14.4 problem 26 A gun is fired with angle of elevation 30°. What is the muzzle speed if the maximum height of the shell is 500m?

We wish to find the muzzle speed $v_m = |\vec{v}(0)|$. We do need to pick a coordinate system. We can do this in just two dimensions, fixing $\hat{\jmath}$ to point up and with the motion of the shell in the positive $\hat{\imath}$ direction, with the gun at the origin. Our strategy will be to find an expression for $\vec{r}(t)$ in terms of v_m, maximize the y component of this (Calc I style) and solve for v_m by setting the maximum equal to 500.
The things we know are:

I The acceleration of the shell is (neglecting air resistance) entirely due to gravity. This gives us

$$\vec{a}(t) = \langle 0, -9.8 \rangle$$

Note that we picked our unit of measure to match the units set by the problem for height.

II We can calculate $\vec{v}(0)$ in terms of the v_m because we know the elevation of the gun.

$$\vec{v}(0) = \langle v_m \cos(30°), v_m \sin(30°) \rangle = \langle \frac{\sqrt{3}}{2} v_m, \frac{1}{2} v_m \rangle$$

III Because of the way we set up the coordinate system, we also have

$$\vec{r}(0) = \langle 0, 0 \rangle$$

This is enough to find an expression for $\vec{r}(t)$. We first calculate

$$\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u)\, du = \langle \frac{\sqrt{3}}{2} v_m, \frac{1}{2} v_m \rangle + \langle 0, -9.8t \rangle$$

and then

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u)\, du = \langle \frac{\sqrt{3}}{2} v_m t, \frac{1}{2} v_m t - 4.9t^2 \rangle$$

The y component of this is $\frac{1}{2} v_m t - 4.9t^2$. To maximize this we set the derivative equal to zero: $\frac{1}{2} v_m - 9.8t = 0$. Solving this we find that the shell reaches its maximum height at $t = \frac{v_m}{19.6}$. By inserting this back into our expression for the y component, we find a maximum height of $\frac{v_m^2}{39.2} - \frac{v_m^2}{78.4} = \frac{v_m^2}{78.4}$. Now we can set $500 = \frac{v_m^2}{78.4}$ and solve to find

$$v_m = \sqrt{78.4 * 500} \approx 198 m/s$$