \infty$. For example, consider $\lambda(x) = x^{-1}$. We now ignore the term

$$\lambda(\ell) \exp \left( - \int_0^\ell \lambda(t) \, dt \right)$$

since the integral diverges to infinity. That is, the infinite number of breaks expected at the left end makes it entirely unlikely to find a left-end fragment of positive length.

For the last term in Eq. 1, we have

$$\frac{1}{1 - \ell} \exp \left( - \int_{1-\ell}^1 \frac{1}{t} \, dt \right) = \frac{1}{1 - \ell} e^{\ln(1-\ell)} = 1.$$ 

Similarly the first integrand in Eq. 1 simplifies to

$$\frac{1}{s(s + \ell)} \exp \left( \ln(s/(s + \ell)) \right) = \frac{1}{(s + \ell)^2}$$

so that we obtain the power law $n(\ell) = 1 + \int_0^{1-\ell} \frac{1}{(s+\ell)^2} \, ds = \frac{1}{\ell}$. Interestingly, $1/\ell$ is a fixed point for the transformation $\lambda(x) \rightarrow n(\ell)$.

Although we have analytically obtained a power law function $n(\ell)$, the calculation seems
Figure 7: Calculated fragment distributions $n(\ell)$ for a) $\lambda(x) = Ax^{-1}$, $A = 1, 10, 100$; b) $\lambda(x) = x^{-\beta}$, $\beta = 1, 0.5, 0, -0.5, -1$.

to rely on specific aspects of the function $1/x$. In fact the transform becomes much more difficult to solve analytically with $\lambda(x) = Ax^\beta$ when either $A$ or $\beta$ is not one. Numerical calculations can offer some insight. Fig. 7a shows $n(\ell)$ for increasing values of $A$. The graphs appear to show some scaling for small $\ell$ parallel to the graph of $\frac{1}{\ell}$. The point where the function crosses $\frac{1}{\ell}$ moves to the left as $A$ increases.

We also show some examples for varying $\beta$. In Fig. 7b, we decrease $\beta$ from 1 to $-1$ by steps of 0.5. The graph flattens out for small $\ell$ as we cross the threshold from $\beta > 0$ to $\beta < 0$ into the regime where there are a finite number of fragments expected.

In Fig. 8 we increase $\beta$ from 1 to 3 in increments of 0.5. The slopes of the tails of $n(\ell)$ steepen as we increase $\beta$. Observing that $\int_0^1 \ell n(\ell) d\ell = 1$, if $n(\ell) \propto \ell^{-\beta}$ ($\beta > 0$) for small $\ell$, then we must have $\beta < 2$ for the previous integral to converge. So the slopes in Fig. 8 presumably have a limit of $-2$, although we have not attempted to prove this analytically. In an actual experiment, there is a lower bound on the lengths of producible fragments, so that a fragment size distribution may appear to have a power law with $\beta > 2$ over a number of decades. However, many experiments fall within a range that we can reproduce (e.g. $\beta \approx 1.5$ in [5]).

The near ubiquity of experimental observations of power laws would suggest a more general derivation than our singularity-dependent calculation; there may be some experiments reporting power laws where some areas of the rod almost always fragment, and produce many small fragments (i.e. where $\lambda(x)$ is approximately singular). However our assumption of a breaking profile $\lambda(x)$ for a rod breaking predominantly at one end seems too restrictive, so we examined a symmetric class of functions of the form $\lambda(x) = A|x - 0.5|^{-\beta}$. This class of functions describes a rod that breaks throughout its length, but is much more likely to break near the center. This seems to us to be a much more plausible scenario. There is some evidence that rods broken in experiments break uniformly along their lengths, but this is
mentioned only briefly in one of the papers we have found [13]. As a matter of practicality, we capped the maximum value of \( \lambda(x) \) (which also allows for the possibility of fragments of lengths greater than 0.5), as can be seen in Fig. 9. We fit parameters \( A \) and \( \beta \) to the curve \( 2\ell^{-1.5} \). The resulting functions \( \lambda(x) = 5.4926|x - 0.5|^{-1.9564} \) and \( n(\ell) \) are shown in Fig. 9.

8 Experimental Comparisons

8.1 Fitting the previously measured distribution

The main impetus for our study was the observation of peaks in the fragment distribution of brittle rods [1], for which we take the source of breaking probability to be the bending moment, proportional to the local curvature \( \kappa \) of the buckled rod [28]. We first consider data collected by breaking San Giorgio #8 spaghetti. The exact form of the buckled pasta is not known. However, the buckled pasta had a wavelength of about 0.31 (normalized by the length of the pasta – 22.5 cm with a wavelength of 7 cm). The amplitudes of the peaks decreased along the length of the pasta, often vanishing after the first few. Based on this, we conjecture that the displacement of the buckled pasta is of the form \( Ae^{-bx}\sin(6\pi x) \). The curvature of this we will call \( \kappa(x; A, b) \). The simplest possibility is that \( \lambda(x) \) is proportional to \( \kappa(x; A, b) \).

