MATH 231
MIDTERM EXAM
March 5, 2008

NAME __________________________
STUDENT NUMBER __________________
INSTRUCTOR __________________
SECTION NUMBER ________________

- There are 10 problems in this exam (5 multiple choice and 5 partial credit problems).
- Circle exactly one answer for the multiple choice problems.
- Present your work clearly for the partial credit problems. No credit will be given for unsupported answers.
- No calculators, books, or notes is permitted in this exam.
- Box your final answers whenever possible.
- Turn off your cell phone before the exam starts.

CHECK THE EXAMINATION BOOKLET BEFORE YOU START. THERE SHOULD BE 10 PROBLEMS ON 8 PAGES (INCLUDING THIS ONE).

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1 (5 pts). If two masses \( m_1 \) and \( m_2 \) are positioned at \( \vec{r}_1 \) and \( \vec{r}_2 \), then the center of mass is

\[
\frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}.
\]

Calculate the center of mass when \( m_1 = 2 \), \( m_2 = 3 \), \( \vec{r}_1 = (2, -7, 3) \) and \( \vec{r}_2 = (2, 3, 3) \).

(a) \( (1, 2, -1) \)

\[
\frac{2 \begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}}{2 + 3} = \frac{1}{5} \left\{ \begin{pmatrix} 4 \\ -14 \\ 6 \end{pmatrix} + \begin{pmatrix} 6 \\ 9 \\ 9 \end{pmatrix} \right\}
\]

\[
= \frac{1}{5} \begin{pmatrix} 10 \\ -5 \\ 15 \end{pmatrix} = (2, -1, 3)
\]

(b) \( (2, -1, 3) \)

(c) \( (0, 2, 3) \)

(d) \( (2, 3, 0) \)

2 (5 pts). Find a unit vector in the same direction as \( \vec{G} = 8\hat{i} + \hat{j} - 3\hat{k} \).

(a) \( \begin{pmatrix} 4/\sqrt{17} \\ 1/\sqrt{34} \\ -1/\sqrt{17} \end{pmatrix} \)

\[
|\vec{G}| = \sqrt{64 + 1 + 9} = \sqrt{74}
\]

(b) \( \begin{pmatrix} 8/\sqrt{74} \\ 1/\sqrt{74} \\ -3/\sqrt{74} \end{pmatrix} \)

\[
\vec{G}_{\text{unit}} = \left( \frac{8}{\sqrt{74}}, \frac{1}{\sqrt{74}}, \frac{-3}{\sqrt{74}} \right)
\]

(c) \( \begin{pmatrix} 4/\sqrt{14} \\ 1/\sqrt{56} \\ -3/\sqrt{56} \end{pmatrix} \)

(d) \( (1, 1, -1) \)

3 (5 pts). Which of the following best describes the surface defined by \( x^2 - 3y^2 + 7z^2 + 4 = 0 \)?

(a) An ellipsoid

(b) A sphere

(c) A hyperboloid of one sheet

(d) A hyperboloid of two sheets

\[ 3y^2 - x^2 = 7z^2 + 4 \]
4 (5 pts). Find the angle between the planes defined by $3x - 16y = 9$ and $x + 2y + (z - 5) = 1$.

(a) $\theta = \cos^{-1} \left( \frac{-29}{\sqrt{265} \sqrt{6}} \right)$

(b) $\theta = \cos^{-1} \left( \frac{35}{\sqrt{265} \sqrt{6}} \right)$

(c) $\theta = \cos^{-1} \left( \frac{-9}{\sqrt{265} \sqrt{6}} \right)$

(d) $\theta = \cos^{-1} \left( \frac{0}{\sqrt{265} \sqrt{6}} \right)$

The normal vectors of the two planes are

$\vec{n}_1 = (3, -16, 0)$

$\vec{n}_2 = (1, 2, 1)$.

