Math 557
Mathematical Logic
Some Homework Problems

Fall 1998

1. Consider the formula

$$(*) \qquad\qquad (p \wedge \sim q) \supset (\sim p \vee r)\ .$$

   (a) Exhibit a formation sequence for $(*)$.
   (b) List the immediate subformulas of $(*)$, their immediate subformulas, etc., i.e. all subformulas of $(*)$.
   (c) Calculate the degrees of $(*)$ and its subformulas.
   (d) Display the formation tree of $(*)$.
   (e) Write the formula $(*)$ according to various notation systems:
      i. F_0–F_4 (Smullyan page 5).
      ii. F_0'–F_2' (page 6).
      iii. F_0''–F_2'' (pages 6–7).
      iv. ordered pairs and triples (page 7).
      v. Polish notation.
      vi. reverse Polish notation.

2. Let X and Y be formulas.
   (a) Show that X is a tautology iff $\sim X$ is not satisfiable.
   (b) Show that X is satisfiable iff $\sim X$ is not a tautology.
   (c) Show that X is truth-functionally implies Y iff $X \supset Y$ is a tautology.
   (d) Show that X is truth-functionally equivalent to Y iff $X \leftrightarrow Y$ is a tautology.

3. Prove the DeMorgan laws:
   (a) $\sim (X \wedge Y) \simeq\ \sim X \vee \sim Y$.
   (b) $\sim (X \vee Y) \simeq\ \sim X \wedge \sim Y$.

4. Show that if X is part of Y and if $X \simeq X_1$ and if Y_1 is obtained from Y by replacing X by X_1, then $Y \simeq Y_1$. (Smullyan, exercise 1, page 13.)

5. Show that any formula is equivalent to a formula in disjunctive normal form. (Exercise 3, page 13.)

6. Show that $\wedge$ is definable from $\sim$, $\vee$, etc. (Exercise 4, page 14.)

7. Show that all binary connectives are definable from the Sheffer stroke $|$, and from joint denial $\downarrow$. (Exercise 5, page 14.)

8. Exercises in writing formulas of predicate calculus.

   (a) Assume the following predicates:

       Hx: x is a human
       Cx: x is a car
       Tx: x is a truck
       Dxy: x drives y

   Write formulas representing the obvious assumptions: no human is a car, no car is a truck, humans exist, cars exist, only humans drive, only cars and trucks are driven, *etc.*

   (b) Write formulas representing the following statements:

      i. Everybody drives a car or a truck.
      ii. Some people drive both.
      iii. Some people don't drive either.
      iv. Nobody drives both.

   (c) Assume in addition the following predicate:

       Ixy: x is identical to y

   Write formulas representing the following statements:

      i. Every car has at most one driver.
      ii. Every truck has exactly two drivers.
      iii. Everybody drives exactly one vehicle (car or truck).

   (d) Assume the following predicates:

       Ixy: $x = y$
       $Pxyz$: $x \cdot y = z$

   Write formulas representing the axioms for a group: axioms for equality, existence and uniqueness of products, associative law, existence of an identity element, existence of inverses.

9. (a) Formulate the following argument as a tautology.

   If it has snowed, it will be poor driving. If it is poor driving, I will be late unless I start early. Indeed, it has snowed. Therefore, I must start early to avoid being late.

(b) Use the tableau method to prove this tautology.

(c) Brown, Jones, and Smith are suspected of a crime. They testify as follows:

Brown: Jones is guilty and Smith is innocent.

Jones: If Brown is guilty then so is Smith.

Smith: I'm innocent, but at least one of the others is guilty.

Let B, J, and S be the statements "Brown is innocent," "Jones is innocent," "Smith is innocent". Express the testimony of each suspect as a propositional formula. Write a truth table for the three testimonies.

(d) Use this truth table to answer the following questions:

i. Are the three testimonies consistent?

ii. The testimony of one of the suspects follows from that of another. Which from which?

iii. Assuming everybody is innocent, who committed perjury?

iv. Assuming all testimony is true, who is innocent and who is guilty?

v. Assuming that the innocent told the truth and the guilty told lies, who is innocent and who is guilty?