

Review Notes on Conservative Forces,  
Stokes Theorem, and Vorticity  
Math 497C                      Fall 2001

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Some brief comments on conservative forces, specifically with gravity in mind, though these things apply to any vector field.

As in class, let's treat a force per unit volume  $\vec{f}$  as an externally applied force field, defined everywhere in our domain  $\Omega \subset \mathbb{R}^3$ . We write this as  $\vec{f}: \Omega \rightarrow \mathbb{R}^3$ . To borrow a notion from physics, the work done against this force (say in lifting something against gravity) is defined to be the force times the distance covered. Thus in moving from a point  $a$  to a point  $b$  along a curve  $C_1$  in  $\Omega$ , a total work  $W$  is done equal to

$$W = \int_{C_1[a,b]} \vec{f} \cdot d\vec{\ell}$$

The force  $\vec{f}$  is called *conservative* if the same amount of work can be gotten back by coming back from  $b$  to  $a$  along another path  $C_2$  (in fact any other path!) Thus

$$\int_{C_2[b,a]} \vec{f} \cdot d\vec{\ell} = -W$$

Thus we say that this work is "conserved". In this case, we combine these two integrals and write

$$\int_{C_1[a,b]} \vec{f} \cdot d\vec{\ell} + \int_{C_2[b,a]} \vec{f} \cdot d\vec{\ell} = 0$$

But we can use these two curves to define a *circuit*, i.e. a simply connected closed curve in  $\Omega$ . Thus we have for a conservative force that

$$\oint_{C_1+C_2} \vec{f} \cdot d\vec{\ell} = 0,$$

which is true for any closed loop in the domain. For simplicity let's restrict ourselves to coplanar curves.

Now, using Stokes Theorem (see your favorite calculus text), we obtain

$$\int_{C_1+C_2} \vec{f} \cdot d\vec{\ell} = \int_D (\nabla \times \vec{f}) \cdot \hat{n} dA = 0$$

where  $D$  is the area enclosed by the two, and  $\hat{n}$  is the normal to the plane which contains  $C_1$  and  $C_2$  (and thus  $D$ ). But this is true for any  $D \subset \Omega$ , so we can conclude that

$$\nabla \times \vec{f} = \text{curl } \vec{f} = 0$$

everywhere in  $\Omega$ .

Now let's return to the original statement of conservative, which is that the work is independent of the path

$$\int_{C_1[a,b]} \vec{f} \cdot d\vec{\ell} = \int_{C_2[a,b]} \vec{f} \cdot d\vec{\ell}$$

(note that here the integration is in the same direction for each curve). Another way of saying this is: the value of this integral depends only on the endpoints, and not on the path. This implied that  $\vec{f}$  must be the exact derivative of a function (read Fundamental Theorem of Calculus), since

$$\int_{C[a,b]} (-\nabla\chi) \cdot d\vec{\ell} = \chi(a) - \chi(b)$$

where  $\chi$  is a scalar function on our domain, i.e.  $\chi : \Omega \rightarrow \mathbb{R}$ . This is just another way of stating the vector identity result that if  $\nabla \times \vec{f} = 0$  then  $\vec{f} = -\nabla\chi$ , since  $\nabla \times (\nabla\chi) = 0$ . The function  $\chi$  is referred to as the potential for the force field.

Remark: Once we know that the curl of a field is zero, we know there is a potential function. But actually obtaining what that potential is for a given function can be difficult!

For the constant gravitational field experienced here on earth,  $\vec{f} = (0, 0, -\rho g)$ , with down being in the  $-\hat{z}$  direction. The integration is straightforward here, and we obtain the gravitational potential  $\chi = \rho g z$ . Check that by taking the gradient of this scalar function you obtain exactly the gravitational force.

All of these results are general for vector fields, and independent of their specific physical meanings. Thus for the velocity field  $\vec{u}$  of our fluid, with vorticity  $\vec{\omega} = \nabla \times \vec{u} = 0$ , we can write the velocity as the gradient of a velocity potential:

$$\vec{u} = \nabla\phi.$$