Hence, the angle $\theta$ between the planes are given by

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{3 - 32 + 0}{\sqrt{9 + 256} \sqrt{1 + 4 + 1}}$$

$$= \frac{-29}{\sqrt{265} \sqrt{6}}.$$

5 (5 pts). Find the symmetric equations for the tangent line to the curve given by $\vec{r}(t) = (e^{2t} \cos t, e^{2t} \sin t, e^{2t})$ at the point $(1, 0, 1)$.

(a) $\frac{x - 1}{-1} = \frac{y}{3} = \frac{z - 1}{2}$

(b) $\frac{x - 2}{1} = \frac{z - 2}{1}, \ y = 1$

(c) $\frac{x - 1}{3} = \frac{y}{3} = \frac{z - 1}{2}$

(d) $\frac{x - 1}{2} = \frac{y}{1} = \frac{z - 1}{2}$

$\vec{r}(t) = (1, 0, 1)$ at $t = 0$.

The tangent direction at $t=0$ is given by

$$\vec{r}'(0) = \left( 2e^{2t} \cos t - e^{2t} \sin t, 2e^{2t} \sin t + 2e^{2t} \cos t, 2e^{2t} \right)_{t=0}$$

$$= (2, 1, 2).$$
6 (15 pts). Let $A, B, C$ be the three points $(3, 4, 1), (5, 2, 1),$ and $(2, 6, 1)$. Calculate the area of the triangle formed by $A, B$ and $C$.

\[
\overrightarrow{AC} = (2, 6, 1) - (3, 4, 1) \\
= (-1, 2, 0)
\]

\[
\overrightarrow{AB} = (5, 2, 1) - (3, 4, 1) \\
= (2, -2, 0)
\]

The area of the parallelogram formed by $\overrightarrow{AC}$ and $\overrightarrow{AB}$ is equal to $|\overrightarrow{AB} \times \overrightarrow{AC}|$.

\[
\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix}
2 & -2 & 0 \\
-1 & 2 & 0
\end{vmatrix} = (0, 0, 2).
\]

Hence $|\overrightarrow{AB} \times \overrightarrow{AC}| = 2$.

Because the area of the triangle is the half of that of the parallelogram,

\[
\text{Area of } \triangle ABC = \frac{1}{2} \cdot 2 = 1.
\]
7 (15 pts). Let $C$ be the curve given by $\vec{r}(t) = (t, t^2, \frac{2}{3}t^3)$. Find:

(a) $\lim_{t \to 2} \vec{r}(t)$.

(b) the arc length of $C$ from $t = 0$ to $t = 1$.

(c) the curvature at the point $(1, 1, \frac{2}{3})$.

(a) $\lim_{t \to 2} \vec{r}(t) = \left( \lim_{t \to 2} t, \lim_{t \to 2} t^2, \lim_{t \to 2} \frac{2}{3} t^3 \right) = (2, 4, \frac{16}{3})$.

(b) $\vec{r}'(t) = (1, 2t, 2t^2)$.

$|\vec{r}'(t)| = \sqrt{1 + 4t^2 + 4t^4} = \sqrt{(1 + 2t^2)^2} = 1 + 2t^2$.

So the arc-length from $t=0$ to $t=1$ is

$S = \int_0^1 |\vec{r}'(t)| \, dt = \int_0^1 (1 + 2t^2) \, dt = \left[ t + \frac{2}{3} t^3 \right]_0^1 = \frac{5}{3}$.

(c) $\vec{r}'(t) = (1, 1, \frac{2}{3}) \quad \text{at} \quad t = 1$.

$\vec{r}'(1) = (1, 2, 2)$, \hspace{1cm} |\vec{r}'(1)| = \sqrt{1 + 4 + 4} = 3$.

$\vec{r}''(1) = (0, 2, 4t)_{t=1} = (0, 2, 4)$.

$\therefore \vec{r}' \times \vec{r}''(1) = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 2 & 4 \end{vmatrix} = (4, -4, 2)$.

Thus, $K(1) = \frac{|\vec{r}' \times \vec{r}''(1)|}{|\vec{r}'|^3} = \frac{\sqrt{16 + 16 + 4}}{27} = \frac{6}{27} = \frac{2}{9}$.
8 (15 pts). The point \( P = (-2, 0, 4) \) is on the sphere \( S \) defined by \((x+2)^2 + (y-3)^2 + z^2 = 25\). Find the point \( Q \) on the sphere \( S \) that is exactly opposite to \( P \) (in other words, the line joining \( P \) and \( Q \) passes through the center of the sphere \( S \)).

