Singular and Nearly Singular SPD Systems Arising from Discretization for DEs

(Research Project for Math 497A, Spring 2008)

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1 Motivation: a simple example

$$A_0 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \in R(A_0)$$

The Gauss-Seidel method for $(A_0 + \varepsilon I)x = b$ (with $x_0 = b$ and stopping criterion: $\|Ax^k - b\| \leq 10^{-8}$):

ε	# of iterations
1.	18
10^{-1}	100
10^{-2}	852
10^{-3}	6982
10^{-4}	54470
0. [singular case]	2

2 More general nearly singular systems

Given A_0 (semi-definite) and D (SPD), consider

$$(A_0 + \varepsilon D)u = f$$

Facts:

- Most methods (such as CG, MG, and DD) converge for any $\varepsilon \geq 0$.

- Convergence becomes slower when ε gets smaller, and, in particular slower than for $\varepsilon = 0$.

Questions:

- Why?
- How to fix the problems?

3 Projects

1. Read relevant papers on SPD problems
2. Extend the results to non-SPD
3. Nonlinear problems
4. Applications to anisotropic diffusion problems
5. Applications to QCD problem