• There are 16 problems in this exam (8 multiple choice and 8 partial credit problems).
• Circle exactly one answer for the multiple choice problems.
• Present your work clearly for the partial credit problems. **No credit will be given for unsupported answers.**
• No calculators, books, or notes is permitted in this exam.
• Box your final answers whenever possible.
• Turn off your cell phone before the exam starts.

**CHECK THE EXAMINATION BOOKLET BEFORE YOU START. THERE SHOULD BE 16 PROBLEMS ON 13 PAGES (INCLUDING THIS ONE).**

<table>
<thead>
<tr>
<th>Question</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 – 8</td>
<td></td>
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<tr>
<td>9</td>
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1. (5 pts) Find the angle between the following two vectors $u$ and $v$.

$$u = \langle 2, -2, 1 \rangle, \quad v = \langle 1, -4, -1 \rangle$$

a) 0  
b) $\frac{\pi}{6}$  
c) $\frac{\pi}{4}$  
d) $\frac{\pi}{3}$

**Solution.** c) $\Box$

2. (5 pts) The spherical coordinates of a point are $(\sqrt{3}, \pi/3, \pi/6)$. Find the rectangular coordinates of the point.

a) $\left( \frac{\sqrt{3}}{4}, \frac{3}{4}, \frac{\sqrt{3}}{2} \right)$  
b) $\left( \frac{3}{4}, \frac{3}{4}, \frac{3}{2} \right)$  
c) $\left( \frac{3}{4}, \frac{3\sqrt{3}}{4}, \frac{3}{2} \right)$  
d) $\left( \frac{\sqrt{3}}{4}, \frac{3}{4}, \frac{3}{2} \right)$

**Solution.** d) $\Box$
3. (5 pts) Find symmetric equations for the tangent line to the curve
\[ \mathbf{r}(t) = (t + 1)\mathbf{i} + (t^2 + 1)\mathbf{j} + e^t\mathbf{k} \] at \((1, 1, 1)\)

a) \(x - 1 = y - 1 = z - 1\)

b) \(x - 1 = z - 1, y = 1\)

c) \(x = z, y = 0\)

d) \(x = y = z\)

_Solution._ b)

---

4. (5 pts) A particle moves with position function
\[ \mathbf{r}(t) = < \cos t, e^{-t}, \sin t >. \]

Find the acceleration of the particle.

a) \(< \cos t, e^{-t}, \sin t >\)

b) \(< -\sin t, -e^{-t}, \cos t >\)

c) \(< -\sin t, e^{-t}, -\cos t >\)

d) \(< -\cos t, e^{-t}, -\sin t >\)

_Solution._ d)
5. (5 pts) Find the domain of the following function.

\[ f(x, y) = \sqrt{4 - x^2 - y^2} \]

a) \( \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \geq 4\} \)

b) \( \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 > 4\} \)

c) \( \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\} \)

d) \( \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4\} \)

**Solution.** c) □

6. (5 pts) Find the following limit.

\[ \lim_{(x, y) \to (0,0)} \frac{x}{x + y^2} \]

a) 0

b) 1

c) \( \infty \)

d) Not Exist

**Solution.** d) □
7. (5 pts) Let \( f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \). Find \( \frac{\partial f}{\partial z} \).

a) \( \frac{z}{\sqrt{x^2 + y^2 + z^2}} \)

b) \( \frac{x + y + z}{\sqrt{x^2 + y^2 + z^2}} \)

c) \( \frac{xyz}{\sqrt{x^2 + y^2 + z^2}} \)

d) 1

Solution. a) □

8. (5 pts) Find the gradient vector \( \nabla f \) of the following function.

\[ f(x, y) = 2\sqrt{x} - y^3 \] at (1, 3)

a) \( <1, -27> \)

b) \( <2, -27> \)

c) −26

d) 26

Solution. a) □
9. (15 pts) Consider the curve given by
\[ \mathbf{r}(t) = \langle \frac{1}{3}t^3, \frac{1}{2}t^2, t \rangle \]
a. Find the unit tangent vector.
b. Find \( \mathbf{r}'(t) \times \mathbf{r}''(t) \).
c. Find the curvature at the point \( \left( \frac{1}{3}, \frac{1}{2}, 1 \right) \).