Let \( Q = (a, b, c) \). Since the center of the sphere \( C = (-2, 3, 0) \) is the midpoint between \( P \) and \( Q \), we have

\[
\frac{P + Q}{2} = C,
\]

that is,

\[
\frac{(-2+a, b, 4+c)}{2} = (-2, 3, 0)
\]

\(\iff\)

\( (-2+a, b, 4+c) = (-4, 6, 0) \)

\(\iff\)

\((a, b, c) = (-2, 6, -4) \).
9 (15 pts). Let \( C \) be the curve given by \( \mathbf{r}(t) = (\cos t, \cos t, \sqrt{2}\sin t) \).

(a) Find the unit normal vector \( \mathbf{N} \) at the point \((1, 1, 0)\).

(b) Find an equation of the normal plane at the point \((1, 1, 0)\).

(a) \[ \mathbf{r}' = (-\sin t, -\sin t, \sqrt{2}\cos t). \]

\[ |\mathbf{r}'| = \sqrt{\sin^2 t + \sin^2 t + 2\cos^2 t} = \sqrt{2}. \]

\[ \therefore \quad \mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|} = \frac{1}{\sqrt{2}} (-\sin t, -\sin t, \sqrt{2}\cos t). \]

\[ \mathbf{T}' = \frac{1}{\sqrt{2}} (\cos t, \cos t, -\sqrt{2}\sin t). \]

Now, \( \mathbf{r}(t) = (1, 1, 0) \) at \( t = 0 \). And

\[ \mathbf{T}'(0) = \frac{1}{\sqrt{2}} (1, 1, 0), \quad |\mathbf{T}'(0)| = \frac{1}{\sqrt{2}} |(1, 1, 0)| = \frac{1}{\sqrt{2}} \sqrt{2} = 1. \]

Therefore

\[ \mathbf{N}(0) = \frac{\mathbf{T}'(0)}{|\mathbf{T}'(0)|} = \mathbf{T}'(0) = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0). \]

(b) The normal plane is normal to the unit tangent vector \( \mathbf{T} \). Hence, the equation for the normal plane at \( t = 0 \) is

\[ \mathbf{T}'(0) \cdot (\mathbf{X} - \mathbf{r}(0)) = 0 \]

\[ \iff \quad \frac{1}{\sqrt{2}} (0, 0, \sqrt{2}) \cdot (x-1, y-1, z) = 0 \]

\[ \iff \quad 3 = 0. \]
10 (15 pts). The acceleration of a particle in motion is $\vec{a}(t) = -10\vec{j}$. Its initial position and velocity vector are $\vec{r}(0) = (0, 100, 0)$ and $\vec{v}(0) = \vec{i} + \vec{j}$. Find the position and velocity vector at $t = 2$.

\[
\vec{v}(t) = \int_0^t \vec{a}(t') \, dt' + \vec{v}(0)
\]
\[
= \int_0^t (0, -10, 0) \, dt' + (1, 1, 0)
\]
\[
= (0, -10t', 0) \bigg|_0^t + (1, 1, 0) = (0, -10t, 0) + (1, 1, 0)
\]
\[
= (1, -10t, 0).
\]

\[
\vec{r}(t) = \int_0^t \vec{v}(t') \, dt' + \vec{r}(0)
\]
\[
= \int_0^t (1, -10t' + 1, 0) \, dt' + (0, 100, 0)
\]
\[
= (t', -5t'^2 + t', 0) \bigg|_0^t + (0, 100, 0)
\]
\[
= (t, -5t^2 + t, 0) + (0, 100, 0)
\]
\[
= (t, -5t^2 + t + 100, 0)
\]

Therefore, $\vec{v}(2) = (1, -19, 0)$ and $\vec{r}(2) = (2, 82, 0)$. \qed