MATH 412: Sample Questions for Midterm 2

1. Say whether each of the following functions $f(x)$ is even or odd - or neither. If neither, write $f(x) = f_e + f_o$, where $f_e(x)$ is even and $f_o(x)$ is odd.

   (a) $f(x) = x^5 \sin(x^2)$

   (b) $f(x) = \exp(x^2)$

   (c) $f(x) = 5x + x^4$

   (d) $f(x) = e^x \log|x|$

2. Given the general statement of Fourier decomposition (the representation of a function as an infinite sum of orthogonal functions)

$$\phi(x) = \sum A_n X_n(x)$$

   derive an abstract expression for A_k using the inner product for this problem.

3. Find the general solution by separation of variables to the following unusual diffusion equation between 0 and L, with Dirichlet boundary conditions:

$$\begin{cases} u_t = Ktu_{xx}, & 0 < x < L, \ t > 0 \\ u(0,t) = 0, & t > 0 \\ u(L,t) = 0, & t > 0, \end{cases}$$

   Note that there is an extra time dependence in the PDE, and that no initial conditions are given.

4. Solve the following problem:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & 0 < x < \ell, \ t > 0 \\ u(0,t) = 0, & t > 0 \\ u(\ell,t) = 0, & t > 0 \\ u(x,0) = x, & \\ u_t(x,0) = 0 & \end{cases}$$

   Note that it is better to apply the second initial condition first!

5. Consider the following semi-infinite diffusion equation with Robin boundary conditions

$$\begin{cases} u_t - ku_{xx} = 0, & x \in (0, +\infty), \\ u(x,0) = \exp(-x^2), \\ u_x(0,t) - 2u(0,t) = 0 \end{cases}$$

(a) Find a change of dependent variable (i.e. change the function $u \to w$) such that this problem can be solved via the Odd Extension method for $w(x,t)$.

(b) Write an equation that relates this odd extension of the initial conditions for w to *some sort* of extension for the original problem in u.

(c) Solve the equation in (b) and thus find the appropriate initial conditions for the equivalent Cauchy problem for $u(x,t)$.

6. Given the following Fourier series expansion for $f(x) = 1$ on $(0, \pi)$

$$1 = \sum_{n \text{ odd}} \frac{4}{n\pi} \sin nx$$

use Parseval's equality to evaluate the series

$$\sum_{n \text{ odd}} \frac{1}{n^2}$$

7. Find what linear function $Ax + B$ is 'closest' in the least squares sense to $f = x^3$ on the interval [0,1]. Assume a real-valued inner product in defining the norm used to define 'least-squares sense' - in other words, find A and B which minimize the norm of the difference between $Ax + B$ and f.