Unfortunately, this is not simply \( \lambda(x) = c\kappa(x; A, b) \), since the argument \( x \) in \( \lambda \) represents length along the curve, and here the argument \( x \) in \( \kappa \) represents a vertical distance \( y \) from the top of the pasta. To be clear, we will write \( \lambda(x) = c\kappa(y(x; A, b); A, b) \). We used MATLAB’s \textit{lsqnonlin} function to optimize the parameters \( A, b, c \) in fitting \( n(\ell) \) to the experimental data for brittle fragmentation reported in [1]. The results are shown in Fig. 10 with \( A = 0.48, b = 18.52 \) and \( c = 4.35 \). The discrepancy between curvature by distance \( y \) as opposed to
Figure 9: $\lambda(x) = 5.4926|x - 0.5|^{-1.9564}$ (above) and the resulting $n(\ell)$ plotted alongside $2\ell^{-1.5}$ (dotted line).

Figure 10: Comparison of the probability model (lines) and experimental data (represented by open circles): (a) the displacement $ae^{-bx}\sin(6\pi x)$; (b) $\lambda(x)$ as a multiple of curvature, arc length dependent; (c),(d) the calculated fragment probability function $n(\ell)$, with data points for San Giorgio #8 spaghetti, from [1]
Table 1: Parameters for the pulses in Eq. 23, used for the curves shown in Fig. 11

<table>
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<th>4</th>
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<td>$\pi/B_m$</td>
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<td>0.07</td>
<td>0.07</td>
<td>0.24</td>
<td>0.75</td>
<td>0.03</td>
</tr>
<tr>
<td>$C_m$</td>
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<td>0.11</td>
<td>0.22</td>
<td>0.37</td>
<td>0.60</td>
<td>0.87</td>
</tr>
</tbody>
</table>

arclength $x$ is visible in Fig. 10a,b where the curvatures appear to be shifted to the left of the actual peaks in the rod. A similar effect appears in the experimental data presented in Fig. 14 (where the x-axis in (a) is arclength, and the x-axis in (b) is distance). This fit does show some sensitivity to initial parameters, and should not be assumed to be identical to the actual probability density of breaking along the rod; however, it is a plausible approximation.

The damped sine wave used for $\lambda$ above does not lend itself well to including the commonly observed breaks near the base of the spaghetti (see Fig. 1). After several attempts with different functional forms for $\lambda(x)$, we found the most flexible to be a superposition of positive pulses

$$\sum_m A_m \cos^2(B_m(x - C_m)) I_{[-\pi/2,\pi/2]}(B_m(x - C_m))$$

(23)

where $I$ is the indicator or characteristic function on the interval $[-\pi/2,\pi/2]$, and $C_m$ is the center of the pulse. We expected a contribution from each of the six peaks in the buckled pasta of the experiment, and indeed found that a good fit could be obtained with only six pulses. After deciding on this functional form and free parameters for $\lambda(x)$, we again optimized these parameters in fitting $n(\ell)$. An example of this is shown in Fig. 11, with the parameters listed in Table 1. This ‘fit’ shows possible breaks near both ends.

For comparison we analyzed a part of the data set shown in Fig. 1 (because the length of the spaghetti varied in some of the experiments, only a part of this data set was used to ensure correct normalization of lengths). In these experiments, the shorter buckling wavelength allows for nine peaks. We therefore used nine pulses in fitting this data. In the fit shown in Fig. 12, however, only five of these were identified as significant.

For both experimental data sets we see a similar general pattern in $\lambda$: several pulses clustered near the impacted end ($x = 0$), with a low, wide pulse further down, and a narrow pulse near $x = 0.9$. The differences in the $\lambda(x)$ shown for the different experiments indicates that it is not possible to determine the form $\lambda(x)$ takes by merely fitting parameters. In fact this inverse problem – finding $\lambda(x)$ from a given $n(\ell)$ – does not generally have a unique solution; for example, $\lambda(x)$ and $\lambda(1-x)$ produce the same number density $n(\ell)$. To actually measure $\lambda(x)$ we performed a new experiment in which we measured the positions of breaks as well as the sizes of fragments obtained.
Figure 11: Comparison of the probability model (lines) and experimental data (represented by open circles): (a) the input to the model, represented by the breaking probability function $\lambda(x)$ plotted on a log scale; (b) the calculated fragment probability function $n(\ell)$, with data points for San Giorgio #8 spaghetti, from [1]; (c) same plot of $n(\ell)$, shown as log-log.
Figure 12: Comparison of the probability model (lines) and experimental data (represented by open circles): (a) the input to the model, represented by the breaking probability function $\lambda(x)$; (b) the calculated fragment probability function $n(\ell)$, with data points for angel hair spaghetti, from [1]; (c) same plot of $n(\ell)$, shown as log-log.

