14.4 problem 26 A gun is fired with angle of elevation 30°. What is the muzzle speed if the maximum height of the shell is 500m?

We wish to find the muzzle speed \( v_m = |\vec{v}(0)| \). We do need to pick a coordinate system. We can do this in just two dimensions, fixing \( \hat{j} \) to point up and with the motion of the shell in the positive \( \hat{i} \) direction, with the gun at the origin. Our strategy will be to find an expression for \( \vec{r}(t) \) in terms of \( v_m \), maximize the y component of this (Calc I style) and solve for \( v_m \) by setting the maximum equal to 500.

The things we know are:

I The acceleration of the shell is (neglecting air resistance) entirely due to gravity. This gives us
\[
\vec{a}(t) = \langle 0, -9.8 \rangle
\]
Note that we picked our unit of measure to match the units set by the problem for height.

II We can calculate \( \vec{v}(0) \) in terms of the \( v_m \) because we know the elevation of the gun.
\[
\vec{v}(0) = \langle v_m \cos(30°), v_m \sin(30°) \rangle = \langle \frac{\sqrt{3}}{2} v_m, \frac{1}{2} v_m \rangle
\]

III Because of the way we set up the coordinate system, we also have
\[
\vec{r}(0) = \langle 0, 0 \rangle
\]

This is enough to find an expression for \( \vec{r}(t) \). We first calculate
\[
\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u) \, du = \langle \frac{\sqrt{3}}{2} v_m, \frac{1}{2} v_m \rangle + \langle 0, -9.8t \rangle
\]
and then
\[
\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u) \, du = \langle \frac{\sqrt{3}}{2} v_m t, \frac{1}{2} v_m t - 4.9t^2 \rangle
\]
The y component of this is \( \frac{1}{2} v_m t - 4.9t^2 \). To maximize this we set the derivative equal to zero: \( \frac{1}{2} v_m - 9.8t = 0 \). Solving this we find that the shell reaches its maximum height at \( t = \frac{v_m}{19.6} \). By inserting this back into our expression for the y component, we find a maximum height of \( \frac{v_m^2}{39.2} - \frac{v_m^2}{78.4} = \frac{v_m^2}{78.4} \). Now we can set 500 = \( \frac{v_m^2}{78.4} \) and solve to find
\[ v_m = \sqrt{78.4 \times 500} \approx 198m/s \]