Solution. a.
\[ \mathbf{r}'(t) = \langle t^2, t, 1 \rangle \]
Thus
\[ \mathbf{T}(1) = \frac{\langle t^2, t, 1 \rangle}{\sqrt{t^2 + t + 1}} \]
b.
\[ \mathbf{r}'(t) = \langle t^2, t, 1 \rangle \]
\[ \mathbf{r}''(t) = \langle 2t, 1, 0 \rangle \]
So
\[ \mathbf{r}'(t) \times \mathbf{r}''(t) = \langle -1, 2t, -t^2 \rangle \]
c. The point \( \left( \frac{1}{3}, \frac{1}{2}, 1 \right) \) corresponds to \( t = 1 \). So,
\[ \mathbf{r}'(1) = \langle 1, 1, 1 \rangle \]
\[ \mathbf{r}'(1) \times \mathbf{r}''(1) = \langle -1, 2, -1 \rangle \]
So,
\[ \kappa = \frac{\left| \mathbf{r}'(1) \times \mathbf{r}''(1) \right|}{\left| \mathbf{r}'(1) \right|^3} = \frac{\sqrt{1 + 4 + 1}}{\sqrt{1 + 1 + 1}} = \frac{\sqrt{6}}{3\sqrt{3}} = \frac{\sqrt{2}}{3} \]
\[ \square \]
10. (10 pts) Let C be a space curve given by

\[ \mathbf{r}(t) = \langle e^t, \cos t, \sin t \rangle \]

Find the arc length of C from \( t = 0 \) to \( t = 1 \).

Solution.

\[ \mathbf{r}'(t) = \langle e^t, -\sin t, \cos t \rangle \]

\[ |\mathbf{r}'(t)| = \sqrt{e^{2t} + \sin^2 t + \cos^2 t} = \sqrt{e^{2t} + 1} \]

So

\[ \int_0^1 \sqrt{e^{2t} + 1} \, dt = \left. \frac{1}{3} (e^{2t} + 1)^{3/2} \right|_0^1 = \frac{(e^2 + 1) \sqrt{e^2 + 1} - 2\sqrt{2}}{3} \]
11. (10 pts) If $xz^2 + y^2z = e^{xyz}$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution.

\[ z^2 + 2xz \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial x} = (yz + xy \frac{\partial z}{\partial x})e^{xyz} \]

\[ (2xz + y^2 - yze^{xyz}) \frac{\partial z}{\partial x} = yze^{xyz} - z^2 \]

\[ \frac{\partial z}{\partial x} = \frac{yze^{xyz} - z^2}{2xz + y^2 - yze^{xyz}} \]

\[ 2xz \frac{\partial z}{\partial y} + 2yz + y^2 \frac{\partial z}{\partial y} = (xz + xy \frac{\partial z}{\partial y})e^{xyz} \]

\[ (2xz + y^2 - yze^{xyz}) \frac{\partial z}{\partial y} = xze^{xyz} - 2yz \]

\[ \frac{\partial z}{\partial y} = \frac{xze^{xyz} - 2yz}{2xz + y^2 - yze^{xyz}} \]

□
12. (15 pts) Let \( f(x, y) = x \cos(\pi y) \).

   a. Calculate \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \).

   b. Write the linearization \( L(x, y) \) of \( f(x, y) \) with respect to the point \((0,1)\).

   c. Use \( L(x, y) \) from part b to approximate \( 0.02 \cos(0.98 \pi) \).

Solution. a.

\[
\frac{\partial f}{\partial x} = \cos(\pi y) \\
\frac{\partial f}{\partial y} = -\pi x \sin(\pi y)
\]

b. Since \( \frac{\partial f}{\partial x} = -1 \) and \( \frac{\partial f}{\partial y} = 0 \) at \((0,1)\),

\[
L(x, y) = -(x - 0) + 0(y - 1) + 0 = -x
\]

c. By part b,

\[
L(0.02, 0.98) = -0.02
\]
13. (15 pts) Let \( f(x, y) = 2xy + 4\sqrt{1+y} \).

a. Find the directional derivative of \( f \) in the direction of \((\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\) at the point \((1, 3)\).

b. Find the maximum rate of change of \( f \) at the point \((1, 3)\).

c. Find the direction in which the maximum rate of change of \( f \) at the point \((1, 3)\) occurs.

Solution.  

a. 
\[
f_x = 2y, \quad f_y = 2x + \frac{2}{\sqrt{1+y}}
\]

So 
\[
D_{\left< \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right>} f(1, 3) = \left< 6, 3 \right> \cdot \left< \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right> = 6\sqrt{2}
\]

b. The maximum rate of change at the point is
\[
|\nabla f| = |\left< 6, 3 \right>| = \sqrt{36 + 9} = 3\sqrt{5}
\]

c. It is the direction of \( \left< 6, 3 \right> \). 

\[\square\]
14. (15 pts) Let \( f(x, y, z) = z - \cos x \sin y \).

a. Calculate \( \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \) at \((x, y, z) = (0, \pi, 1)\).

b. Write an equation for the tangent plane \( P \) of the level surface \( f(x, y, z) = 1 \) at \((x, y, z) = (0, \pi, 1)\).

c. Write an equation for the line passing through the point \((0, \pi, 1)\) and normal to the tangent plane \( P \) described in part b, i.e., find the normal line.