Figure 13: Two simultaneous orthogonal perspectives of the fragmentation of a brittle rod (spaghetti) under the impact of a projectile moving at 4 m/s (see text). The dynamic buckling of the rod before fragmentation leads to a pattern in the location of break, here into 8 fragments. The time from impact to fragmentation is less than 5 ms.
Figure 14: Observed patterns in the probability of breaking: a) measured probability of breaking $\lambda(x)$ over 175 events (see text); b) several cases of the initial buckling contributing to $\lambda(x)$, where the buckling plane was almost entirely in a single plane of view. The error bars are produced by considering the measurements for $\lambda(x)$ as independent experiments for different values of $x$.

### 8.2 Measuring $\lambda(x)$ experimentally

In many of the fragmentation problems studied in engineering or geophysics, data consists of only the fragment distributions $n(\ell)$. However, the experiments described in [1] allow in principle for the measurement of both the probability of breaking along the rod and the resultant fragment distribution. Here we report a preliminary measurement of $\lambda(x)$ for brittle rods in a similar experimental setup, from which we make a comparison between the fragment distribution $n(\ell)$ as calculated from Eq. 1 and the directly measured distribution.

The experiments were performed with spaghetti (San Giorgio angel hair) of length $L = 17 \pm 0.1$ cm diameter $d \approx 0.85$ mm, kept upright by holding one end in an aluminum base. A cylindrical brass weight of 51 g was dropped onto one end of the pasta, impacting with a velocity of about 4 m/s; an example of the resulting fragmentation is shown in Fig. 13. For our experimental parameters, the predicted buckling wavelength is 3.6 cm [1], while observation of the images indicate a wavelength of 3.2 cm (see Fig. 14b).

The experiments were filmed at 11,111 fps using a Phantom v5.0 high-speed digital video camera. In each image the pasta is directly visible, as is a mirror image (see Fig. 13). The angle of the mirror was controlled so that on the line where the pasta sat the images were orthogonal. This was accomplished by sighting along pins set through a piece of graph paper.
Figure 15: The open points are the measured distribution for $\lambda(x)$, while the solid points show values within the 95% confidence interval of these measurements that provide a best approximation to measured values of $n(\ell)$. 

...into a block of wood that was milled flat. Only about 8 cm of the spaghetti is visible in all of the photos, so data collection was restricted to this range. The independence assumption of our model allows us to treat the breaks in this section of the rod as if it was the whole rod: we simply normalize our data to the length of the section (8 cm) instead of by the total length $L$.

We used images of the pasta just after it broke to record the locations of breaks along the length of the spaghetti. This provided us with experimental values of both $\lambda(x)$ and $n(\ell)$ (see Figures 14 and 16). Note that the measured $n(\ell)$ is somewhat different than what would be measured by picking up the pieces afterwards - it does not include any secondary breaks due to fragment collisions after the primary event (including the brass re-impacting the unbroken end of the pasta, or curvature dynamics leading to further breaking [28]).

We compared the measured values of $n(\ell)$ with values calculated from the experimental values of $\lambda(x)$ to see if the relationship is well described by Eq. 1. Moreover, we expanded this comparison to all possible values for $\lambda(x)$ that lie within the 95% confidence interval of the experimental values. Unfortunately, even the closest fit – shown in Figures 15 and 16 – has obvious differences with measured values of $n(\ell)$. The differences do not necessarily indicate that the hypotheses leading to Eq. 1 are incorrect: There are many sources of error, including low statistical numbers and limited optical resolution. What is important is that the fragment distributions compared in Fig. 16 come from the same set of measurements, and that - with the exception of an apparent experimental peak at $\ell \simeq 0.3$ - the features of the experimentally measured fragment distribution are qualitatively captured by Eq. 1.
9 Discussion and Conclusions

The mathematical framework developed here provides an explicit answer to the primary hypothesis put forward to explain the non-power law distribution observed in [1]; it also goes much further, allowing us to map any 1D probability of breaking along the length of a rod to its resultant fragment distribution. Often in fragmentation studies in engineering or geophysical applications, one does not have access to this original probability of breaking, usually because the fragmentation process itself was not observed - this is typically true for accidents, for which determining the cause of the failure, and therefore the original stress distribution, is often an important goal. While the inverse problem is evidently complicated, it can be done numerically, as we have shown in attempting to fit the data shown in Fig. 1. Note however that while the expected total number of fragments is easily obtained in our framework, it remains to connect this quantity to the energy input into the object, via impact or other mechanisms [9].

Our derivation relies on a generalization of the Poisson process, which underlies several complex and accepted models for fragment distributions [25, 6, 22]; however, its utility for describing fragment distributions in one dimension is often dismissed [20, 35] since it is thought to produce only exponential fragment distributions. In modeling our own one dimensional experiment, we have shown the fragment distributions produced by such a Poisson process to be much more rich and complex.
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