Solution. a.

\[
\begin{align*}
x(x) &= \sin x \sin y, \\
y(y) &= -\cos x \cos y, \\
z(z) &= 1
\end{align*}
\]

b. At the point, \( f_x = 0, f_y = 1, f_z = 1 \), so the tangent plane is

\[
\begin{align*}
y - \pi + z - 1 &= 0 \\
y + z &= \pi + 1
\end{align*}
\]

c. The line is

\[
x = 0, y - \pi = z - 1
\]

□
15. (15 pts) Consider
\[ f(x, y) = 3x^2y - 3x^2 + y^3 - 3y^2 + 2 \]

a. Find all critical points of \( f \).

b. Find local maximum, minimum values, and saddle points of \( f \).

c. Find the absolute maximum and minimum values of \( f \) on the closed triangular region in the \( xy \)-plane with vertices \((0, 0), (1, 0), (1, 1)\).

**Solution.**

a.
\[ f_x = 6xy - 6x = 6x(y - 1) = 0 \]
\[ f_y = 3x^2 + 3y^2 - 6y = 0 \]

So, when \( x = 0 \)
\[ 3y^2 - 6y = 4y(y - 2) = 0 \]
\[ y = 0, \quad 2 \]

and when \( y = 1, \)
\[ 3x^2 - 3 = 3(x + 1)(x - 1) = 0 \]
\[ x = 1, \quad -1 \]

Thus the critical points are \((0, 0), (0, 2), (1, 1), (-1, 1)\).

b.
\[ f_{xx} = 6y - 6, \quad f_{yy} = 6y - 6, \quad f_{xy} = 6x, \]
\[ D = (6y - 6)^2 - (6x)^2 \]

So,
\[ (0, 0), \quad D(0, 0) = 36 > 0, f_{xx}(0, 0) = -6 \]
\[ (0, 2), \quad D(0, 2) = 36 > 0, f_{xx}(0, 2) = 6 \]
\[ (\pm 1, 1), \quad D(1, 1) = -36 < 0 \]

Hence \((\pm 1, 1)\) are saddle points, \( f(0, 0) = 2 \) is a local maximum and \( f(0, 2) = -2 \) is a local minimum.

c.
\[ D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\} \]

So, for the points \((x, 0)\) with \( 0 \leq x \leq 1, \)
\[ f(x, 0) = -3x^2 + 2, \quad -1 \leq f(x, y) \leq 2, \]

for the points \((1, y)\) with \( 0 \leq y \leq 1, \)
\[ f(0, y) = y^3 - 3y^2 + 3y - 1 = (y - 1)^3, \quad -1 \leq f(x, y) \leq 0 \]
and for the points \((x, x)\) with \(0 \leq x \leq 1\),
\[
f(x, x) = 3x^3 - 3x^2 + x^4 - 3x^2 + 2 = 4x^3 - 6x^2 + 1 = (x - 1)^2(2x + 1),
\]
\[
0 \leq f(x, y) \leq 1
\]

Thus the absolute maximum is \(2\) at \((0, 0)\) and the absolute minimum is \(-1\) at \((1, 0)\). □
16. (15 pts) Use Lagrange multipliers to find the dimensions of a rectangular box with largest volume if the sum of its length and girth (perimeter of a cross-section perpendicular to the length) is 48 in.

(No credit will be given if Lagrange multipliers are not used.)

Solution. Note $x, y, z > 0$.

\[ V = xyz, \quad x + 2y + 2z = 48 \]

So,

\[ yz = \lambda, \quad xz = 2\lambda, \quad xy = 2\lambda, \]

which imply

\[ 2yz = xz, \quad 2y = x \]
\[ xz = xy, \quad y = z \]

Thus

\[ 2y + 2y + 2y = 6y = 48, \quad y = z = 8, x = 16. \]

The volume is 1024. To show that this is the maximum, try a different values for $x, y, z$, say $x = 44, y = 1, z = 1$. Then we get $V = 44$. Thus $(16, 8, 8)$ gives an absolute maximum of $V$ subject to $x + 2y + 2z = 48$.

\[ \square \]